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# Ricci solitons on spacetimes with the spatially homogeneous rotating metrics

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Abstract. This paper investigates Ricci solitons on spacetimes equipped with spatially homogeneous rotating metrics. We systematically classify all vector fields that generate Ricci solitons within this geometric framework. Special attention is devoted to identifying the precise conditions under which these vector fields assume gradient form. Furthermore, we establish that every vector field associated with a Ricci soliton in this setting exhibits conformal Killing properties, revealing a fundamental connection between these geometric structures. Finally, we present an example of the desired space and examine the Ricci solitons on it.

## 1. Introduction

A fundamental issue within the framework of general relativity involves the derivation of spacetime geometries that fulfill the Einstein field equations. To address this complex problem, researchers commonly implement geometric constraints through the imposition of symmetry conditions or the specification of particular matter configurations. These symmetries are mathematically represented by vector fields that maintain invariance of specific tensor structures. In gravitational physics, the most extensively examined symmetries include Killing, homothetic, and conformal vector fields, alongside Noether symmetries. These symmetry generators bear a significant relationship to conservation principles and prove essential for investigating spacetime singularities, analyzing cosmological evolution, and facilitating the dimensional reduction of Einstein's equations through Lagrangian formulations [1, 14, 5].

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The notion of Ricci flow, initially introduced by Hamilton [12], constitutes a fundamental concept in differential geometry, serving as a mechanism for evolving metrics according to their Ricci curvature. On a pseudo-Riemannian manifold $(M, g(t))$, this evolution is governed by the equation

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{S}_{g(t)},$$

where $\mathrm{S}_{g(t)}$ represents the Ricci curvature tensor associated with the metric $g(t)$. In contrast, Ricci solitons emerge as a distinct geometric entity. Rather than characterizing symmetries of a fixed metric, a Ricci soliton comprises a metric $g$ paired with a vector field $X$ that satisfies

$$\mathcal{L}_X g + S = \lambda g, \tag{1}$$

where $\mathcal{L}_X g$ denotes the Lie derivative of the metric, $S$ is the Ricci tensor, and $\lambda$ is a constant. The soliton is categorized as shrinking ($\lambda > 0$), steady ($\lambda = 0$), or expanding ($\lambda < 0$), based on the sign of $\lambda$. When $X = 0$, the Ricci soliton degenerates to an Einstein metric, whereas if $X = \nabla h$ for some smooth potential function $h$, the pair $(M, g)$ forms a gradient Ricci soliton.

Ricci solitons naturally emerge in the analysis of Ricci flow and are crucial for comprehending singularity development and the geometric evolution of manifolds under this flow. Although initially examined within Riemannian geometry, the theory has subsequently been generalized to pseudo-Riemannian and Lorentzian contexts, particularly in relation to spacetime models in general relativity [9, 17]. The physical and mathematical relevance of Ricci flow and Ricci solitons has been extensively documented in numerous studies [6, 13, 19, 23, 25], owing to their profound connections with self-similar solutions and gravitational dynamics. When the soliton equation simplifies to an Einstein condition, the metric satisfies the Einstein field equations with a cosmological constant, yielding physically significant cosmological models.

In recent years, multiple efforts have been directed toward classifying Ricci solitons across diverse spacetimes. These include investigations of plane-symmetric static geometries [15], LRS Bianchi type V models [18, 24], static spherically symmetric spacetimes, and examinations within modified gravitational frameworks such as $f(T)$, $f(R)$, and $f(R, T)$ gravity [11, 22]. Additional research has explored soliton structures in more specialized geometric contexts, including Kenmotsu manifolds and almost paracontact manifolds, or their relationship with curvature inheritance symmetries [4]. The majority of these investigations depend on direct integration of the soliton equations – a methodology that frequently proves arduous, susceptible to errors, and potentially overlooks special solutions that manifest only under specific differential constraints.

In the present work, we focus on spacetimes equipped with spatially homogeneous rotating (SHR) metrics, which represent a class of Lorentzian manifolds of physical interest. An example of a physically relevant SHR metric is the Gödel–Friedmann metric, which describes a rotating universe and serves as a key example in the study of causality violation. Our aim is to analyze the conditions under which such space-times admit Ricci solitons, to classify the corresponding vector fields, and to determine when these solitons are of gradient type. Furthermore, we explore the relationships among Killing vector fields that arise naturally in this context.

## 2. Preliminaries

Let us consider a SHR spacetime $(M, g)$ with coordinate system $(x, y, z, w)$ endowed with the metric

$$g = -dx^2 + dy^2 + A(y)dz^2 + 2B(y)dxdz + dw^2, \tag{2}$$

where $A(y)$ and $B(y)$ are nowhere zero smooth functions of the coordinate $y$ and $A + B^2 \neq 0$. The metric structure has been systematically studied in the context of spatially homogeneus and cylindrically symmetric spacetimes [20]. This metric represents five spacetimes [10, 16], which can be achieved by choosing particular values of the metric functions A and B. In [21], Shabbir et al. studied SHR spacetimes according to their teleparallel Killing vector fields using a direct integration technique. Let

$$\partial_1 = \partial_x = \frac{\partial}{\partial x}, \;\; \partial_2 = \partial_y = \frac{\partial}{\partial y}, \;\; \partial_3 = \partial_z = \frac{\partial}{\partial z}, \;\; \partial_4 = \partial_w = \frac{\partial}{\partial w}.$$

The matrix representation of the metric $g$ and its inverse is as follows:

$$g = \begin{pmatrix} -1 & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ B & 0 & A & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g^{-1} = \frac{-1}{A + B^2} \begin{pmatrix} A & 0 & -B & 0 \\ 0 & -(A + B^2) & 0 & 0 \\ -B & 0 & -1 & 0 \\ 0 & 0 & 0 & -(A + B^2) \end{pmatrix}.$$

From the formula $\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ for the Christoffel symbol we have $\Gamma^1_{12} = \frac{1}{2} g^{13} \partial_2 g_{13} = \frac{BB'}{2(A+B^2)}$, which is consistent with the general formula for $\nabla_{\partial_x} \partial_y$ where the $x$-component is $\frac{BB'}{2(A+B^2)}$. By calculating the remaining Christoffel symbols similar to above, we will have the following lemma.

**Lemma 1.** *The Levi-Civita connection $\nabla$ linked to the Lorentzian metric $g$ is defined by*

$$\nabla_{\partial_x}\partial_y = \frac{BB'}{2(A+B^2)}\,\partial_x + \frac{B'}{2(A+B^2)}\,\partial_z, \qquad \nabla_{\partial_x}\partial_z = -\frac{B'}{2}\,\partial_y,$$
$$\nabla_{\partial_y}\partial_z = \frac{BA'-AB'}{2(A+B^2)}\,\partial_x + \frac{BB'+A'}{2(A+B^2)}\,\partial_z, \qquad \nabla_{\partial_z}\partial_z = -\frac{A'}{2}\,\partial_y.$$

Using the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ for any vector fields $X, Y, Z$, we compute

$$\begin{aligned} R(\partial_z,\partial_x)\partial_x &= \nabla_{\partial_z}\nabla_{\partial_x}\partial_x - \nabla_{\partial_x}\nabla_{\partial_z}\partial_x - \nabla_{[\partial_z,\partial_x]}\partial_x \\ &= -\nabla_{\partial_x}(-\frac{B'}{2}\,\partial_y) = \frac{B'}{2}\nabla_{\partial_x}\partial_y \\ &= \frac{BB'^2}{4(A+B^2)}\partial_x + \frac{B'^2}{4(A+B^2)}\,\partial_z. \end{aligned}$$

Similarly, we obtain the non–vanishing components of the curvature tensor as follows:

$$\begin{aligned} R(\partial_y,\partial_x)\partial_y &= \frac{2B^3B''-B^2B'^2+2ABB''+AB'^2-BA'B'}{4(A+B^2)^2}\partial_x \\ &\quad + \left(\frac{B''}{2(A+B^2)} - \frac{B'(2BB'+A')}{4(A+B^2)^2}\right)\partial_z, \\ R(\partial_z,\partial_y)\partial_y &= -\left(\frac{(AB'-BA')(A'+2BB')}{4(A+B^2)^2} + \frac{BA''-AB''}{4(A+B^2)}\right)\partial_x \\ &\quad - \frac{-B^2B'^2+AB'^2+2B^2A''-3BA'B'+2AA''-A'^2}{4(A+B^2)}\partial_z \\ &\quad - \frac{BB''}{2}\partial_z, \\ R(\partial_z,\partial_x)\partial_z &= \frac{AB'^2}{4(A+B^2)}\partial_x - \frac{BB'^2}{4(A+B^2)}\partial_z, \end{aligned}$$

$$\begin{aligned} R(\partial_y,\partial_x)\partial_x &= \frac{B'^2}{4(A+B^2)}\partial_y, \\ R(\partial_z,\partial_y)\partial_x &= \left(\frac{B''}{2} - \frac{B'(BB'+A')}{4(A+B^2)}\right)\partial_y, \\ R(\partial_y,\partial_y)\partial_z &= -\left(\frac{B''}{2} - \frac{B'(BB'+A')}{4(A+B^2)}\right)\partial_y, \\ R(\partial_z,\partial_y)\partial_z &= \left(\frac{-AB'^2+2BA'B'+A'^2}{4(A+B^2)} + \frac{A''}{2}\right)\partial_y. \end{aligned}$$

Applying the formula $R_{ijkl} = g(R(\partial_i, \partial_j)\partial_k, \partial_i)$, we obtain the non–vanishing components of Riemann curvature tensor as follows:

$$R_{1313} = -\frac{B'^2}{4}, \quad R_{2323} = \frac{-A'^2 - 2A'BB' + B'^2A + 2A''A + 2A''B^2}{4(A+B^2)},$$
$$R_{1212} = -\frac{A'^2}{4(A+B^2)}, \qquad R_{1223} = \frac{B'A' + BB'^2 - 2AB'' - 2B''B^2}{4(A+B^2)}.$$

The formula $S_{ij} = g^{kl}R_{iklj}$ yields

$$S_{11} = g^{kl}R_{1kl1} = g^{22}R_{1221} + g^{33}R_{1331} = \frac{A'^2}{4(A+B^2)} + \frac{1}{A+B^2}\frac{B'^2}{4} = \frac{1}{2}\frac{B'^2}{A+B^2}.$$

With similar calculations, we have the following lemma.

**Lemma 2.** *The non–vanishing components of the Ricci tensor of $g$ are the following:*

$$S_{11} = \frac{1}{2}\frac{B'^2}{A+B^2},$$
$$S_{13} = -\frac{1}{4}\frac{-B'A' + 2AB'' + 2B''B^2}{A+B^2},$$
$$S_{22} = -\frac{1}{4}\frac{-A'^2 - 4A'BB' - 2B'^2B^2 + 2B'^2A}{(A+B^2)^2} - \frac{2BB'' + A''}{2(A+B^2)},$$
$$S_{33} = -\frac{1}{4}\frac{-A'^2 - 2A'BB' + 2B'^2A + 2A''A + 2A''B^2}{A+B^2},$$

*with respect to the basis $\{\partial_1, \partial_2, \partial_3, \partial_4\}$.*

For the metric $g$ and any vector fields expressed as $X = X^i\partial_i$, where the components $X^i$ are smooth functions defined on $M$, using the formula

$$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y XZ) + g(Y, \nabla_Z X),$$

we compute

$$\begin{aligned}
(\mathcal{L}_X g)_{11} &= (\mathcal{L}_X g)(\partial_1, \partial_1) = g(\nabla_{\partial_1} X, \partial_1) + g(\partial_1, \nabla_{\partial_1} X) \\
&= 2g(\nabla_{\partial_1}(\sum_{k=1}^{4} X^k\partial_k), \partial_1) \\
&= 2\sum_{k=1}^{4} X^k g(\nabla_{\partial_1}\partial_k, \partial_1) + 2\sum_{k=1}^{4} \partial_1 X^k g(\partial_k, \partial_1) \\
&= 2X^2 g(\nabla_{\partial_1}\partial_2, \partial_1) + 2X^3 g(\nabla_{\partial_1}\partial_3, \partial_1) + 2\partial_1 X^1 g_{11} + 2\partial_1 X^3 g_{31} \\
&= -2X^1_x + 2BX^3_x.
\end{aligned}$$

Aslo, using the formula

$$(\mathcal{L}_X S)(Y, Z) = X(S(Y, Z)) - S(\mathcal{L}_X Y, Z) - S(Y, \mathcal{L}_X Z),$$

we have

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= (\mathcal{L}_X S)(\partial_1, \partial_1) \\
&= X(S(\partial_1, \partial_1)) - 2S(\mathcal{L}_X \partial_1, \partial_1) \\
&= \sum_{k=1}^{4} X^k \partial_k (S_{11}) - 2S([\sum_{k=1}^{4} X^k \partial_k, \partial_1], \partial_1) \\
&= X^2 S'_{11} + 2\sum_{k=1}^{4} \partial_1 X^k S_{k1} \\
&= X^2 S'_{11} + 2X^1_x S_{11} + 2X^3_x S_{13}.
\end{aligned}
$$

Therefore, with similar calculations, we obtain the following lemma.

**Lemma 3.** *For the metric $g$ and any vector fields expressed as $X = X^i \partial_i$, where the components $X^i$ are smooth functions defined on $M$, we have*

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= 2BX^3_x - 2X^1_x, \\
(\mathcal{L}_X g)_{12} &= BX^3_y + X^2_x - X^1_y, \\
(\mathcal{L}_X g)_{13} &= BX^3_z + AX^3_x - X^1_z + BX^1_x + B'X^2, \\
(\mathcal{L}_X g)_{14} &= -X^1_w + BX^3_w + X^4_x, \\
(\mathcal{L}_X g)_{22} &= 2X^2_y, \\
(\mathcal{L}_X g)_{23} &= BX^1_y + X^2_z + AX^3_y, \\
(\mathcal{L}_X g)_{24} &= X^2_w + X^4_y, \\
(\mathcal{L}_X g)_{33} &= A'X^2 + 2BX^1_z + 2AX^3_z, \\
(\mathcal{L}_X g)_{34} &= BX^1_w + AX^3_w + X^4_z, \\
(\mathcal{L}_X g)_{44} &= 2X^4_w,
\end{aligned}
$$

*and*

$$
\begin{aligned}
(\mathcal{L}_X S)_{11} &= X^2 S'_{11} + 2X^3_x S_{13} + 2X^1_x S_{11}, \\
(\mathcal{L}_X S)_{12} &= X^2_x S_{22} + X^1_y S_{11} + X^3_y S_{13}, \\
(\mathcal{L}_X S)_{13} &= X^2 S'_{13} + X^1_x S_{13} + X^3_x S_{33} + X^1_z S_{11} + X^3_z S_{13}, \\
(\mathcal{L}_X S)_{14} &= X^1_w S_{11} + X^3_w S_{13}, \\
(\mathcal{L}_X S)_{22} &= X^2 S'_{22} + 2X^2_y S_{22}, \\
(\mathcal{L}_X S)_{23} &= X^1_y S_{13} + X^3_y S_{33} + X^2_z S_{22}, \\
(\mathcal{L}_X S)_{24} &= X^2_w S_{22}, \\
(\mathcal{L}_X S)_{33} &= X^2 S'_{33} + 2X^1_z S_{13}, +2X^3_z S_{33}, \\
(\mathcal{L}_X S)_{34} &= X^1_w S_{13} + X^3_w S_{33}, \\
(\mathcal{L}_X S)_{34} &= 0,
\end{aligned}
$$

*where* $(\mathcal{L}_X g)_{ij} = \mathcal{L}_X g(\partial_i, \partial_j)$*,* $(\mathcal{L}_X S)_{ij} = \mathcal{L}_X S(\partial_i, \partial_j)$ *for* $1 \leq i, j \leq 4$*,* $X^i_x = \partial_x X^i$*,* $X^i_y = \partial_y X^i$*,* $X^i_z = \partial_z X^i$*, and* $X^i_w = \partial_w X^i$*.*

## 3. Ricci solitons on SHR spacetimes

Now, we consider the Ricci soliton equation (1), which leads to a system of ten independent differential equations for the components of the vector field $X$, where $X^1, X^2, X^3, X^4$ are smooth functions of the coordinates $(x, y, z, w)$. By applying (1), the structure $(M, g, X, \lambda)$ qualifies as a Ricci soliton if and only if

$$2BX^3_x - 2X^1_x = -(\lambda + S_{11}), \tag{3}$$
$$BX^3_y + X^2_x - X^1_y = 0, \tag{4}$$
$$BX^3_z + AX^3_x - X^1_z + BX^1_x + B'X^2 = B\lambda - S_{13}, \tag{5}$$
$$X^4_x + BX^3_w - X^1_w = 0, \tag{6}$$
$$2X^2_y = \lambda - S_{22}, \tag{7}$$
$$AX^3_y + X^2_z + BX^1_y = 0, \tag{8}$$
$$X^4_y + X^2_w = 0, \tag{9}$$
$$2AX^3_z + 2BX^1_z + A'X^2 = A\lambda - S_{33}, \tag{10}$$
$$X^4_z + AX^3_w + BX^1_w = 0, \tag{11}$$
$$2X^4_w = \lambda. \tag{12}$$

Currently, we address the aforementioned system of partial differential equations. By integrating equation (12), we derive

$$X^4 = \frac{\lambda}{2} w + F^4(x, y, z), \tag{13}$$

for some smooth function $F^4$. Then equations (9) and (13) lead to

$$X^2 = -F^4_y w + F^2(x, y, z), \tag{14}$$

for some smooth function $F^2$. Equations (7) and (14) yield

$$-2F^4_{yy} w + 2F^2_y = \lambda - S_{22},$$

which is a polynomial with respect to $w$ and $w$ is arbitrary, then $F^4_{yy} = 0$ and $F^2_y = \frac{1}{2}(\lambda - S_{22})$. Therefore

$$F^4 = F^5(x, z)\, y + F^6(x, z), \qquad F^2 = \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22} dy + F^7(x, z),$$

for some smooth functions $F^5$, $F^6$, and $F^7$. Thus we get

$$X^4 = \frac{\lambda}{2} w + F^5(x, z)\, y + F^6(x, z), \tag{15}$$

$$X^2 = -F^5 w + \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22} dy + F^7(x, z). \tag{16}$$

We differentiate equations (10) and (11) with respect to $w$ and $z$, respectively, to deduce that

$$2\left(AX^3_{zw} + BX^1_{zw}\right) + A'X^2_w = 0, \quad \left(AX^3_{wz} + BX^1_{zw}\right) + X^4_{zz} = 0.$$

Combining these equations, we have

$$A'X^2_w - 2X^4_{zz} = 0. \tag{17}$$

By inserting equations (15) and (16) into (17), we find

$$A'F^5 + 2F^5_{zz} y + 2F^6_{zz} = 0. \tag{18}$$

By differentiating equation (3) with respect to $w$, we have

$$BX^3_{xw} - X^1_{xw} = 0. \tag{19}$$

Also, by differentiating equation (6) with respect to $x$, we have

$$X^4_{xx} + BX^3_{wx} - X^1_{xw} = 0. \tag{20}$$

Differentiating equation (20) from equation (19) gives $X^4_{xx} = 0$. Then, with equation (15), we find $F^5_{xx} y + F^6_{xx} = 0$. The last equation is a polynomial with respect to $y$, thus we get $F^5_{xx} = 0$ and $F^6_{xx} = 0$. Therefore, we compute

$$F^5 = F^8(z)x + F^9(z), \qquad F^6 = F^{10}(z)x + F^{11}(z),$$

for some smooth functions $F^8, F^9, F^{10}$, and $F^{11}$. So, we obtain

$$X^4 = \frac{\lambda}{2} w + \left(F^8(z)x + F^9(z)\right) y + F^{10}(z)x + F^{11}(z), \tag{21}$$

$$X^2 = -\left(F^8 x + F^9\right) w + \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22} dy + F^7(x, z). \tag{22}$$

By rearranging equation (18), we arrive at

$$A'\left(F^8 x + F^9\right) + 2\left(F^8_{zz} x + F^9_{zz}\right) y + 2\left(F^{10}_{zz} x + F^{11}_{zz}\right) = 0.$$

Hence, since $x$ is arbitrary, we can write

$$A'F^8 + 2F^8_{zz} y + 2F^{10}_{zz} = 0,$$

and

$$A'F^9 + 2F^9_{zz} y + 2F^{11}_{zz} = 0. \tag{23}$$

By differentiating equations (5), (6), and (11) with respect to $w, z, x$, respectively, we have

$$BX^3_{zw} + AX^3_{xw} - X^1_{zw} + BX^1_{xw} + B'X^2_w = 0,$$
$$BX^3_{wz} - X^1_{zw} = -X^4_{xz},$$
$$AX^3_{xw} + BX^1_{xw} = -X^4_{xz}.$$

Combining the above equations, we conclude

$$-2X^4_{xz} + B'X^2_w = 0.$$

Inserting (21) and (22) in the last equation, we arrive at

$$-2\left[F^8_z y+F^{10}_z\right]+B'\left[-F^8 x-F^9\right]=0$$

and it implies that

$$B'F^8=0,\qquad 2F^8_z y+2F^{10}_z+B'F^9=0. \tag{24}$$

Now, using equations (6) and (11), we get:

$$BX^3_w-X^1_w=-X^4_x,\qquad AX^3_w+BX^1_w=-X^4_z.$$

Combining the last equations, we conclude

$$\left(B^2+A\right)X^3_w=-\left(BX^4_x+X^4_z\right). \tag{25}$$

By rearranging equations (4) and (8), we obtain

$$BX^3_y-X^1_y=-X^2_x,\qquad AX^3_y+BX^1_y=-X^2_z.$$

These equations also give the following result

$$\left(B^2+A\right)X^3_y=-\left(BX^2_x+X^2_z\right). \tag{26}$$

Using equations (25) and (21), we find

$$X^3_w=\frac{-1}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F^8_z x+F^9_z\right)y+F^{10}_z x+F^{11}_z\right],$$

and integrating with respect to $w$, leads to

$$\begin{aligned} X^3 &= \frac{-1}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F^8_z x+F^9_z\right)y+F^{10}_z x+F^{11}_z\right]w \\ &\quad +F^3(x,y,z), \end{aligned} \tag{27}$$

for some smooth functions $F^3$. By taking the $y$-derivative of equation (27) and putting it into (26), we acquire

$$\begin{aligned} &\frac{2BB'+A'}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F^8_z x+F^9_z\right)y+F^{10}_z x+F^{11}_z\right]w \\ &-\left[B'F^8y+BF^8+F^8_z x+F^9_z\right]w+(B^2+A)F^3_y \\ =\ &-\left[B\left(-F^8w+F^7_x\right)-\left(F^8_z x+F^9_z\right)w+F^7_z\right]. \end{aligned} \tag{28}$$

Equation (28) is a polynomial with respect to $w$, so we have the following system:

$$\begin{aligned} &\frac{2BB'+A'}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F^8_z x+F^9_z\right)y+F^{10}_z x+F^{11}_z\right] \\ &\qquad -\left[B'F^8y+2BF^8+2F^8_z x+2F^9_z\right]=0, \end{aligned} \tag{29}$$

and

$$\left(B^2+A\right)F^3_y=-BF^7_x-F^7_z. \tag{30}$$

After integrating equation (30) with respect to $y$, we obtain $F^3$:

$$F^3 = -F_x^7 \int \frac{B}{B^2+A} dy - F_z^7 \int \frac{1}{B^2+A} dy + H^3(x,z), \tag{31}$$

for some smooth function $H^3$. Also equation (29) is a polynomial in terms of $x$, so it can be rewritten as follows:

$$\left\{ \frac{2BB'+A'}{B^2+A} \left( F_z^8 y + F_z^{10} \right) - 2F_z^8 \right\} x$$
$$+ \frac{2BB'+A'}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + F_z^9 y + F_z^{11} \right] - \left[ B' F^8 y + 2BF^8 + 2F_z^9 \right] = 0.$$

Since $x$ is arbitrary, comparing the cofficients shows that

$$\frac{2BB'+A'}{B^2+A} \left( F_z^8 y + F_z^{10} \right) - 2F_z^8 = 0,$$

and

$$\frac{2BB'+A'}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + F_z^9 y + F_z^{11} \right] - \left[ B' F^8 y + 2BF^8 + 2F_z^9 \right] = 0. \tag{32}$$

From the first equation of (24) we have $B'F^8 = 0$, and by substituting it into equation (32), we get

$$\frac{2BB'+A'}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + F_z^9 y + F_z^{11} \right] - \left[ 2BF^8 + 2F_z^9 \right] = 0. \tag{33}$$

Substituting (31) in (27), we arrive at

$$\begin{aligned} X^3 = & \frac{-1}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + \left( F_z^8 x + F_z^9 \right) y + F_z^{10} x + F_z^{11} \right] w \\ & - F_x^7 \int \frac{B}{B^2+A} dy - F_z^7 \int \frac{1}{B^2+A} dy + H^3(x,z). \end{aligned} \tag{34}$$

Using equation (6), we have

$$\begin{aligned} X_w^1 &= BX_w^3 + X_x^4 \\ &= \frac{-B}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + \left( F_z^8 x + F_z^9 \right) y + F_z^{10} x + F_z^{11} \right] \\ &\quad + F^8 y + F^{10}. \end{aligned}$$

Then, by integrating with respect to $w$, we get

$$\begin{aligned} X^1 = & \frac{-B}{B^2+A} \left[ B\left( F^8 y + F^{10} \right) + \left( F_z^8 x + F_z^9 \right) y + F_z^{10} x + F_z^{11} \right] w \\ & + \left[ F^8 y + F^{10} \right] w + F^1(x,y,z), \end{aligned} \tag{35}$$

for some smooth function $F^1$. By substituting (22), (34), and (35) into (4), we find

$$\left[-BF_x^7\int\frac{B}{B^2+A}dy-BF_z^7\int\frac{1}{B^2+A}dy+BH^3-F^1\right]_y$$
$$=-\frac{B'}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F_z^8x+F_z^9\right)y+F_z^{10}x+F_z^{11}\right]w$$
$$-B'F_x^7\int\frac{B}{B^2+A}dy-B'F_z^7\int\frac{1}{B^2+A}dy+B'H^3+2F^8w-F_x^7.$$

The last equation is a polynomial in term $w$, hence

$$-\frac{B'}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F_z^8x+F_z^9\right)y+F_z^{10}x+F_z^{11}\right]+2F^8=0 \quad (36)$$

and

$$\left[-BF_x^7\int\frac{B}{B^2+A}dy-BF_z^7\int\frac{1}{B^2+A}dy+BH^3-F^1\right]_y \quad (37)$$
$$+B'F_x^7\int\frac{B}{B^2+A}dy+B'F_z^7\int\frac{1}{B^2+A}dy-B'H^3+F_x^7=0.$$

We can write equation (37) in a simpler form

$$\frac{A}{B^2+A}F_x^7-\frac{B}{B^2+A}F_z^7-F_y^1=0. \quad (38)$$

Equation (36) is polynomial with respect to $x$, hence comparing the cofficients leads to

$$-\frac{B'}{B^2+A}\left[F_z^8y+F_z^{10}\right]=0 \quad (39)$$

and

$$-\frac{B'}{B^2+A}\left[B\left(F^8y+F^{10}\right)+F_z^9y+F_z^{11}\right]+2F^8=0. \quad (40)$$

By applying $B'F^8=0$ in equations (39) and (40), we get

$$B'F_z^{10}=0, \quad (41)$$

and

$$\frac{-B'}{B^2+A}\left[BF^{10}+F_z^9y+F_z^{11}\right]+2F^8=0. \quad (42)$$

From equation (38), we can conclude that

$$F^1=F_x^7\int\frac{A}{B^2+A}dy-F_z^7\int\frac{B}{B^2+A}dy+H^1(x,z),$$

for some smooth function $H^1$. Then, we have

$$X^1=-\frac{B}{B^2+A}\left[B(F^8y+F^{10})+(F_z^8x+F_z^9)y+F_z^{10}x+F_z^{11}\right]w \quad (43)$$

$$+(F^8y+F^{10})w+F_x^7\int\frac{A}{B^2+A}dy-F_z^7\int\frac{B}{B^2+A}dy+H^1(x,z).$$

Equation (43) yields

$$\begin{aligned}BX_y^1=&\frac{B'B^3+A'B^2-BB'A}{(B^2+A)^2}\left[B(F^8y+F^{10})+(F_z^8x+F_z^9)y\right]w\\&+\frac{B'B^3+A'B^2-BB'A}{(B^2+A)^2}\left[F_z^{10}x+F_z^{11}\right]w\\&-\frac{B^2}{B^2+A}\left[B'(F^8y+F^{10})+BF^8+F_z^8x+F_z^9\right]w+BF^8w\\&+F_x^7\frac{AB}{B^2+A}-F_z^7\frac{B^2}{B^2+A}.\end{aligned}\tag{44}$$

Also, equation (34) leads to

$$\begin{aligned}AX_y^3=&A\frac{2BB'+A'}{(B^2+A)^2}\left[B\left(F^8y+F^{10}\right)+\left(F_z^8x+F_z^9\right)y+F_z^{10}x+F_z^{11}\right]w\\&-\frac{A}{B^2+A}\left[B'(F^8y+F^{10})+BF^8+F_z^8x+F_z^9\right]w\\&-F_x^7\frac{AB}{B^2+A}-F_z^7\frac{A}{B^2+A}.\end{aligned}\tag{45}$$

Equation (22) implies that

$$X_z^2=-(F_z^8x+F_z^9)w+F_z^7.\tag{46}$$

Substituting (44), (45), and (46) into equation (8) gives

$$\begin{aligned}&\frac{A'+B'B}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F_z^8x+F_z^9\right)y+F_z^{10}x+F_z^{11}\right]\\&-B'(F^8y+F^{10})-2(F_z^8x+F_z^9)=0.\end{aligned}\tag{47}$$

Applying $B'F^8=0$ in (47), we get

$$\begin{aligned}&\frac{A'+B'B}{B^2+A}\left[B\left(F^8y+F^{10}\right)+\left(F_z^8x+F_z^9\right)y+F_z^{10}x+F_z^{11}\right]\\&\quad-B'F^{10}-2F_z^8x-2F_z^9=0.\end{aligned}\tag{48}$$

Equation (48) is a polynomial with respect to $x$, thus we arrive at

$$\frac{A'+B'B}{B^2+A}\left(F_z^8y+F_z^{10}\right)-2F_z^8=0,\tag{49}$$

and

$$\frac{A'+B'B}{B^2+A}\left[B\left(F^8y+F^{10}\right)+F_z^9y+F_z^{11}\right]-B'F^{10}-2F_z^9=0.\tag{50}$$

In view of (24), (41), and (49) it follows that

$$\frac{A'}{B^2+A}\left(F_z^8 y + F_z^{10}\right) - 2F_z^8 = 0.$$

Equation (42) yields

$$\frac{BB'}{B^2+A}\left[BF^{10} + F_z^9 y + F_z^{11}\right] = 2BF^8. \tag{51}$$

Substituting equation (51) into equation (33) and using (24) and (41), we find

$$\frac{A'}{B^2+A}\left[B\left(F^8 y + F^{10}\right) + F_z^9 y + F_z^{11}\right] + 2BF^8 - 2F_z^9 = 0. \tag{52}$$

Using equation (51) and (52) in equation (50), we obtain

$$B'F^{10} = 0. \tag{53}$$

Inserting equations (34) and (43) into (3), we deduce

$$\begin{aligned} &-BF_{xx}^7 \int \frac{B}{B^2+A}dy - BF_{zx}^7 \int \frac{1}{B^2+A}dy + BH_x^3 \\ &-F_{xx}^7 \int \frac{A}{B^2+A}dy + F_{zx}^7 \int \frac{B}{B^2+A}dy - H_x^1 = -\frac{1}{2}\left(\lambda + S_{11}\right). \end{aligned} \tag{54}$$

Taking the derivative of the above equation with respect to $y$, we conclude

$$-B'F_{xx}^7 \int \frac{B}{B^2+A}dy - F_{xx}^7 - B'F_{zx}^7 \int \frac{1}{B^2+A}dy + \frac{1}{2}S_{11}' + B'H_x^3 = 0. \tag{55}$$

From equations (43), (34), and (22) we have

$$\begin{aligned} BX_z^1 = &-\frac{B^2}{B^2+A}\left[B(F_z^8 y + F_z^{10}) + (F_{zz}^8 x + F_{zz}^9)y + F_{zz}^{10}x + F_{zz}^{11}\right] w \\ &+ B(F_z^8 y + F_z^{10})w + BF_{zx}^7 \int \frac{A}{B^2+A}dy - BF_{zz}^7 \int \frac{B}{B^2+A}dy + BH_z^1, \end{aligned} \tag{56}$$

$$\begin{aligned} AX_z^3 = &\frac{-A}{B^2+A}\left[B(F_z^8 y + F_z^{10}) + (F_{zz}^8 x + F_{zz}^9)y + F_{zz}^{10}x + F_{zz}^{11}\right] w \\ &- AF_{zx}^7 \int \frac{B}{B^2+A}dy - AF_{zz}^7 \int \frac{1}{B^2+A}dy + AH_z^3, \end{aligned} \tag{57}$$

and

$$A'X^2 = -A'\left(F^8 x + F^9\right) w + \frac{A'}{2}\lambda y - \frac{A'}{2}\int S_{22}dy + A'F^7. \tag{58}$$

Substituting (56), (57), and (58) into (10), we conclude that

$$\begin{aligned} &-\left[(F_{zz}^8 x + F_{zz}^9)y + F_{zz}^{10}x + F_{zz}^{11}\right] w + BF_{zx}^7 \int \frac{A}{B^2+A}dy + BH_z^1 + \frac{A'}{4}\lambda y \\ &- BF_{zz}^7 \int \frac{B}{B^2+A}dy - AF_{zx}^7 \int \frac{B}{B^2+A}dy - AF_{zz}^7 \int \frac{1}{B^2+A}dy + AH_z^3 \end{aligned}$$

$$-\frac{1}{2}A'\left(F^8x+F^9\right)w-\frac{A'}{4}\int S_{22}dy+\frac{1}{2}A'F^7-\frac{1}{2}\left(A\lambda-S_{33}\right)=0.$$

The coefficient of $w$ is given by

$$-\left[(F^8_{zz}x+F^9_{zz})y+F^{10}_{zz}x+F^{11}_{zz}\right]=\frac{1}{2}A'\left(F^8x+F^9\right),$$

and

$$\begin{aligned}&A\left[-F^7_{xz}\int\frac{B}{B^2+A}dy-F^7_{zz}\int\frac{1}{B^2+A}dy+H^3_z\right]\\&+B\left[F^7_{xz}\int\frac{A}{B^2+A}dy-F^7_{zz}\int\frac{B}{B^2+A}dy+H^1_z\right]\\&=-\frac{A'}{4}\lambda y+\frac{A'}{4}\int S_{22}dy-\frac{A'}{2}F^7+\frac{1}{2}\left(A\lambda-S_{33}\right).\end{aligned}\tag{59}$$

Putting (43), (34), and (22) into (5), we deduce

$$\begin{aligned}&B\left[-F^7_{xz}\int\frac{B}{B^2+A}dy-F^7_{zz}\int\frac{1}{B^2+A}dy+H^3_z\right]-2(F^8_zy+F^{10}_z)w\\&-F^7_{xz}\int\frac{A}{B^2+A}dy+F^7_{zz}\int\frac{B}{B^2+A}dy-H^1_z\\&+A\left[-F^7_{xx}\int\frac{B}{B^2+A}dy-F^7_{zx}\int\frac{1}{B^2+A}dy+H^3_x\right]\\&+B\left[F^7_{xx}\int\frac{A}{B^2+A}dy-F^7_{zx}\int\frac{B}{B^2+A}dy+H^1_x\right]\\=\;&B'\left(F^8x+F^9\right)w-\frac{B'}{2}\lambda y+\frac{B'}{2}\int S_{22}dy-B'F^7+B\lambda-S_{13}.\end{aligned}$$

The last equation is a polynomial with respect to $w$, hence comparing the cofficients leads to

$$2(F^8_zy+F^{10}_z)=B'\left(F^8x+F^9\right),\tag{60}$$

and

$$\begin{aligned}&B\left[-F^7_{xz}\int\frac{B}{B^2+A}dy-F^7_{zz}\int\frac{1}{B^2+A}dy+H^3_z\right]\\&-F^7_{xz}\int\frac{A}{B^2+A}dy+F^7_{zz}\int\frac{B}{B^2+A}dy-H^1_z\\&+A\left[-F^7_{xx}\int\frac{B}{B^2+A}dy-F^7_{zx}\int\frac{1}{B^2+A}dy+H^3_x\right]\\&+B\left[F^7_{xx}\int\frac{A}{B^2+A}dy-F^7_{zx}\int\frac{B}{B^2+A}dy+H^1_x\right]\\=\;&-\frac{B'}{2}\lambda y+\frac{B'}{2}\int S_{22}dy-B'F^7+B\lambda-S_{13}.\end{aligned}\tag{61}$$

Checking whether the derivatives of $A$ and $B$ become zero or not, we consider four cases and in each case, checking whether the functions $F^i$ and $H^i$ are smooth, we investigate the existence of Ricci solitons.

Case 1: $A' = 0$ and $B' = 0$.

In this case, we have $A = a$ and $B = b$ for come constants $a, b$. Equation (51) leads to $bF^8 = 0$, then $F^8 = 0$. Equation (60) yields $F^{10}_z = 0$. Then $F^{10} = a_2$ for some constant $a_2$. Equation (33) implies that $F^9_z = 0$. Hence $F^9 = a_3$ for some constant $a_3$. From (55) we have $F^7_{xx} = 0$. Also, (59) implies that

$$-F^7_{zz}y + aH^3_z + bH^1_z = \frac{1}{2}a\lambda,$$

which is a polynomial with respect to $y$, therefore $F^7_{zz} = 0$ and

$$aH^3_z + bH^1_z = \frac{1}{2}a\lambda. \tag{62}$$

Equation (61) gives

$$-2F^7_{xz}y + bH^3_z - H^1_z + aH^3_x + H^1_x = b\lambda,$$

which is a polynomial with respect to $y$, hence $F^7_{zx} = 0$ and

$$bH^3_z - H^1_z + aH^3_x + H^1_x = b\lambda. \tag{63}$$

Thus $F^7 = a_4z + a_5x + k_6$ for some constants $a_4, a_5$, and $k_6$. Using (23), we arrive at $F^{11} = a_6z + a_7$ for some constants $a_6$ and $a_7$. Equation (54) leads to

$$bH^3_x - H^1_x = -\frac{1}{2}\lambda. \tag{64}$$

Applying (62), (63), and (64), we conclude that

$$\begin{aligned} H^1 &= \frac{1}{2}\lambda x + \frac{b}{a+b^2}\left[(\frac{1}{2}a\lambda - ba_8)z + (\frac{1}{2}b\lambda + a_8)x\right] + ba_{10} + a_8z + a_9, \\ H^3 &= \frac{1}{a+b^2}\left[(\frac{1}{2}a\lambda - ba_8)z + (\frac{1}{2}b\lambda + a_8)x\right] + a_{10}, \end{aligned}$$

for some constants $a_8, a_9, a_{10}$. Therefore,

$$\begin{cases} X^1 &= -\frac{b}{a+b^2}\left[ba_2 + a_6\right]w + a_2w + \frac{a_5a}{a+b^2}y - \frac{a_4b}{a+b^2}y + \frac{1}{2}\lambda x + ba_{10} + a_8z + a_9 \\ & \quad + \frac{b}{a+b^2}\left[(\frac{1}{2}a\lambda - ba_8)z + (\frac{1}{2}b\lambda + a_8)x\right], \\ X^2 &= -a_3w + \frac{1}{2}\lambda y + a_4z + a_5x + k_6, \\ X^3 &= -\frac{1}{a+b^2}\left[ba_2 + a_6\right]w - \frac{a_5b}{a+b^2}y - \frac{a_4}{a+b^2}y \\ & \quad + \frac{1}{a+b^2}\left[(\frac{1}{2}a\lambda - ba_8)z + (\frac{1}{2}b\lambda + a_8)x\right] + a_{10}, \\ X^4 &= \frac{\lambda}{2}w + a_3y + a_2x + a_6z + a_7. \end{cases} \tag{65}$$

Case 2: $B' = 0$ and $A' \neq 0$.

In this case, we have $B = b$. Equations (51) and (60) yield $F^8 = 0$ and $F^{10} = a_2$, respectively. Equation (55) leads to $F^7_{xx} = 0$. Hence $F^7 = K^1(z)x + K^2(z)$ for some smooth functions $K^1, K^2$. Applying (54) we get $bH^3_x - H^1_x = -\frac{1}{2}\lambda$. Deriving it with respect to $z$ gives $bH^3_{xz} - H^1_{xz} = 0$. Now, by taking differential of (61) with respect to $x$ and using the last equation we obtain $AH^3_{xx} + bH^1_{xx} = 0$. Then, by taking derivative with respect to $y$, we obtain $H^3_{xx} = 0$ and $H^1_{xx} = 0$. Hence

$$H^1 = bK^3(z)x + \frac{1}{2}\lambda x + K^4(z), \qquad H^3 = K^3(z)x + K^5(z),$$

for some smooth functions $K^3, K^4, K^5$. By substituting the obtained $F^7$, $H^1$, and $H^3$ into equation (59), we arrive at an expression which is a polynomial in terms of $x$. Therefore, by comparing the coefficients of this relation, we obtain

$$\begin{aligned} &AK^5_z + b\left[K^1_z \int \frac{A}{b^2+A}dy - K^2_{zz}\int \frac{b}{b^2+A}dy + K^4_z\right] + \frac{A'}{4}\lambda y - \frac{A'}{4}\int S_{22}dy \\ &+ A\left[-bK^1_z - K^2_{zz}\right]\int \frac{1}{b^2+A}dy = -\frac{A'}{2}K^2 + \frac{1}{2}\left(A\lambda - S_{33}\right), \end{aligned} \tag{66}$$

and

$$A\left[-K^1_{zz}\int \frac{1}{b^2+A}dy + K^3_z\right] + b\left[-K^1_{zz}\int \frac{b}{b^2+A}dy + bK^3_z\right] = -\frac{A'}{2}K^1 x. \tag{67}$$

Also, by putting the obtained $F^7$, $H^1$, and $H^3$ into equation (61), we get

$$\begin{aligned} &-K^1_z y + bK^5_z - K^4_z + A\left[-K^1_z\int \frac{1}{b^2+A}dy + K^3\right] \\ &\quad + b\left[-K^1_z\int \frac{b}{b^2+A}dy + bK^3 + \frac{1}{2}\lambda\right] = b\lambda - S_{13}. \end{aligned} \tag{68}$$

Therefore,

$$\begin{cases} X^1 &= \left[-\frac{b}{b^2+A}\left(ba_2 + F^9_z y + F^{11}_z\right) + a_2\right] w + K^1\int \frac{A}{b^2+A}dy \\ &\quad -(K^1_z x + K^2_z)\int \frac{b}{b^2+A}dy + bK^3 x + \frac{1}{2}\lambda x + K^4, \\ X^2 &= -F^9 w + \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22}dy + K^1 x + K^2, \\ X^3 &= -\frac{1}{A+b^2}\left[ba_2 + F^9_z y + F^{11}_z\right] w - K^1\int \frac{b}{b^2+A}dy \\ &\quad -(K^1_z x + K^2_z)\int \frac{1}{b^2+A}dy + K^3 x + K^5, \\ X^4 &= \frac{\lambda}{2}w + a_2 x + F^9 y + F^{11}, \end{cases} \tag{69}$$

such that equations (23), (52), (66), (67), and (68) hold.

Case 3: $B' \neq 0$ and $A' = 0$.

In this case, $A = a$ and equations (23), (24), (52), and (53) imply that

$$F^8 = 0, \quad F^9 = 0, \quad F^{10} = 0, \quad F^{11} = c_4 z + c_5.$$

Equation (42) leads to $F^{11}_z = 0$, hence $c_4 = 0$. Therefore,

$$\begin{cases} X^1 = F^7_x \int \frac{a}{B^2+a} dy - F^7_z \int \frac{B}{B^2+a} dy + H^1, \\ X^2 = \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22} dy + F^7, \\ X^3 = -F^7_x \int \frac{B}{B^2+a} dy - F^7_z \int \frac{1}{B^2+a} dy + H^3, \\ X^4 = \frac{\lambda}{2} w + c_5, \end{cases} \tag{70}$$

such that equations (54), (59), and (61) hold.

Case 4: $B' \neq 0$ and $A' \neq 0$.

In this case, equations (24) and (53) yield $F^8 = F^9 = F^{10} = 0$. Equation (42) leads to $F^{11} = d_2$, for some constant $d_2$. Therefore,

$$\begin{cases} X^1 = F^7_x \int \frac{A}{B^2+A} dy - F^7_z \int \frac{B}{B^2+A} dy + H^1, \\ X^2 = \frac{1}{2}\lambda y - \frac{1}{2}\int S_{22} dy + F^7, \\ X^3 = -F^7_x \int \frac{B}{B^2+A} dy - F^7_z \int \frac{1}{B^2+A} dy + H^3, \\ X^4 = \frac{\lambda}{2} w + d_2, \end{cases} \tag{71}$$

such that equations (54), (59), and (61) are true.

**Theorem 1.** *SHR spacetime with metric* (2) *is steady, shrinking, expanding Ricci soliton, and the potential vector field $X$ admits one of the relations* (65)*, or* (69)*, or* (70)*, or* (71)*.*

Now, suppose $X = \nabla f$ for some function $f$ with respect to the tensor metric $g$. Then

$$X = -\frac{A}{A+B^2} f_x \partial_x + \frac{B}{A+B^2} f_z \partial_x + f_y \partial_y + \frac{B}{A+B^2} f_x \partial_z + \frac{A}{A+B^2} f_z \partial_z + f_w \partial_w. \tag{72}$$

According to Theorem 1, the potential vector field $X$ of the Ricci soliton concerning $g$ is a gradient vector field as stated in ( 72) if and only if

$$\begin{cases} f_x = \frac{A+B^2}{A^2+B^2}(-AX^1 + BX^3), \\ f_y = X^2, \\ f_z = \frac{A+B^2}{A^2+B^2}(BX^1 + AX^3), \\ f_w = X^4. \end{cases} \tag{73}$$

Since $f_{xw} = f_{wx}$, the system (73) leads to

$$\frac{A+B^2}{A^2+B^2}(-AX^1_w + BX^3_w) = X^4_x.$$

Similarly, we have

$$\left[\frac{A+B^2}{A^2+B^2}(-AX^1 + BX^3)\right]_y = X^2_x,$$
$$-AX^1_z + BX^3_z = BX^1_x + AX^3_x,$$
$$X^2_z = \left[\frac{A+B^2}{A^2+B^2}(BX^1 + AX^3)\right]_y,$$
$$X^2_w = X^4_y,$$
$$\left[\frac{A+B^2}{A^2+B^2}(BX^1 + AX^3)\right]_w = X^4_z.$$

We investigate just Case 1.
If $A' = B' = 0$, then from (65) and $X^2_w = X^4_y$ we obtain $a_3 = 0$. Equations (65) and $\left[\frac{A+B^2}{A^2+B^2}(BX^1 + AX^3)\right]_w = X^4_z$ imply that $(a+a^2+2b^2)a_6 = 0$. Equation $\frac{A+B^2}{A^2+B^2}(-AX^1_w + BX^3_w) = X^4_x$ leads to $2(a^2 + b^2)a_2 = b(a - 1)a_6$. From equation $\left[\frac{A+B^2}{A^2+B^2}(-AX^1 + BX^3)\right]_y = X^2_x$ we find $2(a^2 + b^2)a_5 = b(a-1)a_4$. Also, equation $X^2_z = \left[\frac{A+B^2}{A^2+B^2}(BX^1 + AX^3)\right]_y$ implies that $(a^2 + 2b^2 + a)a_4 = 0$. Using equation $-AX^1_z + BX^3_z = BX^1_x + AX^3_x$, we get $\left(\frac{ab(1-a)}{2(a+b^2)} - b\right)\lambda = \left(\frac{b^2(1-a)}{a+b^2} + a + 1\right)a_8$. Therefore, with these conditions we can write

$$\begin{aligned}
f_x &= \frac{a+b^2}{a^2+b^2}\left\{-\frac{1}{2}a\lambda x - aba_{10} - aa_9 + (b\lambda + a_8)z + ba_{10}\right\}\\
&\quad + \frac{b(1-a)}{a^2+b^2}(\frac{1}{2}b\lambda + a_8)x + a_2 w + a_5 y,\\
f_y &= \frac{1}{2}\lambda y + a_4 z + a_5 x + k_6,\\
f_z &= a_6 w + a_4 y + \frac{a+b^2}{a^2+b^2}\left\{(b\lambda + a_8)x + b^2 a_{10} + ba_9 + \frac{1}{2}a\lambda z + aa_{10}\right\},\\
f_w &= \frac{\lambda}{2}w + a_2 x + a_6 z + a_7.
\end{aligned}$$

By integrating the above relations, it is concluded that

$$f = \frac{\lambda}{4}w^2 + a_2 xw + a_6 zw + a_7 w + \frac{\lambda}{4}y^2 + a_4 zy + a_5 xy + k_6 y$$

$$
\begin{aligned}
&+\frac{a+b^2}{a^2+b^2}\left[b^2a_{10}z+ba_9z+\frac{1}{4}a\lambda z^2+aa_{10}z\right] \qquad (74)\\
&+\frac{a+b^2}{a^2+b^2}\left[-a(\frac{1}{4}\lambda x^2+ba_{10}x+(b\lambda+a_8)zx+a_9x)+ba_{10}x\right]\\
&+\frac{b(1-a)}{a^2+b^2}\left[(\frac{1}{2}b\lambda+a_8)\frac{x^2}{2}\right].
\end{aligned}
$$

**Corollary 1.** *A Ricci solitos on the SHR spacetime of Case* 1 *is a gradient Ricci soliton with potential function* (74)*.*

*Remark* 1. A vector field $X$ is identified as a Killing field provided it satisfies the condition $\mathcal{L}_X g = 0$ [7]. The notion of conformal vector fields extends the concept of Killing vector fields; a conformal vector field X on the manifold $(M^n, g)$ is characterized by the equation $\mathcal{L}_X g = 2\psi g$, where $\psi$ is a smooth function (refer to [8, 26]). Because any Ricci soliton in the SHR spacetime of Case 1 allows for a Ricci soliton equation with $A' = B' = 0$, we can infer that any potential vector field of a Ricci soliton on the SHR spacetime becomes a conformal Killing vector field.

*Remark* 2. A vector field $X$ on a pseudo-Riemannian manifold $(M, g)$ is defined as a Ricci collineation vector field if $\mathcal{L}_X S = 0$. According to Theorem 1, every potential vector field of a Ricci soliton in the SHR spacetime of Case 1 is classified as a Ricci collineation vector field. Also, a vector field $X$ on a pseudo-Riemannian manifold $(M, g)$ is defined as a Ricci bi-collineation vector field [2, 3] if

$$\mathcal{L}_X g = \alpha g + \beta S, \qquad \mathcal{L}_X S = \alpha S + \beta g,$$

where $\alpha$ and $\beta$ represent smooth functions. According to Theorem 1, every potential vector field of a Ricci soliton in the SHR spacetime of Case 1 is classified as a Ricci bi-conformal vector field.

**Example 1.** Here we will discuss Rici solitons of a spatially homogeneous rotating spacetimes. If $A(y) = -(1+3\cosh^2 2y)$ and $B(y) = -2\cosh 2y$, the homogeneous rotating spacetime becomes Reboucas spacetime and we have

$$S = \begin{pmatrix} 8 & 0 & 16\cosh(2y) & 0 \\ 0 & 4 & 0 & 0 \\ 16\cosh(2y) & 0 & -4+36\cosh^2(2y) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

For Reboucas spacetime, equation (24) yields $F^8 = 0$. Equations (53), (24) and (42) lead to $F^9 = 0$, $F^{10} = 0$, and $F^{11} = k_1$, for some constant $k_1$. Equation (54) becomes

$$3F^7_{xx}y - \sinh(2y)F^7_{zx} - 2\cosh(2y)H^3_x - H^1_x = -\frac{1}{2}(\lambda+8).$$

The last equation yields $F^7_{xx} = F^7_{zx} = H^3_x = 0$ and $H^1_x = -\frac{1}{2}(\lambda+8)$. Equation (59) implies that

$$\begin{aligned}
&\frac{1}{2}(5 + 3\cosh^2(2y))\coth(2y)F^7_{zz} - (1 + 3\cosh^2 2y)H^3_z - 2\cosh(2y)H^1_z \\
&= \frac{3}{4}\sinh(4y)(\lambda y - 4y - 2F^7) \\
&\quad + \frac{1}{2}\left[-(1 + 3\cosh^2 2y)\lambda + 4 - 36\cosh^2(2y)\right].
\end{aligned}$$

By multiplying the sides of the above relation by $\sinh(2y)$ and then tending $y$ towards zero, we get $F^7_{zz} = 0$. Therefore, the above equation becomes

$$\begin{aligned}
&-(1 + 3\cosh^2 2y)H^3_z - 2\cosh(2y)H^1_z \\
&= \frac{3}{4}\sinh(4y)(\lambda y - 4y - 2F^7) \\
&\quad + \frac{1}{2}\left[-(1 + 3\cosh^2 2y)\lambda + 4 - 36\cosh^2(2y)\right].
\end{aligned} \tag{75}$$

Equation (75) holds for any $y$, so by taking the derivative of the above equation with respect to $y$ and then tending $y$ towards zero in the resulting equation, we will have $\lambda y - 4y - 2F^7 = 0$, which is a polynomial with respect to $y$. Thus, $\lambda = 4$, $F^7 = 0$, and

$$(1 + 3\cosh^2 2y)H^3_z + 2\cosh(2y)H^1_z = 24\cosh^2(2y). \tag{76}$$

Again, by differentiating with respect to $y$, we have

$$3\cosh(2y)H^3_z + H^1_z = 24\cosh(2y).$$

The last equation gives $H^3_z = 8$ and $H^1_z = 0$. On the other hand, by substituting $H^3_z = 8$ in equation (76), we get $\cosh(2y)H^1_z = -4$. This gives a contradiction. Therefore, the Reboucas spacetime does not admit a Ricci soliton structure because the system of equations derived from the soliton condition leads to a contradiction.

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