ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 30, Number 1, June 2026
Available online at https://ojs.utlib.ee/index.php/ACUTM

AT algorithm for fractal polynomiographs: a study on fast convergence under weak contraction mappings

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Abstract. In the present work, we use the AT algorithm, an iteration consisting of three steps that approximates the fixed point of a weak contraction. The algorithm not only demonstrates faster convergence compared to established methods such as the S, Normal-S, Varat, Mann, Ishikawa, and Picard iterations for weak contraction, but also exhibits strong convergence properties. The paper also explores the AT algorithm's almost stable behavior for weak contraction. We further apply the AT iterative scheme to construct Julia sets and polynomiographs, providing a practical comparison with the AET values for the Normal-S, Mann, Picard, and AT iterations, thereby demonstrating the real-world relevance of our research.

1. Introduction

In nonlinear analysis, fixed point theory provides key methods for determining the existence and uniqueness of solutions. Starting with the Banach contraction principle [20], the theory has expanded to include generalized forms such as weak contraction mappings, which allow for broader applicability in abstract spaces. In 1997, Berinde [4] contributed significantly to this area by introducing generalized contractions which are more general than several well-established mappings, including contraction mappings, those introduced by Kannan [12], Zamfirescu [25], Chatterjea [7] and others.

Starting from fundamental techniques such as the Picard iteration [24], which ensures convergence to fixed points of contraction mappings, to more advanced methodologies including Mann [16], Ishikawa [10], S [1], normal-S

Received December 8, 2025.
2020 *Mathematics Subject Classification.* 47H09,47H10.
Key words and phrases. AT algorithm, fixed point, Julia set, polynomiography, weak contraction.
https://doi.org/10.12697/ACUTM.2026.30.05
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[20], Varat [22] and F^* [2], iterative algorithms serve as the cornerstone of computational approaches in fixed point theory.

Let Q be a complete normed linear space, and let P be a nonempty, closed and convex subset of Q. Suppose R is a self-mapping on P. Further, let $\{a_m\}$, $\{c_m\}$ and $\{d_m\}$ be sequences in (0,1). We have the following algorithms.

Picard iteration

$$s_{m+1} = Rs_m, \qquad m \in \mathbb{Z}_+. \tag{1}$$

Mann iteration

$$s_{m+1} = (1 - a_m)s_m + a_m Rs_m, \qquad m \in \mathbb{Z}_+. \tag{2}$$

Ishikawa iteration

$$\begin{aligned} b_m &= (1 - d_m)s_m + d_m Rs_m, \\ s_{m+1} &= (1 - a_m)s_m + a_m Rb_m, \qquad m \in \mathbb{Z}_+. \end{aligned} \tag{3}$$

S-iteration

$$\begin{aligned} b_m &= (1 - d_m)s_m + d_m Rs_m, \\ s_{m+1} &= (1 - a_m)Rs_m + a_m Rb_m, \qquad m \in \mathbb{Z}_+. \end{aligned} \tag{4}$$

Normal-S iteration

$$s_{m+1} = R((1 - a_m)s_m + a_m Rs_m), \qquad m \in \mathbb{Z}_+. \tag{5}$$

Varat iteration

$$\begin{aligned} b_m &= (1 - d_m)s_m + d_m Rs_m, \\ t_m &= (1 - c_m)s_m + c_m b_m, \\ s_{m+1} &= (1 - a_m)Rt_m + a_m Rb_m, \qquad m \in \mathbb{Z}_+. \end{aligned} \tag{6}$$

F^* iteration

$$\begin{aligned} b_m &= R((1 - a_m)s_m + a_m Rs_m), \\ s_{m+1} &= Rb_m, \qquad m \in \mathbb{Z}_+. \end{aligned} \tag{7}$$

Given the above information, an important question arises that is it possible to construct an iterative algorithm that achieves faster convergence than the F^* iteration (7) and other methods? In response to this, we introduce a three-step iterative algorithm, referred to as the AT iteration, which can be expressed as follows:

$$
\begin{aligned}
s_0 &\in P, \\
s_{m+1} &= R((1-c_m)b_m + c_m R b_m), \\
b_m &= \frac{1}{2}R(Rs_m) + \frac{1}{2}R(Rx_m), \\
x_m &= (1-a_m)s_m + a_m R s_m, \quad m \in \mathbb{Z}_+,
\end{aligned}
\tag{8}
$$

where $\{a_m\}$ and $\{c_m\}$ lie in $(0, 1)$.

Fixed point theory forms the foundation of many iterative methods used in generating fractals and polynomigraphs [17, 21, 26]. Polynomiography, introduced by Kalantari [11], transforms the root-finding process of polynomials into artistic visualizations, highlighting the role of iteration in complex dynamics. Julia sets, similarly, represent boundaries of stability in iterative systems.

2. Preliminaries

In 1922, the concept of contraction mappings was introduced by Banach [20] as follows.

Definition 1 ([3])**.** Consider a normed linear space Q. We say that a mapping $R : Q \to Q$ is a ζ-*contraction* if

$$\|Rp - Rq\| \leq \zeta \|p - q\|, \quad \forall p, q \in Q, \zeta \in [0, 1). \tag{9}$$

Berinde ([4], [5]) established the following theorem regarding the existence and uniqueness of fixed points for the mapping R.

Theorem 1. *Let Q be a complete normed linear space and $R : Q \to Q$ be a weak contraction with the conditions*

$$\|Rp - Rq\| \leq \zeta \|p - q\| + L\|q - Rp\|, \quad \forall p, q \in Q, \tag{10}$$

and

$$\|Rp - Rq\| \leq \zeta \|p - q\| + L\|p - Rp\|, \quad \forall p, q \in Q, \tag{11}$$

with existing constants $\zeta \in (0, 1)$ and $L \geq 0$. Then R has a unique fixed point in Q.

Ostrowski [18] introduced the concept of stability as follows.

Definition 2 ([18])**.** Let $s_0 \in Q$ and $s_{m+1} = g(R, s_m)$ be an iterative method for a function g on a complete normed linear space Q with self-map R having fixed point s. Let $\{r_m\}$ be sequence of an approximation of $\{s_m\}$ in Q and define $\gamma_m = \|r_{m+1} - g(R, r_m)\|$.

Then the iterative method $s_{m+1} = g(R, s_m)$ is known as R-*stable* if

$$\lim_{m\to\infty} \gamma_m = 0 \Longleftrightarrow \lim_{m\to\infty} r_m = s,$$

and it is known as almost *R-stable* if

$$\sum_{m=0}^{\infty} \gamma_m < \infty \Longrightarrow \lim_{m\to\infty} r_m = s.$$

Lemma 1 ([4])**.** *Let $\{u_m\}$ and $\{v_m\}$ be two sequences in $\mathbb{R}_+$ and $0 \leq k < 1$ so that $u_{m+1} \leq ku_m + v_m$ for all $m \geq 0$. If $\lim_{m\to\infty} v_m = 0$, then $\lim_{m\to\infty} u_m = 0$.*

Lemma 2 ([23])**.** *Let $N \in \mathbb{Z}_+$ and let $\{p_m\}$ be a sequence in $\mathbb{R}_+$ satisfying the inequality*

$$p_{m+1} \leq (1-\delta_m)p_m + \delta_m q_m, \quad \forall m \geq N,$$

where $\{\delta_m\}$ is a sequence such that $\delta_m \in (0,1)$ for all $m \in \mathbb{Z}_+$. Assume that the series $\sum_{m=0}^{\infty} \delta_m$ diverges, and let q_m be a sequence with $q_m \geq 0$ for all m. Under these conditions, we can conclude that

$$0 \leq \limsup_{m\to\infty} p_m \leq \limsup_{m\to\infty} q_m.$$

To compare the convergence rates of two iterative algorithms, Berinde [6] introduces the following notions.

Definition 3. Consider the sequences $\{p_m\}$ and $\{q_m\}$ in $\mathbb{R}_+$, which converge to the limits p and q, respectively. Define

$$\ell = \lim_{m\to\infty} \frac{|p_m - p|}{|q_m - q|}.$$

(i) If $\ell = 0$, we say that the sequence $\{p_m\}$ converges to p *more quickly* than the sequence $\{q_m\}$ converges to q.

(ii) If $0 < \ell < \infty$, we say that both sequences $\{p_m\}$ and $\{q_m\}$ converge at *the same rate.*

Definition 4 ([2])**.** Consider two iterative algorithms, denoted by $\{\theta_m\}$ and $\{\eta_m\}$, which both converge to the same point θ. Let the error estimate for these algorithms be:

$$|\theta_m - \theta| \leq p_m,$$
$$|\eta_m - \theta| \leq q_m.$$

If $\lim_{m\to\infty} \frac{p_m}{q_m} = 0$, then the convergence of $\{\theta_m\}$ is *faster* than the convergence of $\{\eta_m\}$.

3. Main results

By AT iteration in a complete normed linear space, we will establish results which are related to weak contraction.

Theorem 2. *Let $R : P \to P$ be a weak contraction with the condition* (11). *Then AT iteration* $\{s_m\}$ (8) *converges to the fixed point s of R which is unique.*

Proof. According to condition (11), we get

$$\begin{aligned}\|Rs_m - s\| &= \|Rs_m - Rs\| \\ &\leq \zeta\|s_m - s\| + L\|s - Rs\| \\ &= \zeta\|s_m - s\|, \quad \forall m \in \mathbb{Z}_+.\end{aligned}$$

Using AT iteration (8), we have

$$\begin{aligned}\|x_m - s\| &= \|(1-a_m)s_m + a_m Rs_m - s\| \\ &\leq (1-a_m)\|s_m - s\| + a_m\|Rs_m - s\| \\ &\leq \|s_m - s\|\end{aligned}$$

and

$$\begin{aligned}\|b_m - s\| &= \frac{1}{2}\|R^2(s_m) + R^2(x_m) - 2s\| \\ &\leq \frac{1}{2}\|R^2(s_m) - s\| + \frac{1}{2}\|R^2(x_m) - s\| \\ &\leq \frac{\zeta^2}{2}\|s_m - s\| + \frac{\zeta^2}{2}\|x_m - s\| \\ &\leq \frac{\zeta^2}{2}\|s_m - s\| + \frac{\zeta^2}{2}\|s_m - s\|,\end{aligned}$$

which turns into

$$\|b_m - s\| \leq \zeta^2\|s_m - s\|. \tag{12}$$

Using inequality (12), we get

$$\begin{aligned}\|s_{m+1} - s\| &= \|R((1-c_m)b_m + c_m Rb_m) - s\| \\ &\leq \zeta\|(1-c_m)b_m + c_m Rb_m - s\| \\ &\leq \zeta\Big((1-c_m)\|b_m - s\| + c_m\|Rb_m - s\|\Big) \\ &\leq \zeta\Big((1-c_m)\|b_m - s\| + c_m\zeta\|b_m - s\|\Big) \\ &\leq \zeta[(1-c_m) + c_m\zeta]\|b_m - s\|.\end{aligned}$$

As $0 < \zeta < 1$ and $c_m \in (0,1)$, it follows that $1 - (1-\zeta)c_m < 1$, and hence

$$\|s_{m+1} - s\| \leq \zeta^3 \|s_m - s\|.$$

Consequently, we get

$$\|s_{m+1} - s\| \leq \zeta^{3(m+1)} \|s_0 - s\| . \tag{13}$$

Since $0 < \zeta < 1$, thus, $\{s_m\}$ converges strongly to s. □

Theorem 3. *Let $R : P \to P$ be a mapping which defines a weak contraction with the condition* (11)*. Then $\{s_m\}$ defined as an AT iteration* (8) *is almost R-stable.*

Proof. Let $\{r_m\}$ be an arbitrary sequence in P, and let the sequence generated by the AT algorithm (8) be given by $s_{m+1} = g(R, s_m)$. Define $\gamma_m = \|r_{m+1} - g(R, r_m)\|$ for all $m \in \mathbb{Z}_+$. We now proceed to prove that

$$\sum_{m=0}^{\infty} \gamma_m < \infty \Longrightarrow \lim_{m\to\infty} r_m = s.$$

Let $\sum_{m=0}^{\infty} \gamma_m < \infty$. Then, by AT algorithm (8), we have

$$\begin{aligned} \|r_{m+1} - s\| &\leq \|r_{m+1} - g(R, r_m)\| + \|g(R, r_m) - s\| \\ &\leq \gamma_m + \zeta^3 (1 - (1-\zeta)c_m) \|r_m - s\| . \end{aligned}$$

Define $u_m = \|r_m - s\|$ and $k = \zeta^3(1 - (1-\zeta)c_m)$. Then $0 \leq k < 1$, and therefore

$$u_{m+1} \leq k u_m + \gamma_m.$$

Thus the conclusion follows by Lemma 1. □

The next theorem establishes that the AT iterative algorithm exhibits a faster rate of convergence compared to the algorithms given in (1)–(7).

Theorem 4. *Let $R : P \to P$ be a weak contraction with the condition* (11)*. Let the sequences $\{s_{1,m}\}, \{s_{2,m}\}, \{s_{3,m}\}, \{s_{4,m}\}, \{s_{5,m}\}, \{s_{6,m}\}, \{s_{7,m}\}$ and $\{s_m\}$ defined by* (1) *to* (8)*, respectively, converge to s which is a fixed point of R. Then the convergence of the AT algorithm is faster towards fixed point compared to* (1) *to* (7)*.*

Proof. According to inequality (13) of Theorem 2, we obtain

$$\|s_{m+1} - s\| \leq \zeta^{3(m+1)} \|s_0 - s\| = \eta_m, m \in \mathbb{Z}_+,$$

and by the [13, Proposition 1] of Khan iteration, we have

$$\|s_{1,m} - s\| \leq \zeta^{(m+1)} \|s_{1,0} - s\| = \eta_{1,m}, m \in \mathbb{Z}_+.$$

Then

$$\frac{\eta_m}{\eta_{1,m}} = \frac{\zeta^{3(m+1)} \|s_0 - s\|}{\zeta^{m+1} \|s_{1,0} - s\|} = \zeta^{2(m+1)} \frac{\|s_0 - s\|}{\|s_{1,0} - s\|}.$$

Since $0 < \zeta < 1$, we have $\frac{\eta_m}{\eta_{1,m}} \to 0$ as $m \to \infty$. It follows that the sequence $\{s_m\}$ converges faster than $\{s_{1,m}\}$ to s. Now, by the normal-S algorithm (5), we have

$$\begin{aligned}\|s_{m+1}-s\| &= \|R\left((1-a_m)\,s_m + a_m R s_m\right) - s\| \\ &\le \zeta\|(1-a_m)s_m + a_m R s_m - s\| \\ &\le \zeta[(1-a_m)\|s_m - s\| + a_m\|Rs_m - s\|] \\ &\le \zeta[(1-a_m)\|s_m - s\| + a_m\zeta\|s_m - s\|] \\ &= \zeta[(1-a_m) + a_m\zeta]\|s_m - s\| \\ &\le \zeta\left\|s_m - s\right\|.\end{aligned}$$

Similarly, we get

$$\|s_{m+1}-s\| \le \zeta^{m+1}\left\|s_0 - s\right\|.$$

Let

$$\|s_{5,m}-s\| \le \zeta^{(m+1)}\left\|s_{5,0}-s\right\| = \eta_{5,m}.$$

Then

$$\frac{\eta_m}{\eta_{5,m}} = \frac{\zeta^{3(m+1)}\left\|s_0 - s\right\|}{\zeta^{m+1}\left\|s_{5,0}-s\right\|} = \zeta^{2(m+1)}\frac{\|s_0 - s\|}{\|s_{5,0}-s\|}.$$

We get $\frac{\eta_m}{\eta_{5,m}} \to 0$ as $m \to \infty$. It follows that the the sequence $\{s_m\}$ converges faster than $\{s_{5,m}\}$ to the fixed point s. Sintunavarat and Pitea [22] have shown that

$$\|s_{6,m}-s\| \le \zeta^{(m+1)}[1-(1-\zeta)e(f-g+gf)]^{(m+1)}\left\|s_{6,0}-s\right\|, m \in \mathbb{Z}_+.$$

Taking the assumption $1-(1-\zeta)e(f-g+gf) \le 1$ into consideration, we obtain

$$\|s_{6,m}-s\| \le \zeta^{m+1}\left\|s_{6,0}-s\right\| = \eta_{6,m}.$$

Then

$$\frac{\eta_m}{\eta_{6,m}} = \frac{\zeta^{3(m+1)}\left\|s_0 - s\right\|}{\zeta^{(m+1)}\left\|s_{6,0}-s\right\|} = \zeta^{2(m+1)}\frac{\|s_0 - s\|}{\|s_{6,0}-s\|}.$$

Thus, we get $\frac{\eta_m}{\eta_{6,m}} \to 0$ as $m \to \infty$. Hence, $\{s_m\}$ converges faster than $\{s_{6,m}\}$ to s. It was shown by Sintunavarat and Pitea [22] that for the class of weak contraction mappings, the Varat algorithm converges more rapidly than the Mann, Ishikawa, and S iteration.

Now, for the F^* iteration, we have

$$\begin{aligned}\|s_{m+1}-s\| &= \left\|R^2\left((1-a_m)s_m + a_m R s_m\right) - s\right\| \\ &\le \zeta\left\|R\left((1-a_m)s_m + a_m R s_m\right) - s\right\| \\ &\le \zeta^2\left\|(1-a_m)s_m + a_m R s_m - s\right\| \\ &\le \zeta^2[(1-a_m)\|s_m - s\| + a_m\|Rs_m - s\|] \\ &\le \zeta^2[(1-a_m)\|s_m - s\| + a_m\zeta\|s_m - s\|] \\ &= \zeta^2[(1-a_m) + a_m\zeta]\|s_m - s\| \\ &\le \zeta^2\|s_m - s\|.\end{aligned}$$

Similarly, we get

$$\|s_{m+1} - s\| \leq \zeta^{2(m+1)} \|s_0 - s\|.$$

Let

$$\|s_{7,m} - s\| \leq \zeta^{2(m+1)} \|s_{7,0} - s\| = \eta_{7,m}.$$

Then

$$\frac{\eta_m}{\eta_{7,m}} = \frac{\zeta^{3(m+1)} \|s_0 - s\|}{\zeta^{2(m+1)} \|s_{7,0} - s\|} = \zeta^{(m+1)} \frac{\|s_0 - s\|}{\|s_{7,0} - s\|}.$$

Thus, we get $\frac{\eta_m}{\eta_{7,m}} \to 0$ as $m \to \infty$. Hence the sequence $\{s_m\}$ converges to the fixed point s faster than the sequence $\{s_{7,m}\}$. Therefore, the AT algorithm converges more rapidly than the other iterative algorithms. □

Example 1. Let $Q = \mathbb{R}$ be a complete normed linear space with the usual norm and let $P = [0, \pi]$ be a subset of Q. Define a self-mapping $R : Q \to Q$ by $R(p) = \cos\left(\frac{p}{2}\right)$ for all $p \in P$. It can be verified that R is a weak contraction with the condition (11). Therefore, R has a fixed point which is unique, namely $s = 0.9$. Choose the sequences $a_m = 0.5 = c_m$. Using Python, it was determined that the AT iterative algorithm, described by (8), converges faster to the fixed point $s = 0.9$ in comparison to the Mann, Varat, F^*, S, Picard and normal-S iteration. Refer to Table 1, Figures 1–2 for details.

Table 1. A comparative analysis of iterations: Example 1.

iteration	AT	F^*	Picard	normal_s	Mann	Varat
0	1.658950	1.658950	1.658950	1.658950	1.658950	1.658950
1	0.893291	0.934867	0.675263	0.725825	1.517125	0.688976
2	0.900422	0.901728	0.943542	0.929411	1.403039	0.939580
3	0.900367	0.900420	0.890765	0.895096	1.310885	0.892081
4	0.900367	0.900369	0.902446	0.901311	1.236186	0.902078
5	0.900367	0.900367	0.899914	0.900198	1.175459	0.900012
6	0.900367	0.900367	0.900466	0.900398	1.125969	0.900441
7	0.900367	0.900367	0.900346	0.900362	1.085556	0.900352
8	0.900367	0.900367	0.900372	0.900368	1.052500	0.900370
9	0.900367	0.900367	0.900366	0.900367	1.025423	0.900367

4. Applications

This section discusses how our proposed method can be utilized in the fields of polynomiography, absolute value equations, and Julia sets.

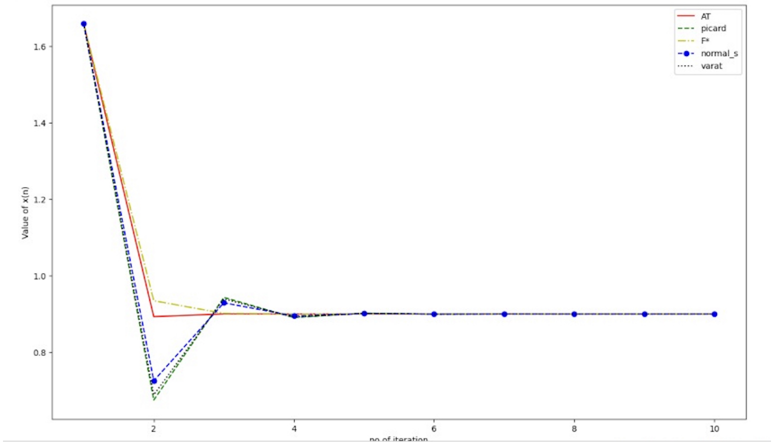

FIGURE 1. Comparisons of iterations of Example 1.

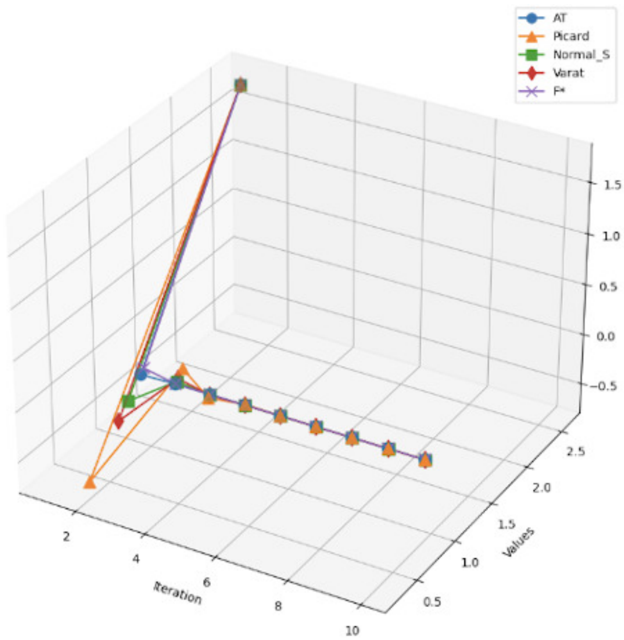

FIGURE 2. Comparison of errors in different iterations with the AT iteration for Example 1.

4.1. Polynomiography. Kalantari introduced the concept of *polynomiography* in [11], describing it as both an artistic and scientific approach to visualizing the roots of complex polynomials. The resulting images, created through this process, are known as polynomiographs. In this section, we conduct a comparative analysis of the AT iteration scheme with several well-established iterative methods including Picard iteration (1), Mann (2), Ishikawa (3), S (4), normal-S (5), and F^* (7). To demonstrate the practical significance of our approach, we apply it to approximate the Newton iteration operator using polynomiography and will show the faster convergence rate of the AT algorithm as an application of our theoretical work.

Let

$$s_{m+1} = s_m - \frac{R(s_m)}{R'(s_m)}, \tag{14}$$

where $s_0 \in \mathbb{C}$ is an initial approximation, and $R(s)$ is a polynomial function. The above formula can be expressed in the form of a fixed-point iteration

$$s_{m+1} = F(s_m), \quad \text{where} \quad F(s) = s - \frac{R(s)}{R'(s)}. \tag{15}$$

Note that this process is equivalent to the Picard iteration [24]. If the sequence approaches to a fixed point $s \in \mathbb{C}$ of the function F, we can determine that

$$s = F(s) = s - \frac{R(s)}{R'(s)}. \tag{16}$$

Thus, s is a root of F. In fact,

$$\frac{R(s)}{R'(s)} = 0 \quad \Leftrightarrow \quad R(s) = 0. \tag{17}$$

We utilize the AT iteration (8) along with other iterative methods in place of the Picard iteration (1) to generate polynomiographs by following the procedure described in Algorithm 1. The algorithm is executed with a maximum number of iterations $N = 15$, over the area $[-2, 2]$, with a tolerance $\epsilon = 0.001$, and it employs the *plasma* colormap. Furthermore, by analyzing the polynomiograph produced via Algorithm 1, we compute the average number of iterations (ANI) [8].

By utilizing the polynomiograph created through Algorithm 1, we can determine the average number of iterations (ANI), as referenced in [8]. The ANI is calculated using the following equation

$$ANI = \frac{1}{|E|} \sum_{z \in E} I(z),$$

where

- E denotes the collection of initial points being analyzed,
- $|E|$ signifies the total count of initial points within the set E,
- $I(z)$ represents the number of iterations needed for the point $z \in E$ to converge,
- $\sum_{z \in E} I(z)$ accounts for the total iterations across all points in E.

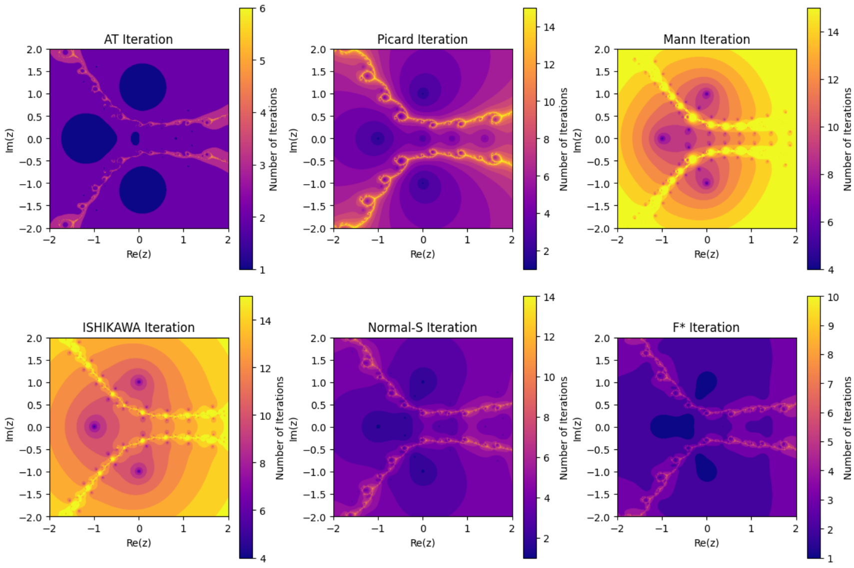

FIGURE 3. Polynomiographs generated by the polynomial $z^3 + z^2 + z + 1$.

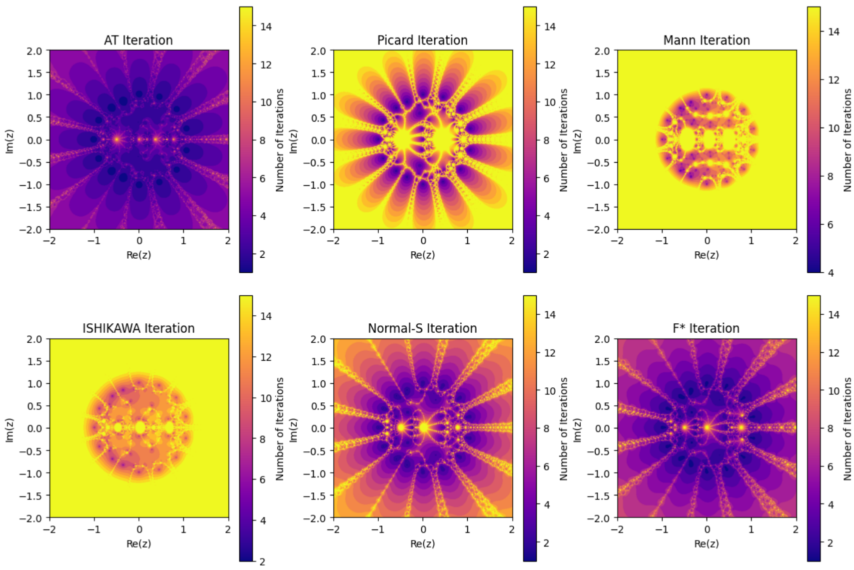

FIGURE 4. Polynomiograph generated by the polynomial $z^{15} + z^2 + z + 1$.

Algorithm 1 Generation of a polynomiograph.

Input: $R \in \mathbb{C}[Z]$, $\deg R \geq 2$ (polynomial); I (iteration process); $A \subset \mathbb{C}$ (area); N (maximum iterations); ϵ (accuracy); colours-(color map).
Output: Polynomiograph for the complex-valued polynomial R within the area A.
for $s_0 \in A$ **do**
  $n = 0$
  while $|Rs_m| > \epsilon$ and $m < N$ **do**
    $s_{m+1} = I(s_m, R)$
    $m = m + 1$
  end while
  Map m to a color from the color map and color s_0.
end for

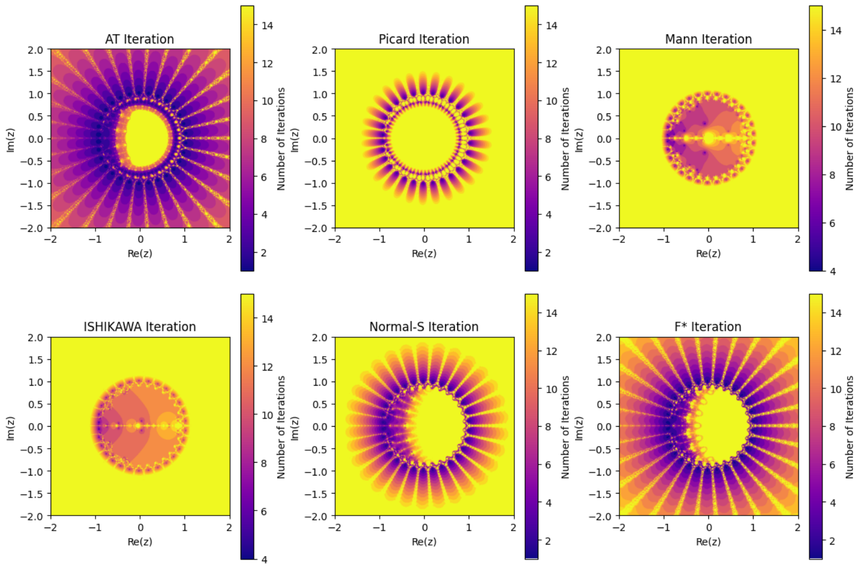

FIGURE 5. Polynomiographs generated by the polynomial $z^{30} + z + 1$.

ANI Analysis: From Figures 3, 4, and 5, we observe the results corresponding to the polynomials $z^3 + z^2 + z + 1$, $z^{15} + z^2 + z + 1$, and $z^{30} + z + 1$, respectively. The minimum ANI values corresponding to the AT iteration for these polynomials are 1.913, 3.679, and 6.801, respectively, indicating that the AT iteration achieves the fastest convergence compared to other iterative methods. These advantages establish the AT iteration as a highly efficient method with faster convergence and better numerical stability.

TABLE 2. Average number of iterations (ANI) for $z^3 + z^2 + z + 1$.

Iteration method	**ANI value**
AT iteration	1.913
F* iteration	2.403
Normal-S iteration	3.727
Picard iteration	5.715
Mann iteration	13.074
Ishikawa iteration	12.251

TABLE 3. Average number of iterations for $z^{15} + z^2 + z + 1$.

Iteration method	**ANI value**
AT iteration	3.679
F* iteration	5.203
Normal-S iteration	7.994
Picard iteration	11.700
Mann iteration	14.371
Ishikawa iteration	14.143

TABLE 4. Average number of iterations for $z^{30} + z + 1$.

Iteration method	**ANI value**
AT iteration	6.801
F* iteration	9.358
Normal-S iteration	12.242
Picard iteration	13.796
Mann iteration	14.183
Ishikawa iteration	14.236

4.2. Absolute value equation. Hashemi [9] presented matrix equations resembling Sylvester's form, given by the expression

$$AXB + C|X|D = E, \tag{18}$$

where the matrices A, B, C, D, and E are in $\mathbb{R}^{n\times n}$, and the matrix $X \in \mathbb{R}^{n\times n}$ is unknown. Now, we present a theorem that demonstrates an application of the AT iteration (8) for exploring absolute value equations.

Theorem 5. *Let $\beta = \|A^{-1}\|\|C\|\|D\|\|B^{-1}\|$, with the condition $0 \leq \beta < 1$, and where A and B are invertible matrices. Then the matrix equation* (18) *possesses a unique solution that can be determined through the AT iteration method.*

Proof. Let us have the matrix equation (18). We can express (18) as

$$AXB = E - C|X|D.$$

Invertibility of the matrices A and B yields

$$X = A^{-1}(E - C|X|D)B^{-1}.$$

We define the operator $R(X)$ as follows:

$$R(X) = A^{-1}(E - C|X|D)B^{-1}.$$

To show that $R(X)$ is a contraction mapping, we need to prove that for any matrices X_1 and X_2, there exists a constant $0 \leq \beta < 1$ such that

$$\|R(X_1) - R(X_2)\| \leq \beta \|X_1 - X_2\|.$$

We compute the difference $R(X_1) - R(X_2) = A^{-1}C(|X_2| - |X_1|)DB^{-1}$.
Taking the norm, we obtain

$$\|R(X_1) - R(X_2)\| \leq \|A^{-1}\|\|C\|\||X_2| - |X_1|\|\|D\|\|B^{-1}\|.$$

Since $\||X_2| - |X_1|\| \leq \|X_2 - X_1\|$, we have

$$\|R(X_1) - R(X_2)\| \leq \beta \|X_2 - X_1\|,$$

where $\beta = \|A^{-1}\|\|C\|\|D\|\|B^{-1}\|$. Since $\beta < 1$, the iteration process described in equation (8) will converge to the unique fixed point of $R(X)$. Therefore, the matrix equation (18) has a unique fixed point. ☐

Example 2. The following example illustrates the iterative solution to find X satisfying $AXB + C|X|D = E$. Given matrices

$$A = \begin{bmatrix} 2.0 & 1.0 \\ 1.0 & 3.0 \end{bmatrix}, \quad B = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.0 \\ 0.0 & 0.5 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix}, \quad E = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix},$$

we aim to find X in the matrix equation (18), where $|X|$ represents the element-wise absolute value of X.

The initial guess for X is

$$X_0 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}.$$

We define an iterative rule based on the operator $g(X)$ as

$$g(X) = A^{-1}\left(E - C|X|D\right)B^{-1}.$$

Then the AT iteration converges towards the unique solution of the equation as

$$X = \begin{bmatrix} 0.4719 & 0.0998 \\ 0.0091 & 0.2904 \end{bmatrix},$$

and the solution X satisfies the matrix equation (18).

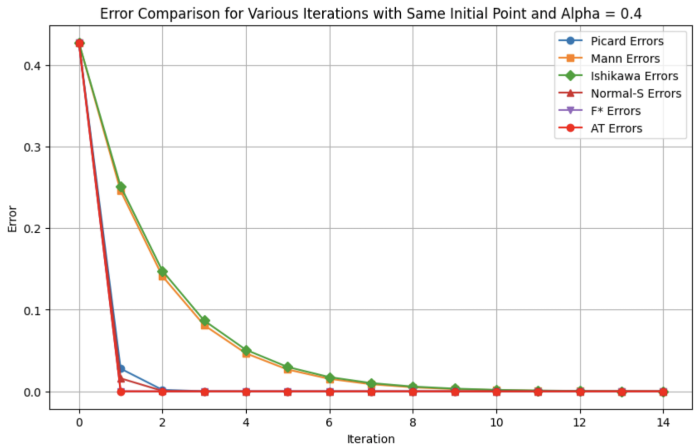

FIGURE 6. Error comparison of iterations for absolute value equation given by Example 2.

TABLE 5. Error comparison for various iteration methods for absolute value equation given by Example 2.

Iteration	Picard	Mann	Ishikawa	Normal-S	F*	AT
0	4.275503e-01	0.427363	0.427295	4.275503e-01	4.275503e-01	4.275503e-01
1	2.788477e-02	0.245668	0.251295	1.594373e-02	1.326553e-04	6.121653e-05
2	1.844690e-03	0.141234	0.147754	6.470726e-04	5.020757e-08	1.039568e-08
3	1.326553e-04	0.081190	0.086838	2.666216e-05	1.902289e-11	1.766881e-12
4	9.590260e-06	0.046655	0.050996	1.101244e-06	7.220132e-15	3.152427e-16
5	6.938651e-07	0.026786	0.029906	4.550244e-08	0.000000e+00	0.000000e+00
6	5.020757e-08	0.015350	0.017496	1.880231e-09	0.000000e+00	0.000000e+00
7	3.633044e-09	0.008765	0.010193	7.769470e-11	0.000000e+00	0.000000e+00
8	2.628894e-10	0.004972	0.005894	3.210472e-12	0.000000e+00	0.000000e+00
9	1.902290e-11	0.002787	0.003364	1.327014e-13	0.000000e+00	0.000000e+00
10	1.376505e-12	0.001527	0.001875	5.477421e-15	0.000000e+00	0.000000e+00
11	9.960920e-14	0.000801	0.000999	3.236829e-16	0.000000e+00	0.000000e+00
12	7.206301e-15	0.000382	0.000483	0.000000e+00	0.000000e+00	0.000000e+00
13	5.095246e-16	0.000140	0.000179	0.000000e+00	0.000000e+00	0.000000e+00
14	0.000000e+00	0.000000	0.000000	0.000000e+00	0.000000e+00	0.000000e+00

Using Python, it is determined that the AT iterative algorithm described by (8) converges faster in comparison to the Picard, Mann, Ishikawa, normal-S, and F^* iterative algorithms for parameters $a_m = c_m = d_m = 0.4$. Refer to Table 5 and Figure 6.

4.3. Julia set. Julia sets are fractal structures generated through the repeated iteration of complex functions and are closely related to fixed point iterative processes [26, 19]. In this section, we will generate Julia sets utilizing the recently introduced AT algorithm (8) and compare their dynamical and graphical characteristics with those obtained from the Picard (1), Mann (2), and Normal-S (4) iterations.

Algorithm 2 Julia set generation

Input: $R(z) = \alpha z^k + c$, where $\alpha, c \in \mathbb{C}$, $k, m \in \mathbb{N}$, $k \geq 2$; $A \subset \mathbb{C}$ — the complex area; K — maximum number of iterations; $a_m, c_m \in (0, 1]$ — parameters for the iteration process; `colormap[0..C-1]` — a colormap with C colors; $c \in \mathbb{C}$ — fixed complex parameter.
Output: Julia set corresponding to the area A and the fixed parameter c.
for each $z_0 \in A$ **do**
  $I \leftarrow 2$
  $m \leftarrow 0$
  $s_0 \leftarrow z_0$
  while $|s_{m+1}| < I$ **and** $m < K$ **do**
    $x_m \leftarrow (1 - a_m)s_m + a_m R s_m$
    $b_m \leftarrow \frac{1}{2}R(R(s_m)) + \frac{1}{2}R(R(x_m))$
    $s_{m+1} \leftarrow R((1 - c_m)b_m + c_m R b_m)$
    $m \leftarrow m + 1$
  end while
  $i \leftarrow \lfloor (C - 1) \cdot \frac{m}{K} \rfloor$
  Color z_0 with `colormap`[i]
end for

We will present visualizations of Julia set for the iterative function using Python:

$$g_{\alpha,c}(z) = \alpha z^k + c.$$

In this context, let k represent the power, with $k = 30$ indicating the maximum number of iterations. The parameters α and c are complex numbers, where α is not equal to zero. The complex grid is defined over the region $[-2, 2] \times [-2, 2]$ with a resolution of 500×500, and the escape radius is set to $I = 2$. The primary colormap used for visualizing these fractals is `turbo`.

Example 3. In this example, the parameters are $k = 3$, $\alpha = 2$, and θ: $= a_m = d_m = c_m = 0.5$. The colormap utilized is `turbo`.

Example 4. In this example, the parameters are $k = 5$, $\alpha = 1$, and $\theta :=$ $a_m = d_m = c_m = 0.5$. The colormap utilized is `turbo`.

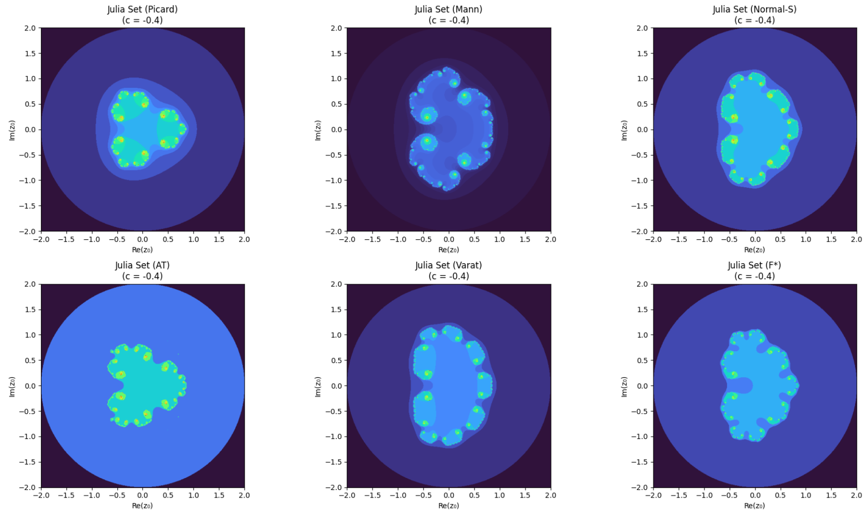

FIGURE 7. Julia set corresponding to $k = 3$.

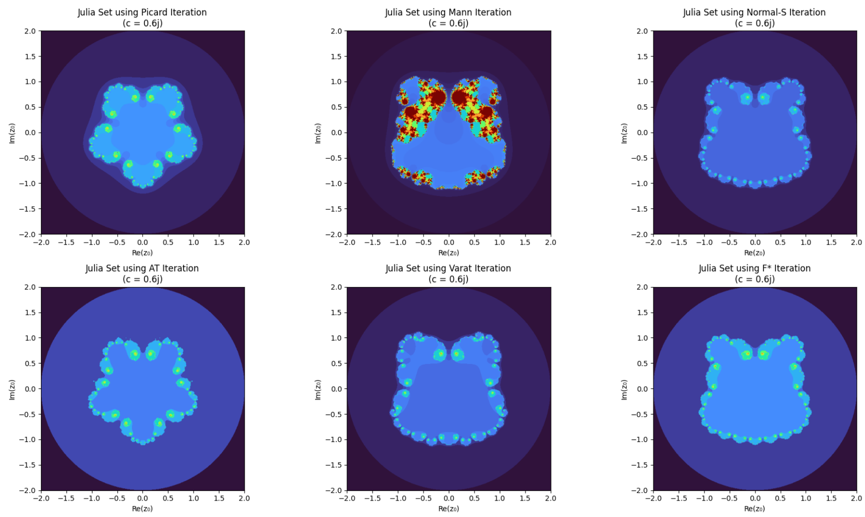

FIGURE 8. Julia set corresponding to $k = 5$.

- **Average escape time (AET)** indicates the average number of iterations required for points to exceed the specified escape radius during the iterative process in the complex plane. The AET metrics were first introduced in [14].

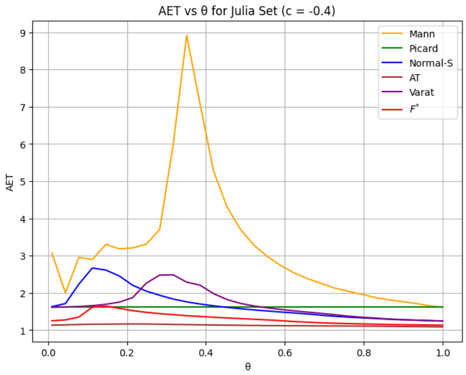

FIGURE 9. AET for $k = 3$.

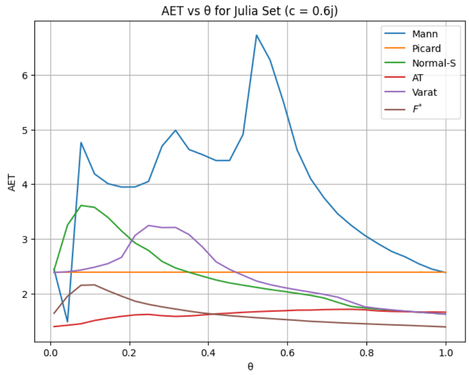

FIGURE 10. AET for $k = 5$.

TABLE 6. AET values for different iteration methods for $k = 3$.

Iteration method	**min AET**	**max AET**
Mann	1.615448	8.923675
Picard	1.615448	1.615448
Normal-S	1.245528	2.668128
Varat	1.245528	2.485544
F*	1.132232	1.645096
AT	1.087096	1.166424

TABLE 7. AET values for different iteration methods for $k = 5$.

Iteration method	**min AET**	**max AET**
Mann	1.483323	6.727068
Picard	2.385416	2.385416
Normal-S	1.624896	3.611688
Varat	1.624896	3.245376
F*	1.390824	2.158888
AT	1.397216	1.712454

AET observation. From Tables 6 and 7, as well as Figures 9 and 10, we observe that the AT iteration method exhibits superior efficiency, with the lowest minimum and maximum AET values among all methods. Its minimal variation in execution time ensures stability, making it the most optimal and reliable choice in terms of AET.

5. Conclusion

We introduced a three-step iterative algorithm aimed at approximating the fixed points of weak contractions. This algorithm shows improved efficiency and faster convergence compared to several traditional iterative methods, as indicated by the results of Theorem 2 and Theorem 3, which confirm the almost R-stability of the AT iterative algorithm. To support our claims, we include an example that demonstrates the effectiveness of our approach. Additionally, we discuss the application of the AT algorithm in proving the convergence of the absolute value equation under certain conditions. We also illustrate the benefits of the AT algorithm when combined with the Newton method, reinforcing our findings through the exploration of polynomiographs and Julia fractals.

In the present work, the AT algorithm is applied as a three-step iterative process to approximate the fixed point of weak contractions. Although the mapping R may be evaluated more times per iteration than in some classical methods such as the F^* iteration, resulting in a slightly higher computational cost, the numerical results demonstrate faster convergence compared to F^*, S, Normal-S, Varat, Mann, Ishikawa, and Picard iterations. Moreover, these analyzes reveal the Average Number of Iterations (ANI) and the Average Escape Time (AET), highlighting the algorithm's efficiency and faster convergence when utilized under suitable conditions.

Acknowledgements

The authors would like to express their sincere thanks to the anonymous referees and the editor for their constructive comments and valuable suggestions, which significantly improved the presentation of the manuscript. The third author is thankful to the DSR, Islamic University of Madinah, KSA.

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