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Cohomologies and linear deformations of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras

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ABSTRACT. The cohomology theory of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras is introduced. The cohomological approach is used to study linear deformations of relative Rota–Baxter operators. In particular, the notion of Nijenhuis elements is introduced to characterize trivial linear deformations.

1. Introduction

Pre-Jacobi–Jordan algebras are recent algebraic structures in the area of non-associative algebras, very close to Jacobi–Jordan algebras and blending ideas from pre-Lie algebras. They are an important subclass of the class of Jacobi–Jordan-admissible algebras studied in [4], where various constructions of these algebras are provided. In recent years, some researchers have become interested in the study of these algebras. In this sense, the cohomology theory of pre-Jacobi–Jordan algebras is introduced [2] and is called a zigzag cohomology, since the cochain complex consists of two sequences of applications.

The notion of Rota–Baxter algebra was first introduced by Baxter in his study of the fluctuation theory in probability [7]. Since then, much further work on this notion has been done. Indeed, Baxter's work was further investigated by Rota and Cartier in [19],[8], respectively, and many excellent results concerning Rota–Baxter algebras are obtained by Guo et al. [14, 15, 16]. Later, Tang, Bai, Guo and Sheng [21] studied deformation theory and cohomology theory of relative Rota–Baxter operators on

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Lie algebras, with applications to Rota–Baxter Lie algebras in mind. Next, relative Rota–Baxter operators on Leibniz algebras were studied in [20] and their deformation and cohomology theories were obtained in [22]. Observe that the deformation of algebraic structures began with the seminal work of Gerstenhaber [12, 13] for associative algebras and it was followed by its extension to Lie algebras by Nijenhuis and Richardson [17, 18]. In general, deformation theory was developed for binary quadratic operads by Balavoine [6]. Deformations of morphisms and $\mathcal{O}$-operators (also called relative Rota–Baxter operators) were developed in [1, 9, 10, 11, 21]. Up to now, there was very little study about (relative) Rota–Baxter operators on pre-Jacobi–Jordan algebras. Thus it is time to move in this direction by approaching such a subject. More precisely, we are interested in the study of linear deformations of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras using the cohomological approach. For this purpose, we first define the cohomology of relative Rota–Baxter operators on these algebras. Then we study linear deformations and introduce the notion of Nijenhuis elements associated to a relative Rota–Baxter operator, which can give rise to trivial linear deformations.

The outline of the paper is as follows. In Section 2, we give some basic notions and concepts, namely some properties of pre-Jacobi–Jordan algebras and their representations. Here, the notion of matched pair of pre-Jacobi–Jordan algebras is provided and a necessary and sufficient condition to obtain it, is given. In Section 3, we introduce the notion of relative Rota–Baxter operator on pre-Jacobi–Jordan algebra with respect to representation. Given a relative Rota–Baxter operator, there is a natural pre-Jacobi–Jordan algebra structure on the representation space. We define the cohomology theory of a relative Rota–Baxter operator on a pre-Jacobi–Jordan algebra in terms of the cohomology of the pre-Jacobi–Jordan algebra structure on the representation space. In Section 4, we study the linear deformation theory of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras. Moreover, we introduce the notion of Nijenhuis elements associated to a relative Rota–Baxter operator, which gives rise to a trivial linear deformation of the relative Rota–Baxter operator. Finally, we build a relationship between linear deformations of relative Rota–Baxter operators and linear deformations of the underlying pre-Jacobi–Jordan algebras.

Throughout this paper, we work over the complex field $\mathbb{C}$ and all the vector spaces are finite-dimensional.

2. Preliminaries and matched pairs of pre-Jacobi–Jordan algebras

In this section, we introduce the notion of a matched pair of pre-Jacobi–Jordan algebras. Next, we prove that a matched pair of a given pre-Jacobi–Jordan algebra gives rise to a matched pair of the corresponding sub-adjacent Jacobi–Jordan algebras.

Definition 1. An algebra $(A,*)$ is said to be a *Jacobi–Jordan algebra* if

$$\begin{aligned} &x*y=y*x \ (\text{ commutativity}),\\ &(x*y)*z+(y*z)*x+(z*x)*y=0, \ \text{for all } x,y,z\in A. \qquad (1)\end{aligned}$$

Observe that (1) is called the Jacobi identity.

Now, we give the definition of a representation of the Jacobi–Jordan algebra.

Definition 2 ([23])**.** A *representation of a Jacobi–Jordan algebra* $(A,*)$ is a couple (V,ρ) where V is a vector space and $\rho: A\to gl(V)$ is a linear map such that the following identity holds:

$$\rho(x*y)=-\rho(x)\rho(y)-\rho(y)\rho(x), \ \text{for all } x,y\in A. \qquad (2)$$

The following result will be used later.

Proposition 1 ([5])**.** *Let $(A,*)$ be a Jacobi–Jordan algebra. Then (V,ρ) is a representation of $(A,*)$ if and only if the direct sum of vector spaces $A\oplus V$ turns into a Jacobi–Jordan algebra with the multiplication defined by*

$$(x+u)\diamond(y+v):=x*y+(\rho(x)v+\rho(y)u).$$

This Jacobi–Jordan algebra is called the *semi-direct product* of A with V and is denoted by $A\ltimes V$.

Definition 3 ([3])**.** Let (V,ρ) be a representation of a Jacobi–Jordan algebra $(A,*)$. A linear operator $T: V\to A$ is called a *relative Rota–Baxter operator* of A with respect to ρ if it satisfies

$$T(u)*T(v)=T\Big(\rho(T(u))v+\rho(T(v))u\Big), \ \text{for all } u,v\in V. \qquad (3)$$

Observe that Rota–Baxter operators on Jacobi–Jordan algebras are relative Rota–Baxter operators with respect to the regular representation.

Proposition 2 ([3])**.** *Let $(A,*)$ be a Jacobi–Jordan algebra and (V,ρ) be a representation of $(A,*)$. Then, $A\oplus V$ is a representation of $(A,*)$ under the maps $\rho_{A\oplus V}: A\to gl(A\oplus V)$ defined by*

$$\rho_{A\oplus V}(a)(b+v):=(a*b)+\rho(a)v.$$

Define a linear map $T: A\oplus V\to A, a+u\mapsto a$. Then T is up to a scalar coefficient, a relative Rota–Baxter operator on A with respect to the representation $(A\oplus V,\rho_{A\oplus V})$.

Definition 4 ([3])**.** A *matched pair of Jacobi–Jordan algebras* consists of two Jacobi–Jordan algebras $(A,*)$ and $(B,\bullet)$ together with representations $\rho_1 : A \to gl(B)$ and $\rho_2 : B \to gl(A)$ such that, for all $x,y \in A$, $a,b \in B$, the following conditions hold:

$$\begin{aligned}&\rho_1(x)(u\bullet v)+(\rho_1(x)u)\bullet v+(\rho_1(x)v)\bullet u\\&\quad+\rho_1(\rho_2(u)x)v+\rho_1(\rho_2(v)x)u=0,\forall(x,u,v)\in A\otimes B\otimes B, \qquad (4)\\&\rho_2(u)(x*y)+(\rho_2(u)x)*y+(\rho_2(u)y)*x\\&\quad+\rho_2(\rho_1(x)u)y+\rho_2(\rho_1(y)u)x=0,\forall(x,y,u)\in A\otimes B\otimes B. \qquad (5)\end{aligned}$$

Such a matched pair is denoted by (A,B,ρ_1,ρ_2).

Theorem 1 ([3])**.** *Let $(A,*)$ and $(B,\bullet)$ be Jacobi–Jordan algebras. Then (A,B,ρ_1,ρ_2) is a matched pair of Jacobi–Jordan algebras if and only if $(A\oplus B,\diamond)$ is a Jacobi–Jordan algebra, where*

$$\begin{aligned}(x+u)\diamond(y+v) \;:=\; & (x*y+\rho_2(u)y+\rho_2(v)x)\\ & +(u\bullet v+\rho_1(x)v+\rho_1(y)u).\end{aligned}$$

Definition 5. Let $(A,\cdot)$ be an algebra.

(i) The *anti-associator* of $(A,\cdot)$ is the map defined by

$$Aasso(x,y,z):=(x\cdot y)\cdot z+x\cdot(y\cdot z),\text{ for all } x,y,z\in A. \qquad (6)$$

(ii) A *left pre-Jacobi–Jordan* (*left skew-symmetric*) *algebra* is an algebra $(A,\cdot)$ satisfying

$$Aasso(x,y,z)=-Aasso(y,x,z),\text{ for all } x,y,z\in A, \qquad (7)$$

i.e., the anti-associator is left skew-symmetric. Actually, (7) is equivalent to

$$(x*y)\cdot z=-x\cdot(y\cdot z)-y\cdot(x\cdot z),\text{ for all } x,y,z\in A, \qquad (8)$$

where $x*y=x\cdot y+y\cdot x$, for all $x,y\in A$.

(iii) If the anti-associator is right skew-symmetric, i.e.,

$$Aasso(x,y,z)=-Aasso(x,z,y)$$

or equivalently

$$x\cdot(y*z)=-(x\cdot y)\cdot z-(x\cdot z)\cdot y,\text{ for all } x,y,z\in A,$$

then the algebra is said to be a *right pre-Jacobi–Jordan* (*right skew symmetric*) *algebra.*

Proposition 3. *Let $(A,\cdot)$ be a left pre-Jacobi–Jordan algebra. Then the product, given by*

$$x*y=x\cdot y+y\cdot x, \qquad (9)$$

defines a Jacobi–Jordan algebra structure on A, which is called the associated (or sub-adjacent) Jacobi–Jordan algebra of $(A,\cdot)$ denoted by A^C.

Proof. For all $x, y, z \in A$, we prove (1) as follows

$$\begin{aligned}J(x,y,z) &= \circlearrowleft_{(x,y,z)} (x * y) * z\\ &= \circlearrowleft_{(x,y,z)} \Big((x \cdot y)\cdot z + (y\cdot x)\cdot z + z\cdot(x\cdot y) + z\cdot(y\cdot x)\Big)\\ &= \circlearrowleft_{(x,y,z)} \Big(Aasso(x,y,z) + Aasso(y,x,z)\Big) = 0 \text{ (by (7))}.\end{aligned}$$

□

Definition 6 ([2]). A *representation of a left pre-Jacobi–Jordan algebra* $(A, \cdot)$ on a vector space V consists of a pair (ρ, μ), where $\rho, \mu : A \to gl(V)$ are linear maps satisfying

$$\rho(x * y) = -\rho(x)\rho(y) - \rho(y)\rho(x), \tag{10}$$

$$\mu(y)\mu(x) + \mu(x\cdot y) = -\mu(y)\rho(x) - \rho(x)\mu(y), \tag{11}$$

for all $x, y \in A$, where $x * y := x \cdot y + y \cdot x$.

Observe that condition (10) means that (V, ρ) is a representation of the subadjacent Jacobi–Jordan of $(A, \cdot)$. One can easily prove that (V, ρ, μ) is a representation of a left pre-Jacobi–Jordan algebra $(A, \cdot)$ if and only if the direct sum $A \oplus V$ of vector spaces turns into a left pre-Jacobi–Jordan algebra under the product

$$\begin{aligned}&(x+u) \circledast (y+v)\\ &:= x\cdot y + (\rho(x)v + \mu(y)u), \text{ for all } x, y \in A \text{ and } u, v \in V.\end{aligned} \tag{12}$$

This left pre-Jacobi–Jordan algebra is called a *semi-direct product* of A and V denoted by $A \ltimes V$.

Let us prove the following.

Proposition 4. *Let (V, ρ, μ) be a representation of a left pre-Jacobi–Jordan algebra $(A, \cdot)$ and $(A, *)$ be its associated Jacobi–Jordan algebra. Then $(V, \rho + \mu)$ is a representation of $(A, *)$.*

Proof. Let (V, ρ, μ) be a representation of a left pre-Jacobi–Jordan algebra $(A, \cdot)$ and $(A, *)$ be its associated Jacobi–Jordan algebra.

We know that $A \ltimes V = (A \oplus V, \circledast)$ is a left pre-Jacobi–Jordan algebra where $\circledast$ is defined by (12). Consider its associated Jacobi–Jordan algebra $(A \oplus V, \star)$, where $\star$ is a Jordan product using $\circledast$. Then we have

$$\begin{aligned}&(x+u)\star(y+v) = (x+u)\circledast(y+v) + (y+v)\circledast(x+u)\\ &= x\cdot y + (\rho(x)v + \mu(y)u) + y\cdot x + (\rho(y)u + \mu(x)v)\\ &= x * y + \Big((\rho+\mu)(x)v + (\rho+\mu)(y)u\Big).\end{aligned}$$

From Proposition 1, we conclude that $(V, \rho + \mu)$ is a representation of $(A, *)$.

□

Definition 7. A *matched pair of left pre-Jacobi–Jordan algebras* consists of two left pre-Jacobi–Jordan algebras $\mathcal{A}_1 := (A_1, \cdot)$ and $\mathcal{A}_2 := (A_2, \top)$ together with representations $\rho_1, \mu_1 : A_1 \to gl(A_2)$ and $\rho_2, \mu_2 : A_2 \to gl(A_1)$ such that for all $x, y, z \in A_1$, $u, v, w \in A_2$ the following conditions hold:

$$\begin{aligned}
\rho_1(x)(u \top v) &= -\rho_1(\rho_2(u)x + \mu_2(u)x)v - (\rho_1(x)u + \mu_1(x)u) \top v \\
&\quad -\mu_1(\mu_2(v)x)u - u \top (\rho_1(x)v), && (13)\\
\mu_1(x)(u \circledast v) &= -u \top (\mu_1(x)v) - v \top (\mu_1(x)u) - \mu_1(\rho_2(u)x)v \\
&\quad -\mu_1(\rho_2(v)x)u, && (14)\\
\rho_2(u)(x \cdot y) &= -\rho_2(\rho_1(x)u + \mu_1(x)u)y - (\rho_2(u)x + \mu_2(u)x) \cdot y \\
&\quad -\mu_2(\mu_1(y)u)x - x \cdot (\rho_2(u)y), && (15)\\
\mu_2(u)(x * y) &= -x \cdot (\mu_2(u)y) - y \cdot (\mu_2(u)x) - \mu_2(\rho_1(x)u)y \\
&\quad -\mu_2(\rho_1(y)u)x, && (16)
\end{aligned}$$

where $*$ is the product of the sub-adjacent Jacobi–Jordan algebra A_1^C and $\circledast$ is the product of the sub-adjacent Jacobi–Jordan algebra A_2^C. Such matched pair is denoted by $(A_1, A_2, \rho_1, \mu_1, \rho_2, \mu_2)$.

Theorem 2. *Let $\mathcal{A}_1 := (A_1, \cdot)$ and $\mathcal{A}_2 := (A_2, \top)$ be left pre-Jacobi–Jordan algebras. Then $(A_1, A_2, \rho_1, \mu_1, \rho_2, \mu_2)$ is a matched pair of left pre-Jacobi–Jordan algebras if and only if $(A_1 \oplus A_2, \diamond)$ is a left pre-Jacobi–Jordan algebra under the product "$\diamond$" given by*

$$\begin{aligned}
(x + u) \diamond (y + v) &:= (x \cdot y + \rho_2(u)y + \mu_2(v)x) \\
&\quad +(u \top v + \rho_1(x)v + \mu_1(y)u). && (17)
\end{aligned}$$

Proof. Pick $x, y, z \in A_1$ and $u, v, w \in A_2$. By straightforward computations, we get

$$\begin{aligned}
&Aasso_{A_1 \oplus A_2}(x + u, y + v, z + w)\\
&= ((x + u) \diamond (y + v)) \diamond (z + w) + (x + u) \diamond ((y + v) \diamond (z + w))\\
&= (x \cdot y) \cdot z + (\rho_2(u)y) \cdot z + (\mu_2(v)x) \cdot z + \rho_2(u \top v)z + \rho_2(\rho_1(x)v)z\\
&\quad + \rho_2(\mu_1(y)u)z + \mu_2(w)(x \cdot y) + \mu_2(w)\rho_2(u)y + \mu_2(w)\mu_2(v)x + (u \top v) \top w\\
&\quad + (\rho_1(x)v) \top w + (\mu_1(y)u) \top w + \rho_1(x \cdot y)w + \rho_1(\rho_2(u)y)w + \rho_1(\mu_2(v)x)w\\
&\quad + \mu_1(z)(u \top v) + \mu_1(z)\rho_1(x)v + \mu_1(z)\mu_1(y)u + x \cdot (y \cdot z) + x \cdot (\rho_2(v)z)\\
&\quad + x \cdot (\mu_2(w)y) + \rho_2(u)(y \cdot z) + \rho_2(u)\rho_2(v)z + \rho_2(u)\mu_2(w)y + \mu_2(v \top w)x\\
&\quad + \mu_2(\rho_1(y)w)x + \mu_2(\mu_1(z)v)x + u \top (v \top w) + u \top (\rho_1(y)w) + u \top (\mu_1(z)v)\\
&\quad + \rho_1(x)(v \top w) + \rho_1(x)\rho_1(y)w + \rho_1(x)\mu_1(z)v + \mu_1(y \cdot z)u + \mu_1(\rho_2(v)z)u\\
&\quad + \mu_1(\mu_2(w)y)u.
\end{aligned}$$

Switching $x+u$ and $y+v$ in the above expression of $Aasso_{A_1 \oplus A_2}(x+u, y+v, z+w)$, we obtain $Aasso_{A_1 \oplus A2}(y+v, x+u, z+w)$. Next, after rearranging terms, we obtain

$$\begin{aligned}
&Aasso_{A_1\oplus A_2}(x+u,y+v,z+w)+Aasso_{A_1\oplus A_2}(y+v,x+u,z+w)\\
&=\Big(Aasso_{A_1}(x,y,z)+Aasso_{A_1}(y,x,z)\Big)+\Big(Aasso_{A_2}(u,v,w)\\
&\quad+Asso_{A_2}(v,u,w)\Big)+\Big(\mu_1(z)\mu_1(y)u+\mu_1(y\cdot z)u+\mu_1(z)\rho_1(y)u\\
&\quad+\rho_1(y)\mu_1(z)u\Big)+\Big(\mu_1(z)\mu_1(x)v+\mu_1(x\cdot z)v+\mu_1(z)\rho_1(x)v\\
&\quad+\rho_1(x)\mu_1(z)v\Big)+\Big(\mu_2(w)\mu_2(v)x+\mu_2(v\top w)x+\mu_2(w)\rho_2(v)x\\
&\quad+\rho_2(v)\mu_2(w)x\Big)+\Big(\mu_2(w)\mu_2(u)y+\mu_2(u\top w)y+\mu_2(w)\rho_2(u)y\\
&\quad+\rho_2(u)\mu_2(w)y\Big)+\Big(\rho_1(x*y)w+\rho_1(x)\rho_1(y)w+\rho_1(y)\rho_1(x)w\Big)\\
&\quad+\Big(\rho_2(u\circledast v)z+\rho_2(u)\rho_2(v)z+\rho_2(v)\rho_2(u)z\Big)+\Big(\rho_1(y)(u\top w)\\
&\quad+\rho_1(\rho_2(u)y+\mu_2(u)y)w+(\rho_1(y)u+\mu_1(y)u)\top w+\mu_1(\mu_2(w)y)u\\
&\quad+u\top(\rho_1(y)w)\Big)+\Big(\rho_1(x)(v\top w)+\rho_1(\rho_2(v)x+\mu_2(v)x)w+(\rho_1(x)v\\
&\quad+\mu_1(x)v)\top w+\mu_1(\mu_2(w)x)v+v\top(\rho_1(x)w)\Big)+\Big(\mu_1(z)(u\circledast v)\\
&\quad+u\top(\mu_1(z)v)+v\top(\mu_1(z)u)+\mu_1(\rho_2(u)z)v+\mu_1(\rho_2(v)z)u\Big)\\
&\quad+\Big(\rho_2(u)(y\cdot z)+\rho_2(\rho_1(y)u+\mu_1(y)u)z+(\rho_2(u)y+\mu_2(u)y)\cdot z\\
&\quad+\mu_2(\mu_1(z)u)y+y\cdot(\rho_2(u)z)\Big)+\Big(\rho_2(v)(x\cdot z)+\rho_2(\rho_1(x)v\\
&\quad+\mu_1(x)v)z+(\rho_2(v)x+\mu_2(b)x)\cdot z+\mu_2(\mu_1(z)v)x+x\cdot(\rho_2(v)z)\Big)\\
&\quad+\Big(\mu_2(w)(x*y)+x\cdot(\mu_2(w)y)+y\cdot(\mu_2(w)x)+\mu_2(\rho_1(x)w)y\\
&\quad+\mu_2(\rho_1(y)w)x\Big).
\end{aligned}$$

From (7), (10) and (11), we get

$$\begin{aligned}
&Aasso(x+u,y+v,z+w)+Aasso(y+v,x+u,z+w)\\
&=\Big(\rho_1(y)(u\top w)+\rho_1(\rho_2(u)y+\mu_2(u)y)w+(\rho_1(y)u+\mu_1(y)u)\top w\\
&\quad+\mu_1(\mu_2(w)y)u+u\top(\rho_1(y)w)\Big)+\Big(\rho_1(x)(v\top w)+\rho_1(\rho_2(v)x+\mu_2(v)x)w\\
&\quad+(\rho_1(x)v+\mu_1(x)v)\top w+\mu_1(\mu_2(w)x)v+\alpha_2(v)\top(\rho_1(x)w)\Big)\\
&\quad+\Big(\mu_1(z)(u\circledast v)+u\top(\mu_1(z)v)+v\top(\mu_1(z)u)+\mu_1(\rho_2(u)z)v
\end{aligned}$$

$$\begin{aligned}
&+ \mu_1(\rho_2(v)z)u\Big) + \Big(\rho_2(u)(y\cdot z) + \rho_2(\rho_1(y)u + \mu_1(y)u)z \\
&+ (\rho_2(u)y + \mu_2(u)y)\cdot z + \mu_2(\mu_1(z)u)y + y\cdot(\rho_2(u)z)\Big) + \Big(\rho_2(v)(x\cdot z) \\
&+ \rho_2(\rho_1(x)v + \mu_1(x)v)z + (\rho_2(v)x + \mu_2(v)x)\cdot z + \mu_2(\mu_1(z)v)x \\
&+ x\cdot(\rho_2(v)z)\Big) + \Big(\mu_2(w)(x*y) + x\cdot(\mu_2(w)y) + y\cdot(\mu_2(w)x) \\
&+ \mu_2(\rho_1(x)w)y + \mu_2(\rho_1(y)w)x\Big).
\end{aligned}$$

Hence (7) holds in $A_1 \oplus A_2$ if and only (13), (14), (15) and (16) hold. ☐

Proposition 5. *Let* $(A_1, A_2, \rho_1, \mu_1, \rho_2, \mu_2)$ *be a matched pair of left pre-Jacobi–Jordan algebras* $\mathcal{A}_1 := (A_1, \cdot)$ *and* $\mathcal{A}_2 := (A_2, \top)$. *Then* $(A_1, A_2, \rho_1 + \mu_1, \rho_2 + \mu_2)$ *is a matched pair of sub-adjacent Jacobi–Jordan algebras* $A_1^C := (A_1, *)$ *and* $A_2^C := (A_2, \circledast)$.

Proof. We know that (A_1, ρ_2, μ_2) and (A_2, ρ_1, μ_1) are representations of left pre-Jacobi–Jordan algebras $(A_2, \top)$ and $(A_1, \cdot)$, respectively. From Proposition 4, $(A_1, \theta_2 := \rho_2 + \mu_2)$ and $(A_2, \theta_1 := \rho_1 + \mu_1)$ are representations of sub-adjacent Jacobi–Jordan algebras A_2^C and A_1^C, respectively. Next, pick $x, y \in A_1$ and $u, v \in A_2$. Then by straightforward computations, after rearranging terms and using (13), (14), we obtain

$$\begin{aligned}
&\theta_1(x)(u \circledast v) + (\theta_1(x)u) \circledast v + (\theta_1(x)v) \circledast u + \theta_1(\theta_2(u)x)v + \theta_1(\theta_2(v)x)u \\
&= \Big(\rho_1(x)(u \top v) + \rho_1(\rho_2(u)x + \mu_2(u)x)v + (\rho_1(x)u + \mu_1(x)u) \top v \\
&\quad + \mu_1(\mu_2(v)x)u + u \top (\rho_1(x)v)\Big) + \Big(\rho_1(x)(v \top u) + \rho_1(\rho_2(v)x + \mu_2(v)x)u \\
&\quad + (\rho_1(x)v + \mu_1(x)v) \top u + \mu_1(\mu_2(u)x)v + v \top (\rho_1(x)u)\Big) + \Big(\mu_1(x)(u \circledast v) \\
&\quad + u \top (\mu_1(x)v) + v \top (\mu_1(x)u) + \mu_1(\rho_2(u)x)v + \mu_1(\rho_2(v)x)u\Big) = 0.
\end{aligned}$$

Hence we have (4). Similarly, we compute

$$\begin{aligned}
&\theta_2(u)(x*y) + (\theta_2(u)x)*y + (\theta_2(u)y)*x + \theta_2(\theta_1(x)u)y + \theta_2(\theta_1(y)u)x \\
&= \Big(\rho_2(u)(x\cdot y) + \rho_2(\rho_1(x)u + \mu_1(x)u)y + (\rho_2(u)x + \mu_2(u)x)\cdot y \\
&\quad + \mu_2(\mu_1(y)u)x + x\cdot(\rho_2(u)y)\Big) + \Big(\rho_2(u)(y\cdot x) + \rho_2(\rho_1(y)u + \mu_1(y)u)x \\
&\quad + (\rho_2(u)y + \mu_2(u)y)\cdot x + \mu_2(\mu_1(x)u)y + y\cdot(\rho_2(u)x)\Big) + \Big(\mu_2(u)(x*y) \\
&\quad + x\cdot(\mu_2(u)y) + y\cdot(\mu_2(u)x) + \mu_2(\rho_1(x)u)y + \mu_2(\rho_1(y)u)x\Big) = 0 \\
&\quad \text{(by (15) and (16)).}
\end{aligned}$$

Thus, we get also (5). ☐

3. Representations and cohomologies of relative Rota–Baxter operators on left pre-Jacobi–Jordan algebras

In this section, we consider the notion of left (resp. right) pre-Jacobi–Jordan algebra first introduced in [5] as left-skew-symmetric (resp. right-skew-symmetric) algebra. We study their relationships with Jacobi–Jordan algebras in terms of relative Rota–Baxter operators of Jacobi–Jordan algebras. Finally, representations and cohomologies theories of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras are introduced and studied.

Definition 8. Let (V,ρ,μ) be a representation of a left pre-Jacobi–Jordan algebra $(A,\cdot)$. A linear map $T:V\to A$ is called a *relative Rota–Baxter operator* of A associated to (V,ρ,μ) if it satisfies

$$T(u)\cdot T(v)=T\Big(\rho(Tu)v+\mu(Tv)u\Big),\ \text{for all } u,v\in V. \tag{18}$$

Note that Rota–Baxter operators on a left pre-Jacobi–Jordan algebra of weight 0 are relative Rota–Baxter operators with respect to the regular representation.

Example 1. Consider the 2-dimensional real left pre-Jacobi–Jordan algebra $(A,\cdot)$ given with respect to a basis $\{e_1,e_2\}$ by

$$e_1\cdot e_1=e_2.$$

Then $T=\begin{pmatrix} x & a\\ y & b\end{pmatrix}$ is a relative Rota–Baxter operator on $(A,\cdot)$ with respect to the regular representation if and only if

$$T(e_i)\cdot T(e_j)=T\left(T(e_i)\cdot e_j+e_i\cdot T(e_j)\right),\ i,j=1,2. \tag{19}$$

From (19), we obtain the following system (S) : $x^2-2xb=xa=a^2=xa-ab=0$. Hence
$(S)\iff(a=x=0,\ b\in\mathbb{R},\ y\in\mathbb{R})$ or $(a=0,\ x=2b,\ y\in\mathbb{R})$.
Then, for all $y,b\in\mathbb{R}$, $T_{y,b}=\begin{pmatrix} 0 & 0\\ y & b\end{pmatrix}$ and $T'_{y,b}=\begin{pmatrix} 2b & 0\\ y & b\end{pmatrix}$ are relative Rota–Baxter operators on $(A,\cdot)$ with respect to the regular representation.

Lemma 1. *Let $(A,\cdot)$ be a left pre-Jacobi–Jordan algebra and (V,ρ,μ) be a representation of $(A,\cdot)$. Then, $A\oplus V$ is a representation of $(A,\cdot)$ under the maps $\rho_{A\oplus V},\mu_{A\oplus V}:A\to gl(A\oplus V)$ defined by*

$$\rho_{A\oplus V}(a)(b+v):=a\cdot b+\rho(a)v, \tag{20}$$

$$\mu_{A\oplus V}(a)(b+v):=b\cdot a+\mu(a)v. \tag{21}$$

Proof. Let $x,y,z\in A$ and $v\in V$. We have

$$\rho_{A\oplus V}(x*y)(z+v)$$

$$\begin{aligned}
&\overset{(20)}{=} (x * y)\cdot z + \rho(x * y)v\\
&= -x\cdot(y\cdot z) - y\cdot(x\cdot z) - \rho(x)\rho(y)v - \rho(y)\rho(x)v\\
&= -x\cdot(y\cdot z) - \rho(x)\rho(y)v - y\cdot(x\cdot z) - \rho(y)\rho(x)v\\
&= -\rho_{A\oplus V}(x)\rho_{A\oplus V}(y)(z+v) - \rho_{A\oplus V}(y)\rho_{A\oplus V}(x)(z+v).
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
&\mu_{A\oplus V}(x\cdot y)(z+v)\\
&\overset{(21)}{=} z\cdot(x\cdot y) + \mu(x\cdot y)v = -(z\cdot x)\cdot y - (x\cdot z)\cdot y - x\cdot(z\cdot y)\\
&\quad - \mu(y)\mu(x)v - \mu(y)\rho(x)v - \rho(x)\mu(y)v \text{ (by (7) and (11))}\\
&= \underbrace{-(z\cdot x)\cdot y - \mu(y)\mu(x)v}\underbrace{-x\cdot(z\cdot y) - \rho(x)\mu(y)v}\\
&\quad \underbrace{-(x\cdot z)\cdot y - \mu(y)\rho(x)v}\\
&= -\mu_{A\oplus V}(y)\mu_{A\oplus V}(x)(z+v) - \rho_{A\oplus V}(x)\mu_{A\oplus V}(y)(z+v)\\
&\quad - \mu_{A\oplus V}(y)\rho_{A\oplus V}(x)(z+v).
\end{aligned}$$

Hence the conclusion follows. ☐

Using the previous lemma and Proposition 2, we give the following example.

Example 2. Let $(A,\cdot)$ be a left pre-Jacobi–Jordan algebra and (V,ρ,μ) be a representation of $(A,\cdot)$. Consider the representation $A\oplus V$ of $(A,\cdot)$ given in the previous lemma and define the linear map $T : A\oplus V \to A, a+v \mapsto a$. Then T is up to a scalar coefficient, a relative Rota–Baxter operator of sub-adjacent Jacobi–Jordan algebra A^C associated to the representation $(A\oplus V, \rho_{A\oplus V} + \mu_{A\oplus V})$.

A relative Rota–Baxter operator of a pre-Jacobi–Jordan algebra turns into a relative Rota–Baxter operator of its associated Jacobi–Jordan algebra as we can see in the following result.

Proposition 6. *Let T be a relative Rota–Baxter operator of a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Then T is a relative Rota–Baxter operator of its associated Jacobi–Jordan algebra $(A,*)$ with respect to the representation $(V,\rho+\mu)$.*

Proof. Suppose that T is a relative Rota–Baxter operator of a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Pick $u,v \in V$. Since T is a relative Rota–Baxter operator of a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ), we have $Tu, Tv \in A$ and

$$Tu * Tv \qquad = Tu\cdot Tv + Tv\cdot Tu$$

$$= T\Big(\rho(Tu)v + \mu(Tv)u\Big) + T\Big(\rho(Tv)u + \mu(Tu)v\Big)$$
$$= T\Big((\rho+\mu)(Tu)v + (\rho+\mu)(Tv)u\Big).$$

Hence the conclusion follows. □

Proposition 7. *A linear map $T : V \to A$ is a relative Rota–Baxter operator of a left pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ) if and only if the graph of T,*

$$G_r(T) := \{(T(v), v) : v \in V\},$$

is a subalgebra of the semi-direct product algebra $A \ltimes V$.

Proof. Let $u, v \in V$. We compute

$$(Tu, u) \circledast (Tv, v) = (Tu \cdot Tv, \rho(Tu)v + \mu(Tv)u).$$

Hence

$$(Tu, u) \circledast (Tv, v) \in Gr(T) \Leftrightarrow Tu \cdot Tv = T(\rho(Tu)v + \mu(Tv)u).$$

□

The next result says that a relative Rota–Baxter operator can be lifted up the Rota–Baxter operator.

Proposition 8. *Let $(A, \cdot)$ be a left pre-Jacobi–Jordan algebra, (V, ρ, μ) be a representation on it and $T : V \to A$ be a linear map. Define $\widehat{T} \in End(A \oplus V)$ by $\widehat{T}(a+v) := T(v)$. Then T is a relative Rota–Baxter operator with respect to (V, ρ, μ) if and only if $\widehat{T}$ is a Rota–Baxter operator on $A \oplus V$.*

Proof. Let $a, b \in A$ and $u, v \in V$. Then we have

$$\widehat{T}(a+u) \circledast \widehat{T}(b+v) = Tu \circledast Tv = Tu \cdot Tv$$

and

$$\begin{aligned}&\widehat{T}(\widehat{T}(a+u) \circledast (b+v) + (a+u) \circledast \widehat{T}(b+v))\\ &= \widehat{T}(Tu \cdot b + \rho(Tu)v + a \cdot Tv + \mu(Tv)u)\\ &= T(\rho(Tu)v + \mu(Tv)u).\end{aligned}$$

Hence

$$\widehat{T}(a+u) \circledast \widehat{T}(b+v) = \widehat{T}(\widehat{T}(a+u) \circledast (b+v) + (a+u) \circledast \widehat{T}(b+v))$$

if and only if

$$Tu \cdot Tv = T(\rho(Tu)v + \mu(Tv)u).$$

□

Proposition 9. *Let $(A, *)$ be a Jacobi–Jordan algebra and (V, ρ) be a representation on it. If T is a relative Rota–Baxter operator with respect to ρ, then $(V, \cdot)$ is a left pre-Jacobi–Jordan algebra, where*

$$u \cdot v := \rho(T(u))v, \text{ for } u, v \in V. \tag{22}$$

Hence there exists an associated Jacobi–Jordan algebra structure on V given by (9) and T is a morphism of Jacobi–Jordan algebras. In addition, $T(V) := \{T(v)|v \in V\} \subset A$ is a Jacobi–Jordan subalgebra of $(A, *)$ and there is an induced left pre-Jacobi–Jordan algebra structure on $T(V)$ given by

$$T(u) \cdot T(v) := T(u \cdot v), \text{ for } u, v \in V.$$

The corresponding associated Jacobi–Jordan algebra structure on $T(V)$ given by (9) is just a Jacobi–Jordan subalgebra of $(A, *)$ and T is a momorphism of left pre-Jacobi–Jordan algebras.

Proof. Let $u, v, w \in V$ and put $u \star v = u \cdot v + v \cdot u$. Note first that $T(u \star v) = T(u) * T(v)$. We compute (8) using (22) and (2) as follows

$$\begin{aligned}(u \star v) \cdot w &= \rho(T(u) * T(v))w = -\rho(Tu)\rho(Tv)w - \rho(Tv)\rho(Tu)w \\ &= -u \cdot (v \cdot w) - v \cdot (u \cdot w).\end{aligned}$$

Therefore, $(V, \cdot)$ is a left pre-Jacobi–Jordan algebra. The other claims follow immediately. □

As an obvious consequence of Proposition 9, we obtain the following construction of a left pre-Jacobi–Jordan algebra in terms of a Rota–Baxter operator (of weight zero) of a Jacobi–Jordan algebra.

Corollary 1. *Let $(A, *)$ be a Jacobi–Jordan algebra and P be a Rota–Baxter operator (of weight zero) on A. Then there is a left pre-Jacobi–Jordan algebra structure on A given by*

$$x \cdot y := P(x) * y, \text{ for all } x, y \in A.$$

Proof. Straightforward. □

Corollary 2. *Let $(A, *)$ be a Jacobi–Jordan algebra. Then there exists a compatible left pre-Jacobi- Jordan algebra structure on A if and only if there exists an invertible relative Rota–Baxter operator of $(A, *)$.*

Proof. Let $(A, \cdot)$ be a left pre-Jacobi–Jordan algebra and $(A, \star)$ be the associated Jacobi–Jordan algebra. Then it is obvious that the identity map $id : A \to A$ is an invertible relative Rota–Baxter operator of $(A, \star)$ with respect to (A, ad).

Conversely, suppose that there exists an invertible relative Rota–Baxter operator T of $(A, *)$ with respect to a representation (V, ρ). Then Proposition 9 says that there is a left pre-Jacobi–Jordan algebra structure on

$T(V) = A$ given by

$$T(u) \cdot T(v) = T(\rho(T(u))v), \text{ for all } u, v \in V.$$

Setting $T(u) = x$ and $T(v) = y$, we get

$$x \cdot y = T(\rho(x)T^{-1}(y)), \text{ for all } x, y \in A.$$

This is a compatible left pre-Jacobi–Jordan algebra structure on $(A, *)$. Indeed,

$$\begin{aligned} x \cdot y + y \cdot x &= T(\rho(x)T^{-1}(y) + \rho(y)T^{-1}(x)) \\ &= T(T^{-1}(x)) * T(T^{-1}(y)) = x * y. \end{aligned}$$

□

Proposition 10. *Let $T : V \to A$ be a relative Rota–Baxter operator on a Jacobi–Jordan algebra $(A, *)$ with respect to the representation (V, ρ). Define a map $\rho_T : V \to gl(A)$ by*

$$\rho_T(u)x := T(u) * x - T(\rho(x)u), \text{ for all } (u, x) \in V \times A.$$

Then the triplet (A, ρ_T) is a representation of the sub-adjacent Jacobi–Jordan algebra $V^C = (V, \star)$ associated with the left pre-Jacobi–Jordan algebra $(V, \cdot)$ defined in Proposition 9.

Proof. Let $u, v \in V,\ x \in A$. Observe that $u \star v = \rho(Tu)v + \rho(Tv)u$ and $T(u \star v) = T(u) * T(v)$. Hence, by straightforward computations, we obtain

$$\rho_T(u \star v)x = (T(u) * T(v)) * x - T\Big(\rho(x)\rho(Tu)v\Big) - T\Big(\rho(x)\rho(Tv)u\Big).$$

In addition, we get

$$\begin{aligned} \rho_T(u)\rho_T(v)x &= Tu * (Tv * x) - Tu * T(\rho(x)v) - T\Big(\rho(Tv * x)u\Big) \\ &\quad + T\Big(\rho(T(\rho(x)v))u\Big) = Tu * (Tv * x) - T\Big(\rho(Tu)\rho(x)v + \rho(T(\rho(x)v))u\Big) \\ &\quad - T\Big(\rho(Tv)\rho(x)u + \rho(x)\rho(Tv)u\Big) + T\Big(\rho(T(\rho(x)v))u\Big) \text{ (by (3) and (2))} \\ &= Tu * (Tv * x) - T\Big(\rho(Tu)\rho(x)v\Big) + T\Big(\rho(Tv)\rho(x)u\Big) + T\Big(\rho(x)\rho(Tv)u\Big). \end{aligned}$$

Switching u and v in the above equation, we come to

$$\begin{aligned} \rho_T(v)\rho_T(u)x &= Tv * (Tu * x) - T\Big(\rho(Tv)\rho(x)u\Big) \\ &\quad + T\Big(\rho(Tu)\rho(x)v\Big) + T\Big(\rho(x)\rho(Tu)v\Big). \end{aligned}$$

Hence, by (1), we obtain

$$-\rho_T(u)\rho_T(v)x - \rho_T(v)\rho_T(u)x = \rho_T(u \star v)x,$$

i.e., (2) holds in (A, ρ_T). □

Proposition 11. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to (V,ρ,μ). Define*

$$u\cdot_T v=\rho(Tu)v+\mu(Tv)u,\ \textit{for all } u,v\in V. \tag{23}$$

Then $(V,\cdot_T)$ is a left pre-Jacobi–Jordan algebra.

Proof. Denote by $Aasso_T$ the anti-associator with respect to $\cdot_T$ and pick u, v and w in V. We have

$$\begin{aligned}
&Aasso_T(u,v,w)+Aasso_T(v,u,w)\\
&=(u\cdot_T v)\cdot_T w+u\cdot_T(v\cdot_T w)+(v\cdot_T u)\cdot_T w+v\cdot_T(u\cdot_T w)\\
&\overset{(23)}{=}\rho(T(u\cdot_T v))w+\mu(Tw)(u\cdot_T v)+\rho(Tu)(v\cdot_T w)+\mu(T(v\cdot_T w))u\\
&\quad+\rho(T(v\cdot_T u))w+\mu(Tw)(v\cdot_T u)+\rho(Tv)(u\cdot_T w)+\mu(T(u\cdot_T w))v\\
&\overset{(18),(23)}{=}\rho(Tu\cdot Tv)w+\mu(Tw)\rho(Tu)v+\mu(Tw)\mu(Tv)u+\rho(Tu)\rho(Tv)w\\
&\quad+\rho(Tu)\mu(Tw)v+\mu(Tv\cdot Tw)u+\rho(Tv\cdot Tu)w+\mu(Tw)\rho(Tv)u\\
&\quad+\mu(Tw)\mu(Tu)v+\rho(Tv)\rho(Tu)w+\rho(Tv)\mu(Tw)u+\mu(Tu\cdot Tw)v\\
&=\Big(\rho(Tu*Tv)w+\rho(Tu)\rho(Tv)w+\rho(Tv)\rho(Tu)w\Big)+\Big(\mu(Tv\cdot Tw)u\\
&\quad+\mu(Tw)\mu(Tv)u+\mu(Tw)\rho(Tv)u+\rho(Tv)\mu(Tw)u\Big)+\Big(\mu(Tu\cdot Tw)v\\
&\quad+\mu(Tw)\mu(Tu)v+\mu(Tw)\rho(Tu)v+\rho(Tu)\mu(Tw)v\Big)\\
&\overset{(10),(11)}{=}0.
\end{aligned}$$

Therefore $(V,\cdot_T)$ is a left pre-Jacobi–Jordan algebra. ☐

Corollary 3. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Then T is a morphism from the left pre-Jacobi–Jordan algebra $(V,\cdot_T)$ to the initial left pre-Jacobi–Jordan algebra $(A,\cdot)$.*

Proof. It follows from Proposition 11 and (18). ☐

Theorem 3. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Define*

$$\rho_T(v)x=Tv\cdot x-T(\mu(x)v),\ \textit{for all } (x,v)\in A\times V, \tag{24}$$

$$\mu_T(v)x=x\cdot Tv-T(\rho(x)v),\ \textit{for all } (x,v)\in A\times V. \tag{25}$$

Then (A,ρ_T,μ_T) is a representation of the left pre-Jacobi–Jordan algebra $(V,\cdot_T)$.

Proof. If $u,v\in V$ and $x\in A$, then we have

$$\rho_T(u\cdot_T v+v\cdot_T u)x+\rho_T(u)\rho_T(v)x+\rho_T(v)\rho_T(u)x$$

$$\begin{aligned}
&\overset{(24),(23)}{=} (Tu\cdot Tv)\cdot x - T\Big(\mu(x)\rho(Tu)v + \mu(x)\mu(Tv)u\Big) + (Tv\cdot Tu)\cdot x\\
&\quad - T\Big(\mu(x)\rho(Tv)u + \mu(x)\mu(Tu)v\Big) + Tu\cdot(Tv\cdot x) - T\Big(\rho(Tu)\mu(x)v\\
&\quad + \mu(T(\mu(x)v))u + \mu(Tv\cdot x)u - \mu(T(\mu(x)v))u\Big) + Tv\cdot(Tu\cdot x)\\
&\quad - T\Big(\rho(Tv)\mu(x)u + \mu(T(\mu(x)u))v + \mu(Tu\cdot x)v - \mu(T(\mu(x)u))v\Big)\\
&= (Tu\cdot Tv)\cdot x + Tu\cdot(Tv\cdot x) + (Tv\cdot Tu)\cdot x + Tv\cdot(Tu\cdot x)\\
&\quad - T\Big(\mu(x)\rho(Tu)v + \mu(x)\mu(Tv)u\Big) - T\Big(\mu(x)\rho(Tv)u + \mu(x)\mu(Tu)v\Big)\\
&\quad - T\Big(\rho(Tu)\mu(x)v + \mu(Tv\cdot x)u\Big) - T\Big(\rho(Tv)\mu(x)u + \mu(Tu\cdot x)v\Big)\\
&\overset{(6)}{=} Aasso(Tu,Tv,x) + Aasso(Tv,Tu,x)\\
&\quad - T\Big(\mu(Tv\cdot x)u + \mu(x)\mu(Tv)u + \mu(x)\rho(Tv)u + \rho(Tv)\mu(x)u\Big)\\
&\quad - T\Big(\mu(Tu\cdot x)v + \mu(x)\mu(Tu)v + \mu(x)\rho(Tu)v + \rho(Tu)\mu(x)v\Big)\\
&\overset{(7),(11)}{=} 0.
\end{aligned}$$

Thus we deduce that

$$\rho_T(u\cdot_T v + v\cdot_T u)x + \rho_T(u)\rho_T(v)x + \rho_T(v)\rho_T(u)x = 0.$$

We have also

$$\begin{aligned}
&\mu_T(u\cdot_T v)x + \mu_T(v)\mu_T(u)x + \mu_T(v)\rho_T(u)x + \rho_T(u)\mu_T(v)x\\
&= x\cdot(Tu\cdot Tv) - T\Big(\rho(x)\rho(Tu)v + \rho(x)\mu(Tv)u\Big) + (x\cdot Tu)\cdot Tv\\
&\quad - T\Big(\rho(T(\rho(x)u))v + \mu(Tv)\rho(x)u + \rho(x\cdot Tu)v - \rho(T(\rho(x)u))v\Big)\\
&\quad + (Tu\cdot x)\cdot Tv - T\Big(\rho(T(\mu(x)u))v + \mu(Tv)\mu(x)u + \rho(Tu\cdot x)v\\
&\quad - \rho(T(\mu(x)u))v\Big) + Tu\cdot(x\cdot Tv) - T\Big(\rho(Tu)\rho(x)v + \mu(T(\rho(x)v))u\\
&\quad + \mu(x\cdot Tv)u - \mu(T(\rho(x)v))u\Big)\\
&= (x\cdot Tu)\cdot Tv + x\cdot(Tu\cdot Tv) + (Tu\cdot x)\cdot Tv + Tu\cdot(x\cdot Tv)
\end{aligned}$$

$$
\begin{aligned}
&- T\Big(\rho(x)\rho(Tu)v + \rho(x)\mu(Tv)u\Big) - T\Big(\mu(Tv)\rho(x)u + \rho(x\cdot Tu)v\Big)\\
&- T\Big(\mu(Tv)\mu(x)u + \rho(Tu\cdot x)v\Big) - T\Big(\rho(Tu)\rho(x)v + \mu(x\cdot Tv)u\Big)\\
&\overset{(6)}{=} Aasso(x,Tu,Tv) + Aasso(Tu,x,Tv) - T\Big(\rho(Tu\cdot x + x\cdot Tu)v\\
&+ \rho(Tu)\rho(x)v + \rho(x)\rho(Tu)v\Big) - T\Big(\mu(x\cdot Tv)u + \mu(Tv)\mu(x)u\\
&+ \mu(Tv)\rho(x)u + \rho(x)\mu(Tv)u\Big)\\
&\overset{(7),(10),(11)}{=} 0.
\end{aligned}
$$

Therefore (A,ρ_T,μ_T) is a representation of the left pre-Jacobi–Jordan algebra $(V,\cdot_T)$. $\square$

Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Consider the new pre-Jacobi–Jordan algebra $(V,\cdot_T)$ and its representation (A,ρ_T,μ_T). Based on the zigzag complex developed in [2] for pre-Jacobi–Jordan algebras, we obtain the following zigzag cochain complex

$$(\mathbf{C}^\bullet(V,A) = \bigoplus_{k=0}^{+\infty} C^k(V,A), \mathbf{A}^\bullet(V,A) = \bigoplus_{k=0}^{+\infty}\mathbf{A}^k(V,A), d^\bullet_{\rho_T}, \delta^\bullet_{\rho_T})$$

of the algebra $(V,\cdot_T)$ with coefficients in the representation (A,ρ_T,μ_T) where, for all integers $n\geq 1$,

$$\mathbf{C}^n(V,A) := \{f : V^{\otimes n}\to A : f \text{ is linear}\}$$

and

$$\mathbf{A}^n(V,A)$$

is given by $f\in\mathbf{A}^n(V,A)$ if and only if $f\in Hom(\wedge^{n-1}V\otimes V, A)$ and

$$
\begin{aligned}
&f(u*_T v, u_1,\cdots,u_{n-1}, w\cdot_T u_n) + f(v*_T w, u_1,\cdots,u_{n-1}, u\cdot_T u_n)\\
&+f(w*_T u, u_1,\cdots,u_{n-1}, v\cdot_T u_n) = 0,\\
&\text{ for all } (u,v,w,u_1,\cdots,u_n)\in V^{\otimes(n+3)},
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^0(V,A) &= \mathbf{C}^0(V,A)\\
&:= \{x\in A : \rho_T(u\cdot_T v)x + \rho_T(u)\circ\rho_T(v)x = 0, \forall u,v\in V\}.
\end{aligned}
$$

The two sequences of differential maps are given by linear maps

$$
\begin{aligned}
d^n_T &: \mathbf{C}^n(V,A)\to\mathbf{C}^{n+1}(V,A),\\
\delta^n_T &: \mathbf{A}^n(V,A)\to\mathbf{C}^{n+1}(V,A),
\end{aligned}
$$

where

$$\begin{aligned} d_T^n f(u_1, \cdots, u_{n+1}) &= \sum_{i=1}^{n} \rho_T(u_i) f(u_1, \cdots, \widehat{u_i}, \cdots, u_{n+1}) \\ &\quad + \sum_{i=1}^{n} \mu_T(u_{n+1}) f(u_1, \cdots, \widehat{u_i}, \cdots, u_n, u_i) \\ &\quad + \sum_{i=1}^{n} f(u_1, \cdots, \widehat{u_i}, \cdots, u_n, x_i \cdot_T u_{n+1}) \\ &\quad + \sum_{1 \le i < j \le n} f(u_i *_T u_j, u_1, \cdots, \widehat{u_i}, \cdots, \widehat{u_j}, \cdots, u_{n+1}), \end{aligned}$$

$$\begin{aligned} \delta_T^n f(u_1, \cdots, u_{n+1}) &= \sum_{i=1}^{n} \rho_T(u_i) f(u_1, \cdots, \widehat{u_i}, \cdots, u_{n+1}) \\ &\quad + \sum_{i=1}^{n} \mu_T(u_{n+1}) f(u_1, \cdots, \widehat{u_i}, \cdots, u_n, u_i) \\ &\quad - \sum_{i=1}^{n} f(u_1, \cdots, \widehat{u_i}, \cdots, u_n, u_i \cdot_T u_{n+1}) \\ &\quad - \sum_{1 \le i < j \le n} f(u_i *_T u_j, u_1, \cdots, \widehat{u_i}, \cdots, \widehat{u_j}, \cdots, u_{n+1}), \end{aligned}$$

with

$$\delta_T^0 x(u) = d_T^0 x(u) := \rho_T(u)x + \mu_T(u)x, \text{ for all } (x, u) \in A \times V. \tag{26}$$

Proposition 12 ([2]). *With the above notations, for any integer $n \geq 1$, it holds that*

$$d_{\rho_T}^n \circ \delta_{\rho_T}^{n-1} = 0.$$

Definition 9. Let T be a relative Rota–Baxter operator of the pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). The cochain complex $(C^\bullet(V, A), \mathbf{A}^\bullet(V, A), d_{\rho_T}^\bullet, \delta_{\rho_T}^\bullet)$ is called the *cohomology complex* for the relative Rota–Baxter operator T.

The k^{th} cohomology space of V with values/coefficients in A is given by the quotient

$$H^k(V, A) := Z^k(V, A)/B^k(V, A).$$

4. Linear deformations of relative Rota–Baxter operators on pre-Jacobi–Jordan algebras

In this section, we use the cohomological approach to study linear deformations of relative Rota–Baxter operators. In particular, the notion of

Nijenhuis elements is introduced to characterize trivial linear deformations.

First, let us give the following definition.

Definition 10. Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ) and $J : V \to A$ be a linear map. If $T_t = T + tJ$ is still a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to (V,ρ,μ) for all $t \in \mathbb{C}$, we say that J generates a *linear deformation of the relative Rota–Baxter operator T*.

Observe that J generates a lineair deformation $T_t = T + tJ$ of the relative Rota–Baxter operator T if and only if, for all $u, v \in V$,

$$\begin{aligned} Ju \cdot Jv &= J(\rho(Ju)v + \mu(Jv)u), && (27)\\ Tu \cdot Jv + Ju \cdot Tv &= T(\rho(Ju)v + \mu(Jv)u) \\ &\quad + J(\rho(Tu)v + \mu(Tv)u). && (28) \end{aligned}$$

The equality (27) means that J is a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to (V,ρ,μ). The equality (28) is equivalent to

$$\rho_T(u)J(v) + \mu_T(v)J(u) - J(u \cdot_T v) = 0,$$

i.e., $J \in Ker(\delta^1_T)$.

Let us recall the following definition.

Definition 11 ([2])**.** Let $(A,\cdot)$ be a left pre-Jacobi–Jordan algebra and $\omega : A \times A \to A$ be a linear map. If, for any $t \in \mathbb{C}$, the multiplication $\cdot_t$ defined by

$$x \cdot_t y = x \cdot y + t\omega(x,y), \forall x, y \in A, \tag{29}$$

also gives a left pre-Jacobi–Jordan algebra structure, we say that ω generates a *linear deformation of the left pre-Jacobi–Jordan algebra $(A,\cdot)$*.

The two types of linear deformations are related as follows.

Proposition 13. *If J generates a linear deformation of a relative Rota–Baxter operator T on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ), then the product ω_J defined by*

$$\omega_J(u,v) = \rho(Ju)v + \rho(Jv)u, \text{ for all } u, v \in V,$$

generates a linear deformation of the left pre-Jacobi–Jordan algebra $(V,\cdot_T)$.

Proof. Suppose that J generates a linear deformation T_t of the relative Rota–Baxter operator T and denote by $\cdot_t$ the corresponding left pre-Jacobi–Jordan algebra structure associated to T_t. Then we have

$$u \cdot_t v = \rho(T_t u)v + \mu(T_t v)u = u \cdot_T v + t\omega_J(u,v), \text{ for all } u, v \in V.$$

It follows that ω_J generates a linear deformation of $(V, \cdot_T)$. $\square$

Definition 12. Let T and T' be two relative Rota–Baxter operators on a left pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). A *morphism* from T' to T consists of a left pre-Jacobi–Jordan algebras morphism $\phi_A : A \to A$ and a linear map $\phi_V : V \to V$ such that

$$T \circ \phi_V = \phi_A \circ T', \tag{30}$$

$$\phi_V \circ \rho(x) = \rho(\phi_A(x)) \circ \phi_V, \text{ for all } x \in A, \tag{31}$$

$$\phi_V \circ \mu(x) = \mu(\phi_A(x)) \circ \phi_V, \text{ for all } x \in A. \tag{32}$$

In particular, if both ϕ_A and ϕ_V are invertible, then (ϕ_A, ϕ_V) is called an *isomorphism* from T' to T.

Proposition 14. *Let T and T' be two relative Rota–Baxter operators on a left pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ) and (ϕ_A, ϕ_V) a morphism (resp. an isomorphism) from T' to T. Then ϕ_V is a morphism (resp. an isomorphism) from the left pre-Jacobi–Jordan algebra $(V, \cdot_{T'})$ to $(V, \cdot_T)$.*

Proof. Let $u, v \in V$. By (30)–(32), we have

$$\begin{aligned}
\phi_V(u \cdot_{T'} v) &= \phi_V(\rho(T'u)v + \mu(T'v)u) \\
&= \phi_V(\rho(T'u)v) + \phi_V(\mu(T'v)u) \\
&= \rho(\phi_A(T'u))\phi_V(v) + \mu(\phi_A(T'v))\phi_V(u) \\
&= \rho(T(\phi_V(u)))\phi_V(v) + \mu(T(\phi_V(v)))\phi_V(u) \\
&= \phi_V(u) \cdot_T \phi_V(v).
\end{aligned}$$

Thus ϕ_V is a morphism from the left pre-Jacobi–Jordan algebra $(V, \cdot_{T'})$ to $(V, \cdot_T)$. $\square$

Definition 13. Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). Two linear deformations $T_t^1 = T + tJ_1$ and $T_t^2 = T + tJ_2$ of T are said to be *equivalent* if there exists $x \in C^0(V, A)$ such that $(Id_A + tL_x + tR_x, Id_V + t\rho(x) + t\mu(x))$ is a morphism from T_t^2 to T_t^1. In particular, a linear deformation $T_t = T + tJ$ of a relative Rota–Baxter operator T is said to be *trivial* if there exists $x \in C^0(V, A)$ such that $(Id_A + tL_x + tR_x, Id_V + t\rho(x) + t\mu(x))$ is a morphism from T_t to T.

Suppose that there exists $x \in C^0(V, A)$ such that $(Id_A + tL_x + tR_x, Id_V + t\rho(x) + t\mu(x))$ is a morphism from T_t^2 to T_t^1. Then $Id_A + tL_x + tR_x$ is a left pre-Jacobi–Jordan algebras morphism of $(A, \cdot)$ and (30)-(32) hold. Now $(Id_A + tL_x + tR_x)(y \cdot z) = (Id_A + tL_x + tR_x)(y) \cdot (Id_A + tL_x + tR_x)(z)$ for all $y, z \in A$, if and only if x satisfies

$$(x \cdot y) \cdot (x \cdot z) + (x \cdot y) \cdot (z \cdot x)$$

$$+(y\cdot x)\cdot(x\cdot z)+(y\cdot x)\cdot(z\cdot x)=0 \text{ for all } y,z\in A, \tag{33}$$

$$\begin{aligned}&(x\cdot y)\cdot z+(y\cdot x)\cdot z+y\cdot(x\cdot z)+y\cdot(z\cdot x)\\&=x\cdot(y\cdot z)+(y\cdot z)\cdot x \text{ for all } y,z\in A.\end{aligned} \tag{34}$$

By (30), we get
$(T+tJ_1)\circ(Id_V+t\rho(x)+t\mu(x))(v)=(Id_A+tL_x+tR_x)\circ(T+tJ_2)(v),\forall v\in V$, which holds if and only if x satisfies

$$J_2-J_1=T\circ\rho(x)+T\circ\mu(x)-L_x\circ T-R_x\circ T=-\partial_T(x), \tag{35}$$

$$J_1\circ\rho(x)+J_1\circ\mu(x)=L_x\circ J_2+R_x\circ J_2. \tag{36}$$

Also, by (31), we obtain
$(Id_V+t\rho(x)+t\mu(x))\circ\rho(y)=\rho(y+tL_x(y)+tR_x(y))\circ(Id_V+t\rho(x)+t\mu(x))$, which holds if and only if x satisfies

$$\begin{aligned}&\rho(x\cdot y)\rho(x)+\rho(y\cdot x)\rho(x)\\&\quad+\rho(x\cdot y)\mu(x)+\rho(y\cdot x)\mu(x)=0 \text{ for all } y\in A,\end{aligned} \tag{37}$$

$$\begin{aligned}&\rho(x\cdot y)+\rho(y\cdot x)+\rho(y)\rho(x)+\rho(y)\mu(x)\\&=\rho(x)\rho(y)+\mu(x)\rho(y) \text{ for all } y\in A.\end{aligned} \tag{38}$$

Finally, (32) gives

$$\begin{aligned}&(Id_V+t\rho(x)+t\mu(x))\circ\mu(y)\\&=\mu(y+tL_x(y)+tR_x(y))\circ(Id_V+t\rho(x)+t\mu(x)) \text{ for all } y\in A,\end{aligned}$$

which holds if and only if x satisfies

$$\begin{aligned}&\mu(x\cdot y)\mu(x)+\mu(y\cdot x)\mu(x)\\&\quad+\mu(x\cdot y)\rho(x)+\mu(y\cdot x)\rho(x)=0 \text{ for all } y\in A,\end{aligned} \tag{39}$$

$$\begin{aligned}&\mu(x\cdot y)+\mu(y\cdot x)+\mu(y)\mu(x)+\mu(y)\rho(x)\\&=\mu(x)\mu(y)+\rho(x)\mu(y) \text{ for all } y\in A.\end{aligned} \tag{40}$$

Note that (35) means that

$$J_2-J_1=-\partial_T(x).$$

Thus we have the following result.

Theorem 4. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). If two linear deformations $T_t^1=T+tJ_1$ and $T_t^2=T+tJ_2$ are equivalent, then J_1 and J_2 are in the same cohomology class of $H^1(V,A)$.*

Definition 14. Let T be a relative Rota–Baxter operator on a pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). An element $x\in C^0(V,A)$ is called a *Nijenhuis element* associated to T if x satisfies (33), (34), (37), (38), (39), (40) and the equation

$$(L_x+R_x)\circ T\circ(\rho(x)+\mu(x))-(L_x+R_x)\circ(L_x+R_x)\circ T=0. \tag{41}$$

Denote by $N_{ij}(T)$ the set of Nijenhuis elements associated to a relative Rota–Baxter operator T.

The following lemma is very useful.

Lemma 2. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ). Let $\phi_A : A \to A$ be a pre-Jacobi–Jordan algebras isomorphism and $\phi_V : V \to V$ an isomorphism of vector spaces such that* (31)-(32) *hold. Then, $\varphi = \varphi_A^{-1}\circ T\circ\varphi_V$ is a relative Rota–Baxter operator on the left pre-Jacobi–Jordan algebra $(A,\cdot)$ with respect to a representation (V,ρ,μ).*

Proof. First, observe that

$$\varphi = \varphi_A^{-1}\circ T\circ\varphi_V \iff \varphi_A\circ\varphi = T\circ\varphi_V. \tag{42}$$

Next, if $u,v\in V$, then we obtain

$$\begin{array}{rcl}
\varphi(u)\cdot\varphi(v) & = & \Big(\varphi_A^{-1}\circ T\circ\varphi_V\Big)(u)\cdot\Big(\varphi_A^{-1}\circ T\circ\varphi_V\Big)(v) \\
 & & (\text{ by the definition of } \varphi) \\
 & = & \varphi_A^{-1}\Big(T(\varphi_V(u))\Big)\cdot\varphi_A^{-1}\Big(T(\varphi_V(v))\Big) \\
 & = & \varphi_A^{-1}\Big(T(\varphi_V(u))\cdot T(\varphi_V(v))\Big) \text{ (since } \varphi_A^{-1} \text{ is a morphism)} \\
 & = & \varphi_A^{-1}\Big(T\Big(\rho(T(\varphi_V(u)))\varphi_V(v)+\mu(T(\varphi_V(v)))\varphi_V(u)\Big)\Big) \\
 & & \text{(by (18))} \\
 & = & (\varphi_A^{-1}\circ T)\Big(\rho(T(\varphi_V(u)))\varphi_V(v)+\mu(T(\varphi_V(v)))\varphi_V(u)\Big) \\
 & = & (\varphi_A^{-1}\circ T)\Big(\rho((T\circ\varphi_V)(u))\varphi_V(v)+\mu((T\circ\varphi_V)(v))\varphi_V(u)\Big) \\
 & = & (\varphi_A^{-1}\circ T)\Big(\rho((\varphi_A\circ\varphi)(u))\varphi_V(v)+\mu((\varphi_A\circ\varphi)(v))\varphi_V(u)\Big) \\
 & & \text{(by (42))} \\
 & = & (\varphi_A^{-1}\circ T)\Big(\rho((\varphi_A(\varphi(u)))\varphi_V(v)+\mu((\varphi_A(\varphi(v)))\varphi_V(u)\Big) \\
 & = & (\varphi_A^{-1}\circ T)\Big(\varphi_V\circ\rho(\varphi(u))v+\varphi_V\circ\mu(\varphi(v))u\Big) \text{ (by (31)-(32))} \\
 & = & (\varphi_A^{-1}\circ T\circ\varphi_V)\Big(\rho(\varphi(u))v+\mu(\varphi(v))u\Big) \\
 & = & \varphi\Big(\rho(\varphi(u))v+\mu(\varphi(v))u\Big).
\end{array}$$

Therefore $\varphi = \varphi_A^{-1} \circ T \circ \varphi_V$ is a relative Rota–Baxter operator on the pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). □

By (33)-(40), it is obvious that a trivial linear deformation gives rise to a Nijenhuis element. Conversely, the following theorem shows that a Nijenhuis element can also generate a trivial linear deformation.

Theorem 5. *Let T be a relative Rota–Baxter operator on a left pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). Then, for any $x \in N_{ij}(T)$, $T_t = T + tJ$ is a trivial linear deformation of the relative Rota–Baxter operator T, where $J = -\partial_T(x)$.*

Proof. Let T be a Rota–Baxter relative operator on a pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to a representation (V, ρ, μ). Pick $x \in N_{ij}(T)$ and $v \in V$. Then we have

$$\begin{aligned} Jv \quad &= -\partial_T(x)v \\ &= -\rho_T(v)x - \mu_T(v)x \text{ (by (26))}, \end{aligned}$$

i.e., by (24)–(25), we obtain

$$J = T \circ \rho(x) + T \circ \mu(x) - L_x \circ T - R_x \circ T. \tag{43}$$

Therefore

$$\begin{aligned} & L_x \circ J + R_x \circ J \\ = \; & L_x \circ T \circ \rho(x) + L_x \circ T \circ \mu(x) - L_x \circ L_x \circ T - L_x \circ R_x \circ T \\ & + R_x \circ T \circ \rho(x) + R_x \circ T \circ \mu(x) - R_x \circ L_x \circ T - R_x \circ R_x \circ T \\ = \; & (L_x + R_x) \circ T \circ (\rho(x) + \mu(x)) - (L_x + R_x) \circ (R_x + L_x) \circ T, \end{aligned}$$

i.e., by (41), the following holds:

$$L_x \circ J + R_x \circ J = 0. \tag{44}$$

From the relation $T_t = T + tJ$ we compute

$$L_x \circ T_t + R_x \circ T_t \quad = \quad L_x \circ T + R_x \circ T + t(L_x \circ J + R_x \circ J),$$

i.e., by (44), we get

$$L_x \circ T_t + R_x \circ T_t = L_x \circ T + R_x \circ T. \tag{45}$$

By $T_t = T + tJ$, we have also

$$\begin{aligned} T_t - T \quad &= \quad tJ \\ &= \quad t\Big(T \circ \rho(x) + T \circ \mu(x) - L_x \circ T - R_x \circ T\Big) \text{ (by (43))} \\ &= \quad t\Big(T \circ \rho(x) + T \circ \mu(x) - L_x \circ T_t - R_x \circ T_t\Big) \text{ (by (45))}. \end{aligned}$$

Hence

$$T_t + tL_x \circ T_t + tR_x \circ T_t \quad = \quad T + tT \circ \rho(x) + tT \circ \mu(x),$$

and therefore the following holds:

$$\Big(Id_A + tL_x + tR_x\Big) \circ T_t = T \circ \Big(Id_V + t\rho(x) + t\mu(x)\Big). \tag{46}$$

Now, let $y, z \in A$ and $t \in \mathbb{C}$. Then, by (33) and (34), we have

$$\begin{aligned} & y \cdot z + tx \cdot (y \cdot z) + t(y \cdot z) \cdot x = y \cdot z + ty \cdot (x \cdot z) + ty \cdot (z \cdot x) \\ & \quad +t(x \cdot y) \cdot z + t^2(x \cdot y) \cdot (x \cdot z) + t^2(x \cdot y) \cdot (z \cdot x) + t(y \cdot x) \cdot z \\ & \quad +t^2(y \cdot x) \cdot (x \cdot z) + t^2(y \cdot x) \cdot (z \cdot x), \end{aligned}$$

i.e.,

$$\begin{aligned} & \Big(Id_A + tL_x + tR_x\Big)(y \cdot z) \\ & = \Big(Id_A + tL_x + tR_x\Big)(y) \cdot \Big(Id_A + tL_x + tR_x\Big)(z). \end{aligned}$$

Next, by (37) and (38), we have

$$\begin{aligned} & \rho(y) + t\rho(x)\rho(y) + t\mu(x)\rho(y) = \rho(y) + t\rho(y)\rho(x) + t\rho(y)\mu(x) \\ & +t\rho(x \cdot y) + t^2\rho(x \cdot y)\rho(x) + t^2\rho(x \cdot y)\mu(x) + t\rho(y \cdot x) \\ & +t^2\rho(y \cdot x)\rho(x) + t^2\rho(y \cdot x)\mu(x), \end{aligned}$$

i.e.,

$$\begin{aligned} & \big(Id_V + t\rho(x) + t\mu(x)\big) \circ \rho(y) \\ & = \rho\big(y + tL_x(y) + tR_x(y)\big) \circ \big(Id_V + t\rho(x) + t\mu(x)\big). \end{aligned}$$

Also, by (39) and (40), we compute

$$\begin{aligned} & \mu(y) + t\rho(x)\mu(y) + t\mu(x)\mu(y) = \mu(y) + t\mu(y)\rho(x) + t\mu(y)\mu(x) \\ & \quad +t\mu(x \cdot y) + t^2\mu(x \cdot y)\rho(x) + t^2\mu(x \cdot y)\mu(x) + t\mu(y \cdot x) \\ & \quad +t^2\mu(y \cdot x)\rho(x) + t^2\mu(y \cdot x)\mu(x), \end{aligned}$$

i.e.,

$$\begin{aligned} & \big(Id_V + t\rho(x) + t\mu(x)\big) \circ \mu(y) \\ & = \mu\big(y + tL_xy + tR_xy\big) \circ \big(Id_V + t\rho(x) + t\mu(x)\big). \end{aligned}$$

For t sufficiently small, we see that $Id_A + tL_x + tR_x$ is a pre-Jacobi–Jordan algebras isomorphism and $Id_V + t\rho(x) + t\mu(x)$ is an isomorphism of vector spaces. Thus, by (46) we have

$$T_t = \Big(Id_A + tL_x + tR_x\Big)^{-1} \circ T \circ \Big(Id_V + t\rho(x) + t\mu(x)\Big).$$

By Lemma 2, we deduce that T_t is a relative Rota–Baxter operator on the pre-Jacobi–Jordan algebra $(A, \cdot)$ with respect to the representation (V, ρ, μ), for $|t|$ sufficiently small. Hence J given by the equality (43) satisfies conditions (27) and (28). It follows that, for any $x \in N_{ij}(T)$, $T_t = T + tJ$ with $J = -\partial_T(x)$ is a trivial linear deformation of the relative Rota–Baxter operator T. □

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