ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 30, Number 1, June 2026
Available online at https://ojs.utlib.ee/index.php/ACUTM

Abundant families of exact wave solutions for the time-fractional Estevez–Mansfield–Clarkson equation

Nauman Ahmed, Muhammad Z. Baber, Armando Gallegos, Jorge E. Macías-Díaz, Naveed Shahid and Shazia Umer

Abstract. The Estevez–Mansfield–Clarkson (EMC) equation is analytically modified to incorporate conformable time-fractional derivatives. This equation serves as a significant model in mathematical physics, optics, and the study of shape evolution in liquid droplets. In this work, the EMC equation is solved using the Jacobi elliptic function expansion method. Various solitary wave solutions such as dark and bright solitons, and multi-wave solutions are derived in terms of rational, hyperbolic, and trigonometric functions. The physical behavior of these solutions is illustrated graphically through contour plots, as well as 2D and 3D visualizations. To confirm the accuracy of the solutions, numerical simulations are conducted. The study concludes with a discussion of the results and final remarks.

1. Introduction

Research on nonlinear partial differential equations (NLPDEs) has advanced through various computational and theoretical approaches with diverse applications. The study of these kinds of equations is crucial to understanding a wide range of physical phenomena in a wide range of scientific fields, such as electrical circuits [34, 39], biological mathematics [36, 37], optical fiber technology [28, 23], plasma physics and fluid mechanics [27, 26], and more. For example, Abdelrahman and Alkhidhr [1] developed a robust solver providing closed-form solutions for various classes of NLPDEs, with applications spanning physics, fluid mechanics, biology, engineering,

---

Received January 8, 2026.

2020 *Mathematics Subject Classification.* 35A24, 35A25, 35R11.

Key words and phrases. Exact solution, Estevez–Mansfield–Clarkson equation, Jacobi elliptic function expansion.

https://doi.org/10.12697/ACUTM.2026.30.07

Corresponding author: Armando Gallegos

economics, astrophysics, and laser technology. Various methods exist to get NLPDE results. Among these methods, we include the modified direct algebraic methodology [8], the modified Khater scheme [4], the generalized Kudryashov scheme [7], the innovative (G'/G)-expansion approach [15], and the extended mapping scheme [29]. However, the extension of NLPDEs to the fractional case has drawn attention to include memory or long-range effects, for which different methods have been developed to obtain their solution. Several numerical methods can be found [14], among which homotopy analysis stands out [10]. It is also possible to find analytical methods to find exact solutions to the fractional NLPDEs. More specifically, the sub-equation method effectively solves nonlinear time fractional PDEs by employing fractional complex transformations to convert them into ordinary differential equations, with fractional derivatives defined in Jumarie's modified Riemann–Liouville sense [6], the improved (G'/G)-expansion function method demonstrates efficiency in solving time- and space-fractional derivative equations, including the foam drainage equation and nonlinear KdV equation [12]. Similarly, the (G'/G)-expansion method successfully generates exact solutions for time-space fractional differential equations, with applications to fractional Burgers' equation [40]. Additionally, a transformation-based approach converts nonlinear fractional parabolic PDEs into systems of fractional differential equations, enabling exact solution generation for equations generalizing Burgers, Murray, and Fisher equations [11].

In particular, the Jacobi elliptic function expansion (JEFE) method has proven to be a powerful tool for solving NLPDEs and constructing exact and periodic solutions, including shock wave solutions and solitary wave solutions [19, 31]. This method has been used to obtain exact solutions of mKdV and Benjamin–Bona–Mahoney (BBM) equations [20]. An extended version of JEFE has been applied to construct exact solutions of nonlinear wave equations [41, 30]. Another extension of the method is proposed to derive three types of periodic wave solutions, including Jacobi elliptic sine, cosine, and third elliptic function solutions [13]. More recently, the JEFE has been used to obtain optical solitons and other solutions for a coupled system of nonlinear Biswas–Milovic equation with Kudryashov's law of refractive index [17]. Of course, the JEFE method has also been applied in the case of fractional NLPDEs, for example, in the fractional extension of KdV and BBM equations [42], also in the sense of Jumarie's modified Riemann–Liouville derivative [33], Rahman et al. [25] applied the method to space-time fractional equal-width and Wazwaz–Benjamin–Bona–Mahony models using fractional beta-derivatives, obtaining diverse structural solutions with applications in nonlinear optics, solid states, fluid mechanics, and shallow water dynamics.

In this work, we are interested in the Estevez–Mansfield–Clarkson (EMC) equation, which was originally proposed in 1997 [21] by reducing the (2+1)-dimensional shallow water wave equation. The EMC equation is a significant mathematical model with applications in mathematical physics, optics, shape formation in liquid drops, optical fiber communications, and image processing, where the multiple exp-function technique has been used to obtain wave solitons, including 1-soliton, 2-soliton, and 3-soliton structures [16]. Some techniques have been used to solve the EMC equation, such as sine-Gordon expansion [18] or Lie Symmetry Analysis [3]. The integrable (1+1)-dimensional version of the EMC equation can be written as [21]

$$D_{txxx}Y + aD_xYD_{tx}Y + aD_{xx}YD_tY + D_{tt}Y = 0, \tag{1}$$

where D is the usual partial derivative operator, x and t are the spatial and temporal variables respectively, here a is a free real parameter. Recent research has extensively investigated fractional variants of the EMC equation. Multiple studies have employed different analytical methods to derive exact wave solutions for time-fractional EMC equations with various fractional derivative definitions. Ahmed et al. [2] utilized the Sardar sub-equation method to obtain soliton solutions, including dark-bright, combined dark-singular, and periodic singular solitons for the conformable fractional derivative version. Muhammad et al. [22] applied the modified F-expansion method and logarithmic transformation to the beta fractional form, extracting bright, dark, mixed, singular, and combined solitons, along with multi-wave structures and breather waves. Qawaqneh and Alrashedi [24] employed the modified extended direct algebraic method and improved (G'/G)-expansion method for the truncated M-fractional EMC equation, obtaining kink, bright, and singular wave solutions with stability analysis. Yasmeen et al. [38] focused on optical fiber applications, deriving exponential and trigonometric solutions for wave propagation in communication systems.

On the other hand, the conformable fractional derivative has emerged as an important alternative to classical fractional derivatives like Riemann–Liouville and Caputo derivatives. Çenesiz and Kurt [9] demonstrated the practical utility of conformable fractional derivatives by successfully solving two and three-dimensional time fractional wave equations, showing that this approach is simpler and more effective than classical fractional derivative methods for complicated solutions. Toplama [35] extended the theory by introducing a new directional derivative approach using conformable fractional derivatives, reformulating fractional partial derivatives and defining fractional gradient, curl, and divergence operators. Additionally, Shpakivskyi [32] generalized the concept to finite-dimensional commutative associative algebras, establishing connections between φ-monogenic and monogenic functions while proposing new fractional derivative definitions for algebra-valued functions. Recently, a time-conformal fractional version of the EMC equation has been solved by the Sardar sub-equation method [2].

In this work, we focus on finding a set of exact solutions of the (1+1)-dimensional integrable EMC equation in its conformal time fractional extension by the JEFE method, taking (1) as a basis, said equation can be expressed in the form

$$D_t^{\beta} D_{xxx} Y + a D_x Y D_t^{\beta} D_x Y + a D_{xx} Y D_t^{\beta} Y + D_t^{2\beta} Y = 0, \tag{2}$$

where D^{β} is a conformal fractional differential operator and β is the corresponding fractional order derivative. The manuscript is organized as follows. Section 2 presents the application of the Jacobi elliptic function expansion method to general time-conformal fractional NLPDEs, including a brief review of the properties of Jacobi elliptic functions. Section 3 shows the definition of the conformal fractional derivative and its most important properties. Section 4 obtains a set of hyperbolic and trigonometric solutions to the conformal fractional EMC equation by applying the JEFE method. Section 5 shows a sample of the graphical behavior of some of the obtained solutions in terms of contour plots and 2D and 3D visualizations. Finally, a brief conclusion is given.

2. The Jacobi elliptic function expansion method

In this section, we present the JEFE method applied to general time-conformal fractional NLPDEs, which can be expressed as

$$F(Y, D_t^{\beta} Y, D_x Y, D_t^{2\beta} Y, D_{xx} Y, ...) = 0, \tag{3}$$

where F is a polynomial of Y. Subsequently, we propose the transformation

$$Y(x,t) = M(\chi), \qquad \text{where} \quad \chi = kx + \frac{ct^{\beta}}{\beta}. \tag{4}$$

Here β is a fractional order derivative, k and c are real parameters. By putting (4) into (3) and applying the fractional conformal derivative properties (see Section 3), we get

$$B(M, M', M'', M''', ...) = 0, \tag{5}$$

where B is a polynomial of M.

The basic concept underlying this strategy is to increase the possibility of solving the auxiliary ODE (5), in order to generate as many Jacobian elliptic solutions as possible for the main problem. Let us consider the following nonlinear ODE

$$\psi'(\chi) = \sqrt{P\psi(\chi)^4 + Q\psi(\chi)^2 + R}, \tag{6}$$

where $\psi' = \frac{d\psi}{d\chi}$, $\chi = \psi(x,t)$ and $R, Q,$ and P are constants. Different solutions of the equation (6) can be consulted in Table 1. These solutions are expressed in terms of the Jacobi elliptic functions: $\mathrm{sn}\chi = \mathrm{sn}(\chi, \kappa)$, $\mathrm{cn}\chi = \mathrm{cn}(\chi, \kappa)$ and $\mathrm{dn}\chi = \mathrm{dn}(\chi, \kappa)$, their corresponding combinations: $\mathrm{ij}(\chi) = \frac{\mathrm{in}(\chi)}{\mathrm{jn}(\chi)}$ where i, j = s, c, d, and their inverses $\mathrm{ni}(\chi) = \frac{1}{\mathrm{in}(\chi)}$, here κ $(0 < \kappa < 1)$ is the modulus. In addition, the following identities can be useful:

$$\begin{aligned}
\mathrm{sn}^2\chi + \mathrm{cn}^2\chi &= 1,\\
\mathrm{dn}^2\chi + \kappa^2\mathrm{sn}^2\chi &= 1,\\
\frac{d}{d\chi}\mathrm{sn}\chi &= \mathrm{cn}\chi\mathrm{dn}\chi,\\
\frac{d}{d\chi}\mathrm{cn}\chi &= -\mathrm{sn}\chi\mathrm{dn}\chi,\\
\frac{d}{d\chi}\mathrm{dn}\chi &= -\kappa^2\mathrm{sn}\chi\mathrm{cn}\chi.
\end{aligned}$$

When the restrictions are imposed on κ, as $\kappa \longrightarrow 0$ and $\kappa \longrightarrow 1$, the Jacobi elliptic functions are transformed into trigonometric and hyperbolic functions which are listed in Table 2.

Now, let us define the function $M(\chi)$ as the finite series of the Jacobi elliptic functions

$$M(\chi) = \sum_{j=0}^{N} \omega_j \psi(\chi)^j. \tag{7}$$

In this way, the nonlinear ordinary equation (5) is satisfied by $M(\chi)$, where the constants to be obtained are ω_j $(j = 0, 1, 2, \cdots, N)$. The integer N in (7) can be computed by taking into account the highest-order linear terms

$$O\left(\frac{d^v M}{dz^v}\right) = N + v, \quad v = 0, 1, 2, 3, \cdots,$$

and the highest-order nonlinear terms

$$O\left(M^r \frac{d^v M}{dz^v}\right) = (r+1)N + v, \quad v = 0, 1, 2, 3, \cdots, \quad r = 1, 2, 3, \cdots.$$

This process is known as the balancing rule.

Taking into account equations (5) and (7) and setting all of the power coefficients $\psi(\chi)$ to zero, the set of nonlinear algebraic equations for ω_j $(j = 0, 1, 2, 3, \cdots, N)$ is thus obtained. The system of equations is solved by setting all of the values for Q, R, and P in (6) according to Table 1. Then the exact solution of equation (3) is obtained.

3. Fractional-order derivative

In this section, we define the conformal fractional derivative and mention some of its most important properties [5].

	P	Q	R	ψ
1	κ^2	$-(1+\kappa^2)$	1	sn,cd
2	$-\kappa^2$	$2\kappa^2-1$	$1-\kappa^2$	cn
3	-1	$2-\kappa^2$	κ^2-1	dn
4	1	$-(1+\kappa^2)$	κ^2	ns,dc
5	$1-\kappa^2$	$2\kappa^2-1$	$-\kappa^2$	nc
6	κ^2-1	$2-\kappa^2$	-1	nd
7	$1-\kappa^2$	$2-\kappa^2$	1	sc
8	$-\kappa^2(1-\kappa^2)$	$2\kappa^2-1$	1	sd
9	1	$2-\kappa^2$	$1-\kappa^2$	cs
10	1	$2\kappa^2-1$	$-\kappa^2(1-\kappa^2)$	$-$ds
11	$\frac{-1}{4}$	$\frac{\kappa^2+1}{2}$	$\frac{-(1-\kappa^2)^2}{4}$	$\kappa\mathrm{cn}\mp\mathrm{dn}$
12	$\frac{1}{4}$	$\frac{-2\kappa^2+1}{2}$	$\frac{1}{4}$	$\mathrm{ns}\mp\mathrm{cs}$
13	$\frac{1-\kappa^2}{4}$	$\frac{\kappa^2+1}{2}$	$\frac{1-\kappa^2}{4}$	$\mathrm{nc}\mp\mathrm{sc}$
14	$\frac{1}{4}$	$\frac{\kappa^2-2}{2}$	$\frac{\kappa^4}{4}$	$\mathrm{ns}\mp\mathrm{ds}$
15	$\frac{\kappa^2}{4}$	$\frac{\kappa^2-2}{2}$	$\frac{\kappa^2}{4}$	$\mathrm{sn}\mp i\mathrm{cn}$, $\frac{\mathrm{sn}}{\sqrt{1-\kappa^2\mathrm{sn}\mp\mathrm{cn}}}$
16	$\frac{1}{4}$	$\frac{1-2\kappa^2}{2}$	$\frac{1}{4}$	$\kappa\mathrm{cn}\mp i\mathrm{dn}$, $\frac{\mathrm{sn}}{1\mp\mathrm{cn}}$
17	$\frac{\kappa^2}{4}$	$\frac{\kappa^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{1\mp\mathrm{dn}}$
18	$\frac{\kappa^2-1}{4}$	$\frac{\kappa^2+1}{2}$	$\frac{\kappa^2-1}{4}$	$\frac{\mathrm{dn}}{1\mp\kappa\mathrm{sn}}$
19	$\frac{1-\kappa^2}{4}$	$\frac{\kappa^2+1}{2}$	$\frac{-\kappa^2+1}{4}$	$\frac{\mathrm{cn}}{1\mp\mathrm{sn}}$
20	$\frac{(1-\kappa^2)^2}{4}$	$\frac{\kappa^2+1}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{\mathrm{dn}\mp\mathrm{cn}}$
21	$\frac{\kappa^4}{4}$	$\frac{\kappa^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{cn}}{\sqrt{1-\kappa^2\mp\mathrm{dn}}}$

TABLE 1. Solutions of equation (6) for the specifically selected values of P, Q, and R.

	case	$\kappa \longrightarrow 1$	$\kappa \longrightarrow 0$
1	$\mathrm{sn}\chi$	$\tanh\chi$	$\sin\chi$
2	$\mathrm{cn}\chi$	$\mathrm{sech}\ \chi$	$\cos\chi$
3	$\mathrm{dn}\chi$	$\mathrm{sech}\ \chi$	1
4	$\mathrm{cd}\chi$	1	$\cos\chi$
5	$\mathrm{sd}\chi$	$\sinh\chi$	$\sin\chi$
6	$\mathrm{nd}\chi$	$\cosh\chi$	1
7	$\mathrm{dc}\chi$	1	$\sec\chi$
8	$\mathrm{nc}\chi$	$\cosh\chi$	$\sec\chi$
9	$\mathrm{sc}\chi$	$\sinh\chi$	$\tan\chi$
10	$\mathrm{ns}\chi$	$\coth\chi$	$\csc\chi$
11	$\mathrm{ds}\chi$	$\mathrm{csch}\ \chi$	$\csc\chi$
12	$\mathrm{cs}\chi$	$\mathrm{csch}\ \chi$	$\cot\chi$

TABLE 2. Jacobi elliptic functions for $\kappa \to 1$ and $\kappa \to 0$.

Definition 1. Let $q : [0, \infty] \to R$ be a differentiable function. The definition of *conformable derivative of order* β is

$$D^{\beta}(q)(t) = \lim_{\epsilon \to 0} \frac{q\left(\epsilon t^{1-\beta} + t\right) - q(t)}{\epsilon},$$

for all $t > 0$, and $\beta \in (0, 1]$.

Theorem 1. *Let* ζ *and* η *are* β *differentiable functions at a point* $t > 0$ *with* $\beta \in (0, 1]$. *Then*

- $D^{\beta}(\zeta * p + \eta * q) = p * D^{\beta}(\zeta) + q * D^{\beta}(\eta), \forall p, q \in R;$
- $D^{\beta}(t^{u}) = u * t^{r-\beta}, \forall u \in R;$
- $D^{\beta}(\zeta . \eta) = \zeta * D^{\beta}(\eta) + \eta * D^{\beta}(\zeta);$
- $D^{\beta} \frac{\zeta}{\eta} = \frac{\eta * D^{\beta}(\zeta) - \zeta * D^{\beta}(\eta)}{\eta^{2}};$
- $D^{\beta}(p) = 0$, *for all constant functions* $q(t) = p;$
- *if* q *is differentiable, then* $D^{\beta}(\zeta)(t) = t^{1-\beta} * \frac{d\zeta(t)}{dt}$.

4. Solution of the model

Now, we apply the JEFE method to the conformal fractional version of the EMC equation (2). By substituting the transformation (4) into (2) we get the ODE

$$c^{2}M'' + 2ack^{2}M'M'' + k^{3}cM^{iv} = 0.$$

If we integrate this equation and set the integration constant as zero, the resulting ODE is

$$c^{2}M' + ack^{2}M'^{2} + k^{3}cM''' = 0.$$

Now let $M' = H$. Then

$$c^{2}H + ack^{2}H^{2} + k^{3}cH'' = 0. \tag{8}$$

If we apply the balancing procedure to the equation (8), where the solution H is written in terms of the sum (7), the result is

$$H(\chi) = \omega_0 + \omega_1 \psi(\chi) + \omega_2 \psi(\chi)^{2}, \tag{9}$$

where ω_0, ω_1, ω_2 are constants. Substituting (9) and its derivatives in (8), we get the following set of algebraic equations:

$$\begin{aligned}
ack^{2}\omega_0^{2} + c^{2}\omega_0 + 2ck^{3}R\omega_2 &= 0,\\
2ack^{2}\omega_0\omega_1 + c^{2}\omega_1 + ck^{3}Q\omega_1 &= 0,\\
ack^{2}\omega_1^{2} + 2ack^{2}\omega_0\omega_2 + c^{2}\omega_2 + 4ck^{3}Q\omega_2 &= 0,\\
2ack^{2}\omega_1\omega_2 + 2ck^{3}P\omega_1 &= 0,\\
ack^{2}\omega_2^{2} + 6ck^{3}P\omega_2 &= 0.
\end{aligned}$$

The corresponding solution is:

$$\begin{aligned}
\omega_0 &= -\frac{2\left(\sqrt{a^2k^2\left(Q^2-3PR\right)}+akQ\right)}{a^2},\\
\omega_1 &= 0,\\
\omega_2 &= -\frac{6kP}{a},\\
c &= \frac{4k^2\sqrt{a^2k^2\left(Q^2-3PR\right)}}{a}.
\end{aligned}$$

With these results and the solutions for different values of P, Q and R from Table 1, it is possible to obtain an abundant set of solutions of type (9).

In the following cases, we construct hyperbolic or trigonometric solutions according to Table 2.

Case 1. When $P=\kappa^2$, $Q=-\left(\kappa^2+1\right)$, $R=1$, $\psi(\chi)=\mathrm{sn}(\chi,\kappa)$, if $\kappa=1$, then $\mathrm{sn}(\chi,\kappa)=\tanh(\chi)$ and the corresponding solution is

$$Y_1(x,t)=-\frac{2}{a^2}\left(\sqrt{a^2k^2}-2ak\right)-\frac{6k}{a}\left[\tanh^2\left(kx+4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 2. When $P=-\kappa^2$, $Q=2\kappa^2-1$, $R=1-\kappa^2$, $\psi(\chi)=\mathrm{cn}(\chi,\kappa)$, if $\kappa=1$, then $\mathrm{cn}(\chi,\kappa)=\mathrm{sech}(\chi)$ and the corresponding solution is

$$Y_2(x,t)=-\frac{2}{a^2}\left(\sqrt{a^2k^2}+ak\right)+\frac{6k}{a}\left[\mathrm{sech}^2\left(kx+4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 3. When $P=-1$, $Q=2-\kappa^2$, $R=\kappa^2-1$, $\psi(\chi)=\mathrm{dn}(\chi,\kappa)$, if $\kappa=1$, then $\mathrm{dn}(\chi,\kappa)=\mathrm{sech}(\chi)$ and the corresponding solution is

$$Y_3(x,t)=-\frac{2}{a^2}\left(\sqrt{a^2k^2}+ak\right)+\frac{6k}{a}\left[\mathrm{sech}^2\left(kx+4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 4. When $P=1$, $Q=-\left(\kappa^2+1\right)$, $R=\kappa^2$, $\psi(\chi)=\mathrm{ns}(\chi,\kappa)$, if $\kappa=1$, then $\mathrm{ns}(\chi,\kappa)=\coth(\chi)$ and the corresponding solution is

$$Y_4(x,t)=-\frac{2}{a^2}\left(\sqrt{a^2k^2}-2ak\right)-\frac{6k}{a}\left[\coth^2\left(kx+4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 5. When $P=1$, $Q=-\left(\kappa^2+1\right)$, $R=\kappa^2$, $\psi(\chi)=\mathrm{ns}(\chi,\kappa)$, if $\kappa=0$, then $\mathrm{ns}(\chi,\kappa)=\csc(\chi)$ and the corresponding solution is

$$Y_5(x,t)=-\frac{2}{a^2}\left(\sqrt{a^2k^2}-ak\right)-\frac{6k}{a}\left[\csc^2\left(kx+4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 6. When $P = 1-\kappa^2$, $Q = 2\kappa^2-1$, $R = -\kappa^2$, $\psi(\chi) = \mathrm{nc}(\chi,\kappa)$, if $\kappa = 0$, then $\mathrm{nc}(\chi,\kappa) = \sec(\chi)$ and the corresponding solution is

$$Y_6(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} - ak\right) - \frac{6k}{a}\left[\sec^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 7. When $P = 1-\kappa^2$, $Q = 2-\kappa^2$, $R = 1$, $\psi(\chi) = \mathrm{sc}(\chi,\kappa)$, if $\kappa = 0$, then $\mathrm{sc}(\chi,\kappa) = \tan(\chi)$ and the corresponding solution is

$$Y_7(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} + 2ak\right) - \frac{6k}{a}\left[\tan^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 8. When $P = 1$, $Q = 2-\kappa^2$, $R = 1-\kappa^2$, $\psi(\chi) = \mathrm{cs}(\chi,\kappa)$, if $\kappa = 1$, then $\mathrm{cs}(\chi,\kappa) = \mathrm{csch}(\chi)$ and the corresponding solution is

$$Y_8(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} + ak\right) - \frac{6k}{a}\left[\mathrm{csch}^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 9. When $P = 1$, $Q = 2-\kappa^2$, $R = 1-\kappa^2$, $\psi(\chi) = \mathrm{cs}(\chi,\kappa)$, if $\kappa = 0$, then $\mathrm{cs}(\chi,\kappa) = \cot(\chi)$ and the corresponding solution is

$$Y_9(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} + 2ak\right) - \frac{6k}{a}\left[\cot^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 10. When $P = 1$, $Q = 2\kappa^2-1$, $R = -\kappa^2\left(1-\kappa^2\right)$, $\psi(\chi) = -\mathrm{ds}(\chi,\kappa)$, if $\kappa = 1$, then $\mathrm{ds}(\chi,\kappa) = \mathrm{csch}(\chi)$ and the corresponding solution is

$$Y_{10}(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} + ak\right) - \frac{6k}{a}\left[\mathrm{csch}^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 11. When $P = 1$, $Q = 2\kappa^2-1$, $R = \kappa^2\left(1-\kappa^2\right)$, $\psi(\chi) = -\mathrm{ds}(\chi,\kappa)$, if $\kappa = 0$, then $\mathrm{ds}(\chi,\kappa) = \csc(\chi)$ and the corresponding solution is

$$Y_{11}(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} - ak\right) - \frac{6k}{a}\left[\csc^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 12. When $P = -\frac{1}{4}$, $Q = \frac{1}{2}\left(\kappa^2+1\right)$, $R = -\frac{1}{4}\left(1-\kappa^2\right)^2$, $\psi(\chi) = \kappa\mathrm{cn}(\chi,\kappa) + \mathrm{dn}(\chi,\kappa)$, if $\kappa = 1$, then $\mathrm{cn}(\chi,\kappa) = \mathrm{sech}(\chi)$, $\mathrm{dn}(\chi,\kappa) = \mathrm{sech}(\chi)$ and the corresponding solution is

$$Y_{12}(x,t) = -\frac{2}{a^2}\left(\sqrt{a^2k^2} + ak\right) + \frac{6k}{a}\left[\mathrm{sech}^2\left(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 13. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(1 - 2\kappa^2\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \mathrm{cs}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa)$, if $\kappa = 1$, then $\mathrm{cs}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa) = \coth(\chi) + \mathrm{csch}(\chi)$ and the corresponding solution is

$$\begin{aligned} Y_{13}(x,t) = &- \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) \\ &- \frac{3k}{2a}\left[\coth\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + \mathrm{csch}\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right]^2. \end{aligned}$$

Case 14. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(1 - 2\kappa^2\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \mathrm{cs}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa)$, if $\kappa = 0$, then $\mathrm{cs}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa) = \cot(\chi) + \csc(\chi)$ and the corresponding solution is

$$\begin{aligned} Y_{14}(x,t) = &- \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} + \frac{ak}{2}\right) \\ &- \frac{3k}{2a}\left[\cot\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + \csc\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right]^2. \end{aligned}$$

Case 15. When $P = \frac{1}{4}\left(1 - \kappa^2\right)$, $Q = \frac{1}{2}\left(\kappa^2 + 1\right)$, $R = \frac{1}{4}\left(1 - \kappa^2\right)$, $\psi(\chi) = \mathrm{nc}(\chi, \kappa) + \mathrm{sc}(\chi, \kappa)$, if $\kappa = 0$, then $\mathrm{nc}(\chi, \kappa) + \mathrm{sc}(\chi, \kappa) = \sec(\chi) + \tan(\chi)$ and the corresponding solution is

$$\begin{aligned} Y_{15}(x,t) = &- \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} + \frac{ak}{2}\right) \\ &- \frac{3k}{2a}\left[\tan\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + \sec\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right]^2. \end{aligned}$$

Case 16. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(\kappa^2 - 2\right)$, $R = \frac{\kappa^4}{4}$, $\psi(\chi) = \mathrm{ds}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa)$, if $\kappa = 1$, then $\mathrm{ds}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa) = \coth(\chi) + \mathrm{csch}(\chi)$ and the corresponding solution is

$$\begin{aligned} Y_{16}(x,t) = &- \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) \\ &- \frac{3k}{2a}\left[\coth\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + \mathrm{csch}\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right]^2. \end{aligned}$$

Case 17. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(\kappa^2 - 2\right)$, $R = \frac{\kappa^4}{4}$, $\psi(\chi) = \mathrm{ds}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa)$, if $\kappa = 0$, then $\mathrm{ds}(\chi, \kappa) + \mathrm{ns}(\chi, \kappa) = \csc(\chi) + \csc(\chi)$ and the corresponding solution is

$$Y_{17}(x,t) = - \frac{2}{a^2}\left(\sqrt{a^2k^2} - ak\right) - \frac{6k}{a}\left[\csc^2(kx + 4\frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta})\right].$$

Case 18. When $P = \frac{\kappa^2}{4}$, $Q = \frac{1}{2}\left(\kappa^2 - 2\right)$, $R = \frac{\kappa^2}{4}$, $\psi(\chi) = \operatorname{sn}(\chi, \kappa) + i\operatorname{cn}(\chi, \kappa)$, if $\kappa = 1$, then $\operatorname{sn}(\chi, \kappa) + i\operatorname{cn}(\chi, \kappa) = \tanh(\chi) + i\operatorname{sech}(\chi)$ and the corresponding solution is

$$\begin{aligned} Y_{18}(x,t) = &- \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) \\ &- \frac{3k}{2a}\left[\tanh\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + i\operatorname{sech}\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right]^2. \end{aligned}$$

Case 19. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(1 - 2\kappa^2\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \kappa\operatorname{cn}(\chi, \kappa) + i\operatorname{dn}(\chi, \kappa)$, if $\kappa = 1$, then $\kappa\operatorname{cn}(\chi, \kappa) + i\operatorname{dn}(\chi, \kappa) = \operatorname{sech}(\chi) + i\operatorname{sech}(\chi)$ and the corresponding solution is

$$Y_{19}(x,t) = - \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) - \frac{3ik}{a}\left[\operatorname{sech}^2\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)\right].$$

Case 20. When $P = \frac{1}{4}$, $Q = \frac{1}{2}\left(1 - 2\kappa^2\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \kappa\operatorname{cn}(\chi, \kappa) + i\operatorname{dn}(\chi, \kappa)$, if $\kappa = 0$, then $\kappa\operatorname{cn}(\chi, \kappa) + i\operatorname{dn}(\chi, \kappa) = i$ and the corresponding solution is

$$Y_{20}(x,t) = - \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} + \frac{ak}{2}\right) - \frac{3k}{2a}.$$

Case 21. When $P = \frac{\kappa^2}{4}$, $Q = \frac{1}{2}\left(\kappa^2 - 2\right)$,$R = \frac{1}{4}$, $\psi(\chi) = \frac{\operatorname{sn}(\chi,\kappa)}{\operatorname{dn}(\chi,\kappa)+1}$, if $\kappa = 1$, then $\operatorname{dn}(\chi, \kappa) = \operatorname{sech}(\chi)$, $\operatorname{sn}(\chi, \kappa) = \tanh(\chi)$ and the corresponding solution is

$$Y_{21}(x,t) = - \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) - \frac{3k}{2a}\left[\frac{\tanh\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)}{\operatorname{sech}\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + 1}\right]^2.$$

Case 22. When $P = \frac{1}{4}\left(1 - \kappa^2\right)$, $Q = \frac{1}{2}\left(\kappa^2 + 1\right)$, $R = \frac{1}{4}\left(1 - \kappa^2\right)$, $\psi(\chi) = \frac{\operatorname{cn}(\chi,\kappa)}{\operatorname{sn}(\chi,\kappa)+1}$, if $\kappa = 0$, then $\operatorname{cn}(\chi, \kappa) = \cos(\chi)$, $\operatorname{sn}(\chi, \kappa) = \sin(\chi)$ and the corresponding solution is

$$Y_{22}(x,t) = - \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} + \frac{ak}{2}\right) - \frac{3k}{2a}\left[\frac{\cos\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)}{\sin\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right) + 1}\right]^2.$$

Case 23. When $P = \frac{1}{4}\left(1 - \kappa^2\right)^2$, $Q = \frac{1}{2}\left(\kappa^2 + 1\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \frac{\operatorname{sn}(\chi,\kappa)}{\operatorname{cn}(\chi,\kappa)+\operatorname{dn}(\chi,\kappa)}$, if $\kappa = 0$, then $\operatorname{cn}(\chi, \kappa) = \cos(\chi)$, $\operatorname{sn}(\chi, \kappa) = \sin(\chi)$, $\operatorname{dn}(\chi, \kappa) = 1$ and the corresponding solution is

$$Y_{23}(x,t) = - \frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} + \frac{ak}{2}\right) - \frac{3k}{2a}\left[\frac{\sin\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right)}{\cos(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}) + 1}\right]^2.$$

Case 24. When $P = \frac{\kappa^4}{4}$, $Q = \frac{1}{2}\left(\kappa^2 - 2\right)$, $R = \frac{1}{4}$, $\psi(\chi) = \frac{\mathrm{cn}(\chi,\kappa)}{\sqrt{\mathrm{dn}(\chi,\kappa) - \kappa^2 + 1}}$, if $\kappa = 1$, then $\mathrm{cn}(\chi,\kappa) = \mathrm{sech}(\chi)$, $\mathrm{dn}(\chi,\kappa) = \mathrm{sech}(\chi)$ and the corresponding solution is

$$Y_{24}(x,t) = -\frac{2}{a^2}\left(\frac{1}{4}\sqrt{a^2k^2} - \frac{ak}{2}\right) - \frac{3k}{2a}\mathrm{sech}\left(kx + \frac{k^2\sqrt{a^2k^2}t^\beta}{a\beta}\right).$$

5. Graphical behaviour

In this section, we examine the graphical behavior of some previously obtained exact solutions. In order to establish various physical characteristics, we must draw the required answers in 3D, 2D and contour plots. The plots appear in this order in each graph. These graphs provide more trustworthy information about the behaviors of the solutions. Essentially, of the 24 cases studied we can find three types of behavior: the dark soliton type, that is, a solitary traveling wave with a well-defined minimum, exemplified by the case $Y_1(x,t)$ in Figure 1; the bright soliton type, which corresponds to a solitary traveling wave with a well-defined maximum, as in the cases $Y_3(x,t)$, $Y_{12}(x,t)$ and $Y_{18}(x,t)$, see Figures 2, 5 and 7; and the multi-traveling-wave type as in the cases $Y_5(x,t)$, $Y_7(x,t)$ and $Y_{15}(x,t)$, see Figures 3, 4 and 6.

6. Conclusion

In this work, we employ the JEFE method to obtain exact wave solutions of a time-conformable fractional EMC equation. The study focuses on a nonlinear partial differential equation (NLPDE) model that incorporates fractional-order time derivatives, defined in the conformable sense. By applying a fractional traveling-wave transformation, the model is reduced to an ordinary differential equation (ODE). The JEFE method is then used to derive a variety of exact solutions, including hyperbolic and trigonometric wave structures. This approach has proven to be highly reliable and effective for solving a broad class of nonlinear differential equations. The analytical solutions are further validated and visualized using Mathematica through 3D, 2D, and contour plots, which illustrate their physical significance and provide deeper insight into the dynamical behavior of the solutions.

Acknowledgements

The authors thank the anonymous referees for their very thorough and insightful reviews.

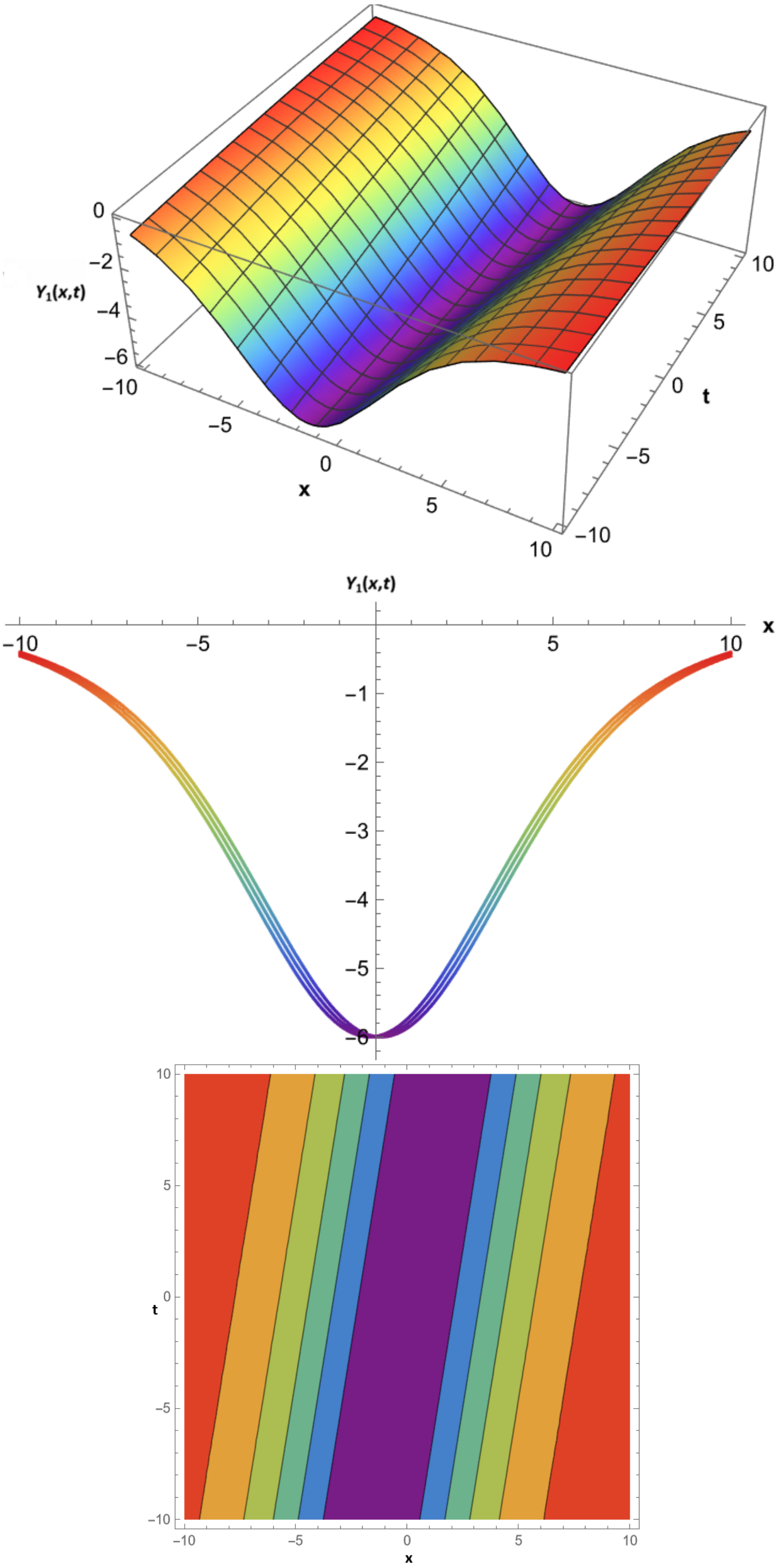

Figure 1. Graphical behavior of $Y_1(x,t)$ with parameters $a = 0.2$, $\beta = 1$, $k = -0.2$.

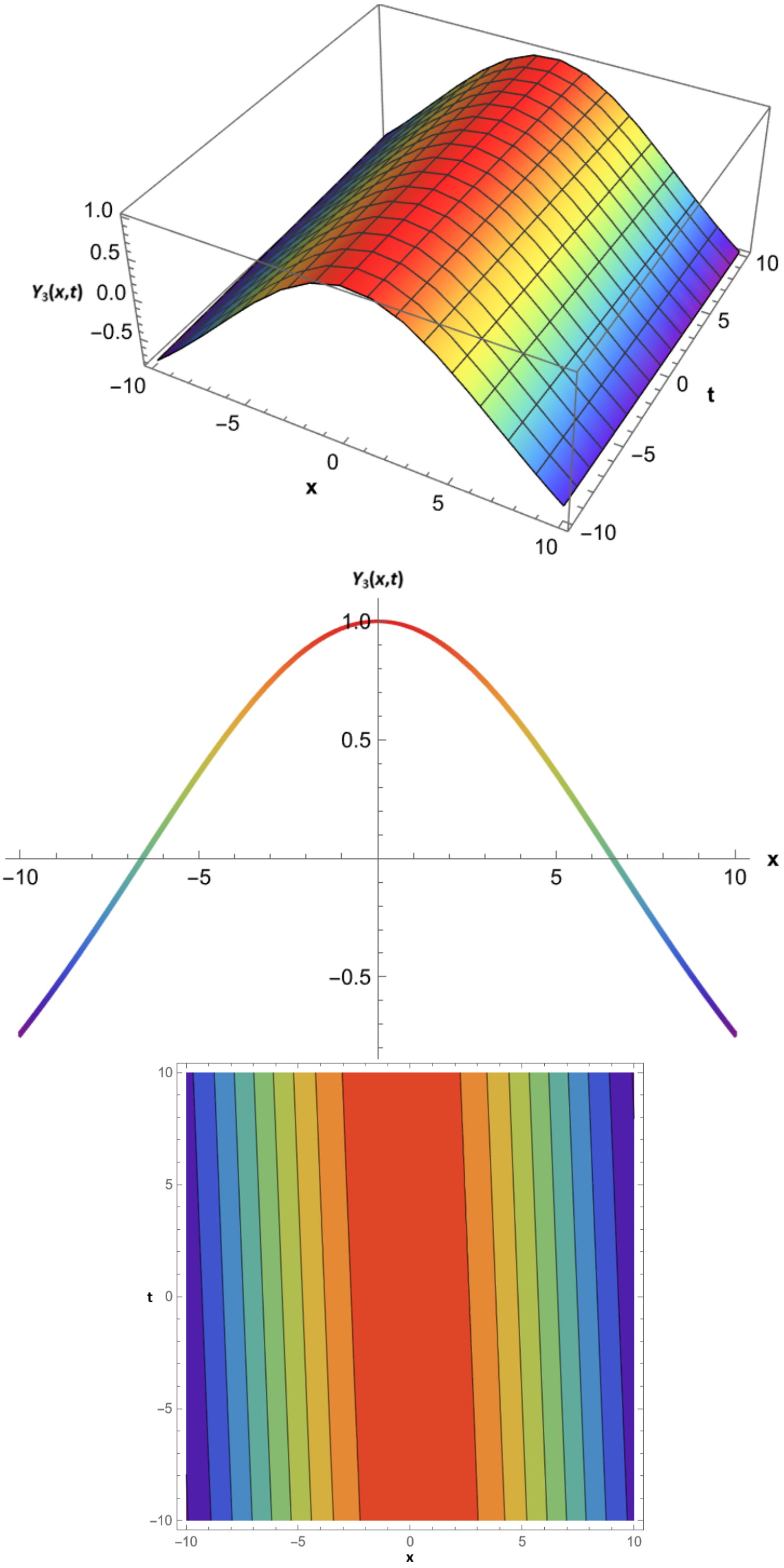

FIGURE 2. Graphical behavior of $Y_3(x,t)$ with parameters $a = 0.2$, $\beta = 0.5$, $k = 0.1$.

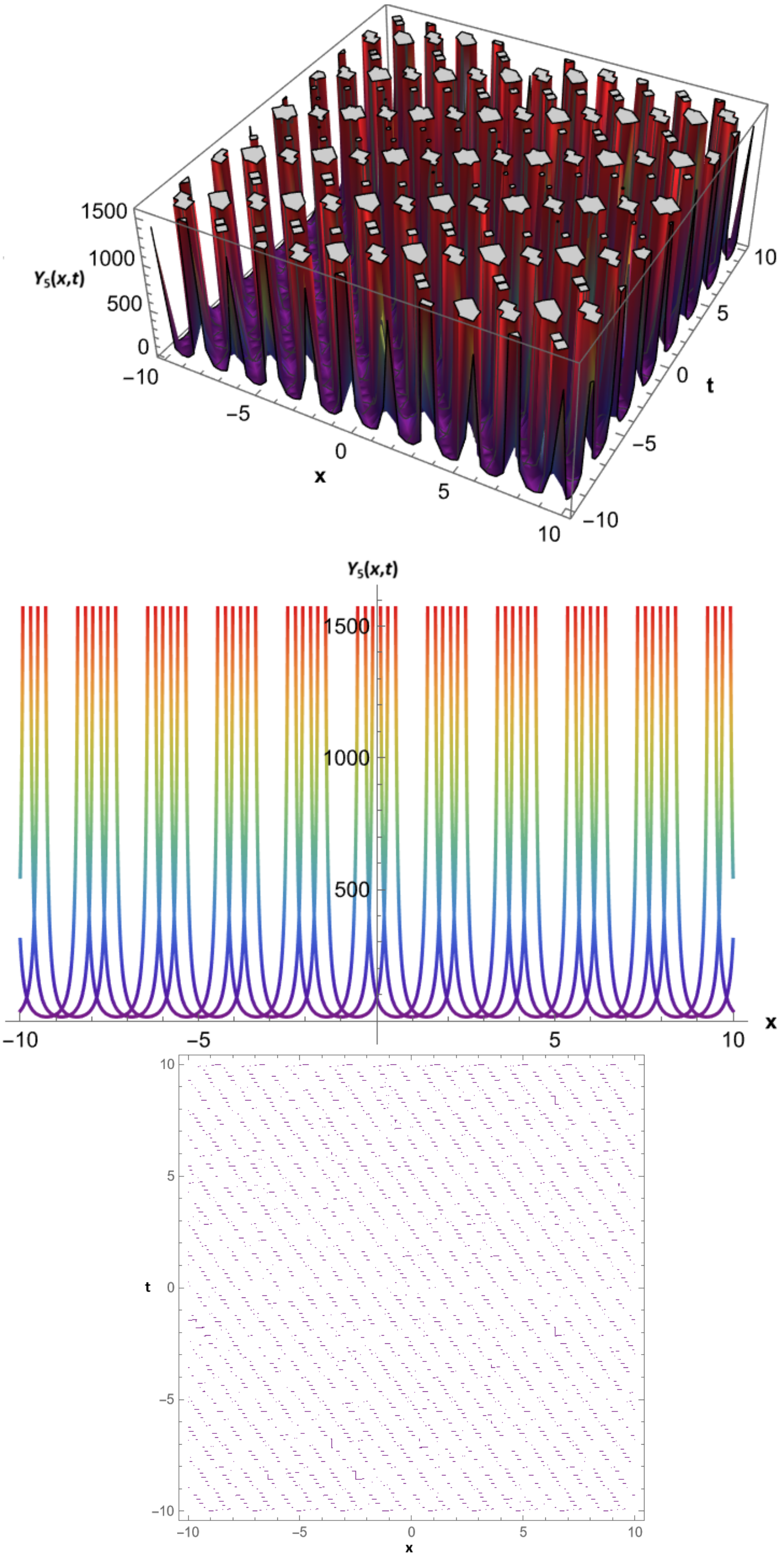

FIGURE 3. Graphical behavior of $Y_5(x,t)$ with parameters $a = 0.2$, $\beta = 1$, $k = -1.6$.

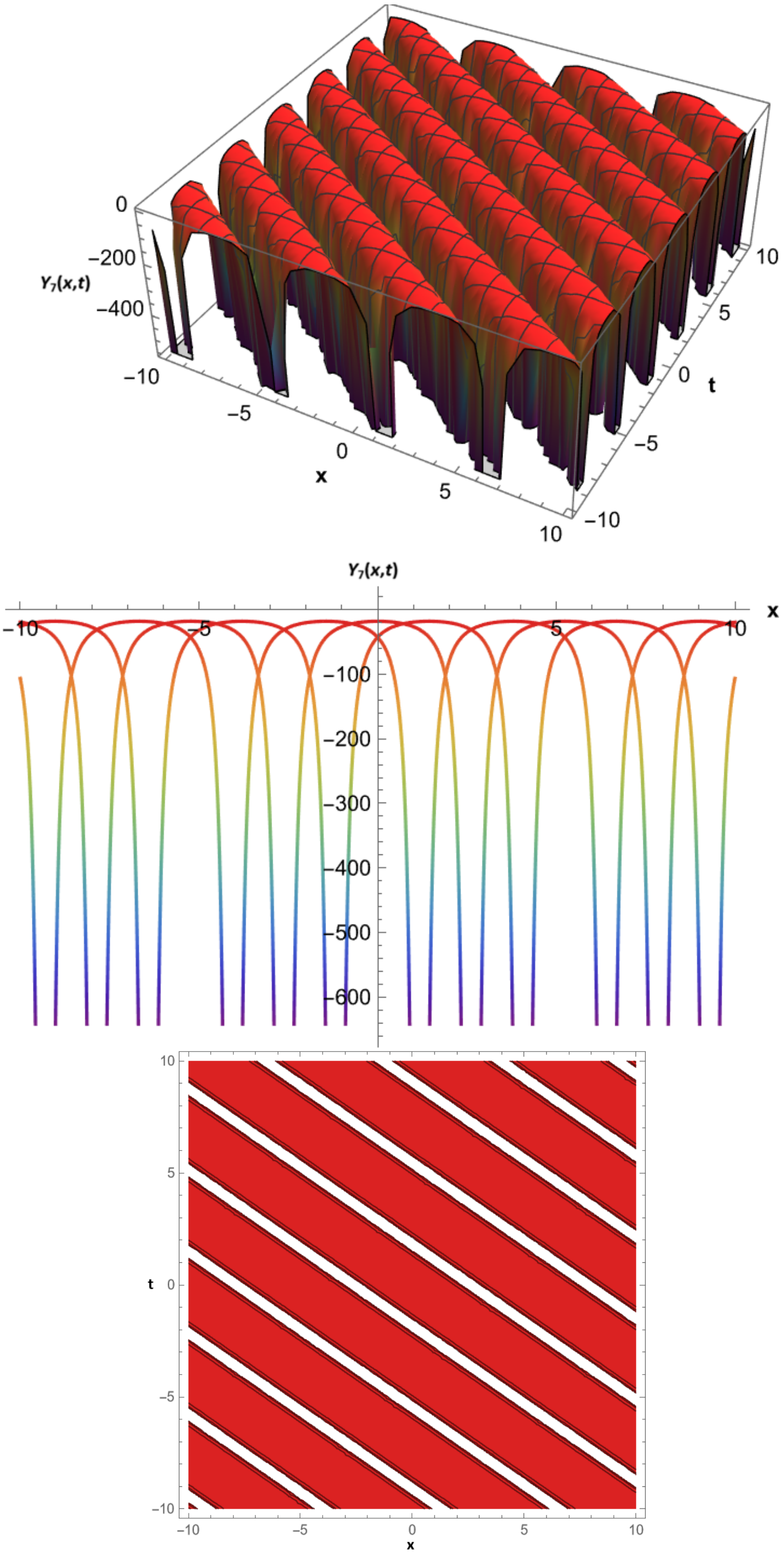

FIGURE 4. Graphical behavior of $Y_7(x,t)$ with parameters $a = 0.2$, $\beta = 1$, $k = 0.6$.

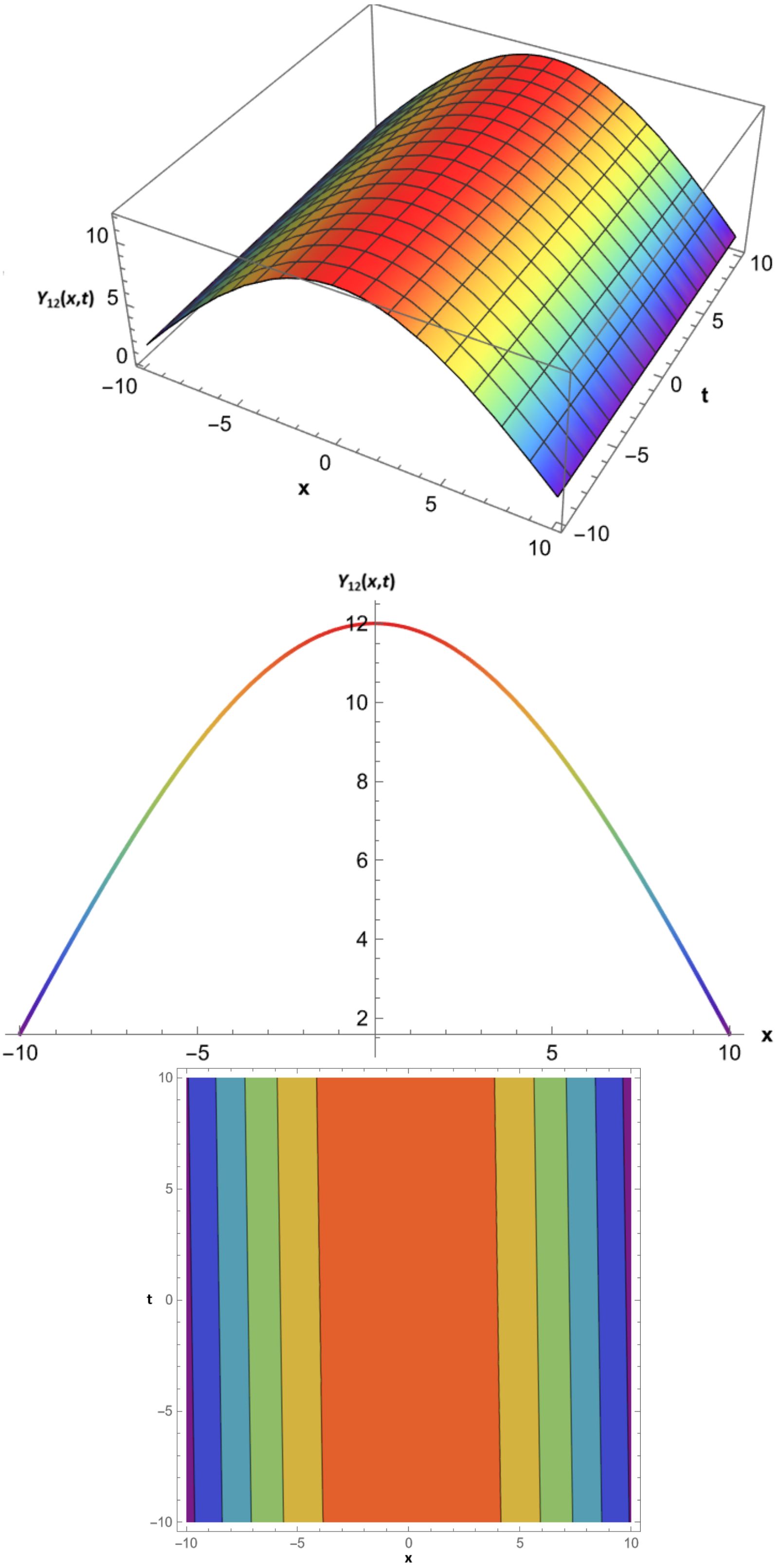

FIGURE 5. Graphical behavior of $Y_{12}(x,t)$ with parameters $a = 0.01$, $\beta = 1$, $k = 0.06$.

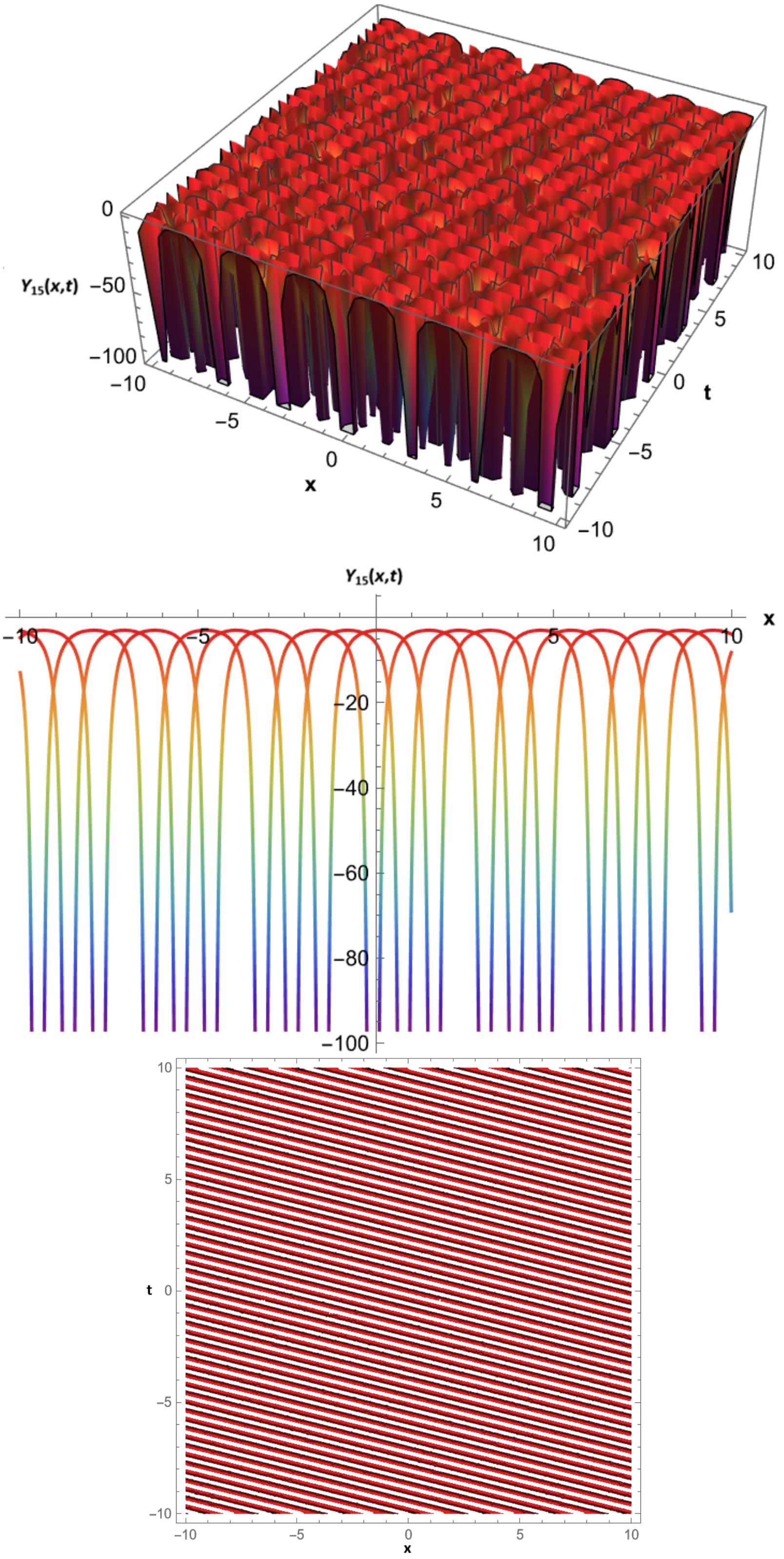

FIGURE 6. Graphical behavior of $Y_{15}(x,t)$ with parameters $a = 1$, $\beta = 1$, $k = 2$.

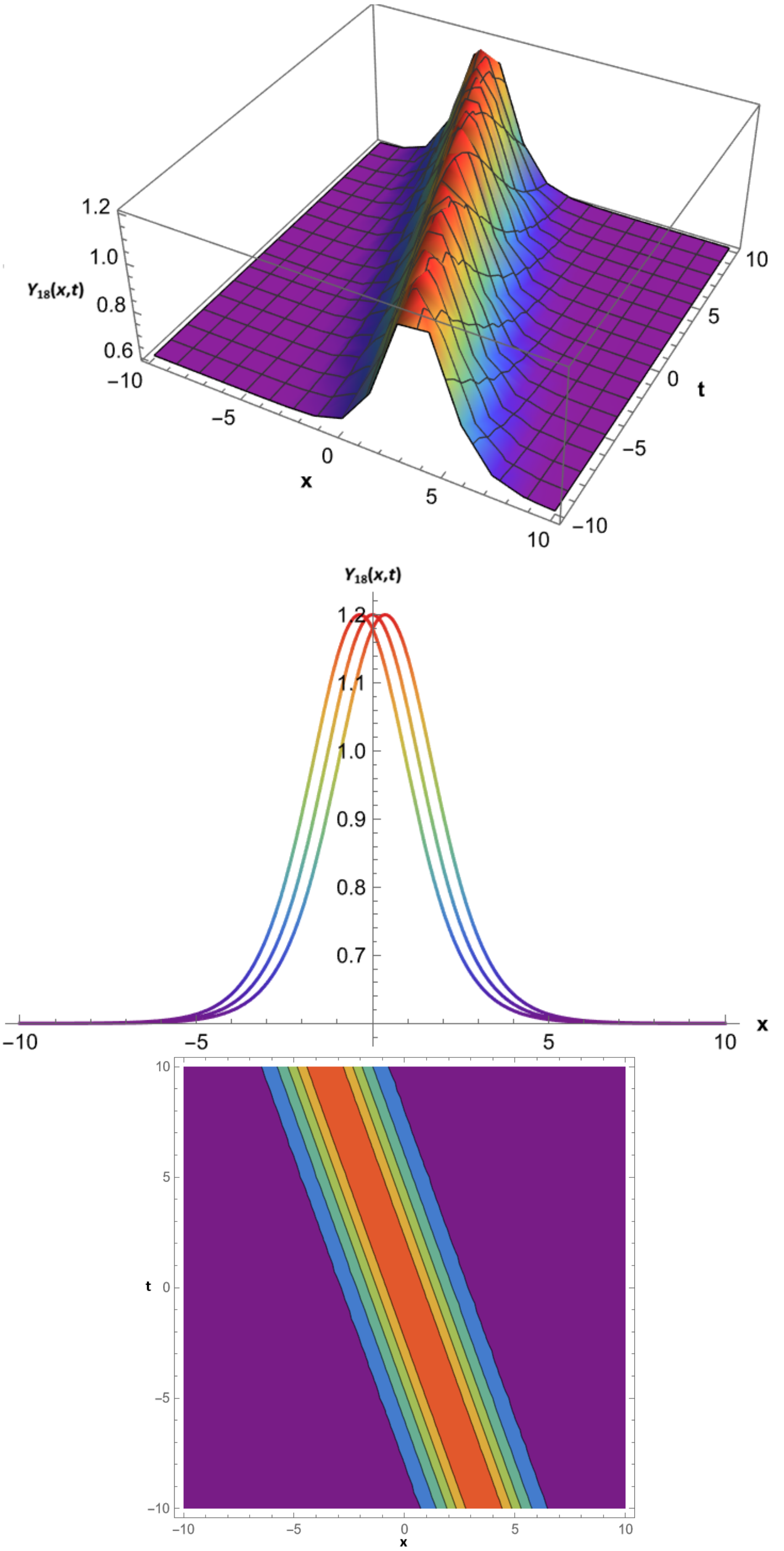

Figure 7. Graphical behavior of $Y_{18}(x,t)$ with parameters $a = 1$, $\beta = 1$, $k = 0.6$.

References

[1] M. A. Abdelrahman and H. A. Alkhidhr, *A robust and accurate solver for some nonlinear partial differential equations and tow applications*, Phys. Scr. **95** (2020), 065212, 11 pp.

[2] N. Ahmed, J. E. Macías-Díaz, S. Umer, M. Z. Baber, M. Jawaz, and H. Vargas-Rodríguez, *On some solitary wave solutions of the Estevez–Mansfield–Clarkson equation with conformable fractional derivatives in time*, Open Phys. **22** (2024), 20240109, 10 pp.

[3] A. I. Aliyu, J. S. Yusuf, M. M. Nauman, D. U. Ozsahin, B. G. Agaie, J. H. Zaini, and H. Umar, *Lie Symmetry Analysis and Explicit Solutions to the Estevez–Mansfield–Clarkson Equation*, Symmetry **16** (2024), 1194, 13 pp.

[4] B. Alshahrani, H. A. Yakout, M. M. Khater, A. H. Abdel-Aty, E. E. Mahmoud, D. Baleanu, and H. Eleuch, *Accurate novel explicit complex wave solutions of the* $(2+1)$*-dimensional Chiral nonlinear Schrödinger equation*, Results Phys. **23** (2021), 104019, 9 pp.

[5] E. Balci, I. Öztürk, and S. Kartal, *Dynamical behaviour of fractional order tumor model with Caputo and conformable fractional derivative*, Chaos Solitons Fractals **123** (2019), 43–51.

[6] A. Bekir, E. Aksoy, and A. C. Cevikel, *Exact solutions of nonlinear time fractional partial differential equations by sub-equation method*, Math. Methods Appl. Sci. **38** (2015), 2779–2784.

[7] S. Bibi, N. Ahmed, U. Khan, and S. T. Mohyud-Din, *Some new exact solitary wave solutions of the van der Waals model arising in nature*, Results Phys. **9** (2018), 648–655.

[8] M. Bilal and J. Ahmad, *New exact solitary wave solutions for the 3D-FWBBM model in arising shallow water waves by two analytical methods*, Results Phys. **25** (2021), 104230, 15 pp.

[9] Y. Çenesiz and A. Kurt, *The new solution of time fractional wave equation with conformable fractional derivative definition*, J. New Theory **7** (2015), 79–85.

[10] M. Dehghan, J. Manafian, and A. Saadatmandi, *Solving nonlinear fractional partial differential equations using the homotopy analysis method*, Numer. Methods Partial Differential Equations **26** (2010), 448–479.

[11] M. M. El-Borai, *Exact solutions for some nonlinear fractional parabolic partial differential equations*, Appl. Math. Comput. **206** (2008), 150–153.

[12] K. A. Gepreel and S. Omran, *Exact solutions for nonlinear partial fractional differential equations*, Chinese Physics B **21** (2012), 110204, 7 pp.

[13] X. Gui-Qiong and L. Zhi-Bin, *Applications of Jacobi elliptic function expansion method for nonlinear differential-difference equations*, Commun. Theor. Phys. **43** (2005), 385–388.

[14] B. Guo, X. Pu, and F. Huang, *Fractional Partial Differential Equations and Their Numerical Solutions*, World Scientific, Singapore, 2015.

[15] M. G. Hafez, M. N. Alam, and M. A. Akbar, *Exact traveling wave solutions to the Klein–Gordon equation using the novel (G'/G)-expansion method*, Results Phys. **4** (2014), 177–184.

[16] S. S. Kazmi, A. Jhangeer, and M. B. Riaz, *Analyzing sensitivity and multi-soliton solutions in the Estevez–Mansfield–Clarkson equation: Insights into dynamics of bifurcation and chaos*, Partial Differ. Equ. Appl. Math. **11** (2024), 100826, 14 pp.

[17] T. A. Khalil, N. Badra, H. M. Ahmed, and W. B. Rabie, *Optical solitons and other solutions for coupled system of nonlinear Biswas-Milovic equation with Kudryashov's*

law of refractive index by Jacobi elliptic function expansion method, Optik **253** (2022), 168540.

[18] P. R. Kundu, M. R. A. Fahim, M. E. Islam, and M. A. Akbar, *The sine-Gordon expansion method for higher-dimensional NLEEs and parametric analysis*, Heliyon **7** (2021), e06459, 8 pp.

[19] S. Liu, Z. Fu, S. Liu, and Q. Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett. A **289** (2001), 69–74.

[20] H. C. Ma, Z. P. Zhang, and A. P. Deng, *A new periodic solution to Jacobi elliptic functions of MKdV equation and BBM equation*, Acta Math. Appl. Sin. Engl. Ser. **28** (2012), 409–415.

[21] E. L. Mansfield and P. A. Clarkson, (1997). *Symmetries and exact solutions for a 2+ 1-dimensional shallow water wave equation*, Math. Comput. Simulation **43** (1997), 39–55.

[22] J. Muhammad, N. Nasreen, E. Hussain, U. Younas, and A. S. Alsubaie, *On the study of analytical soliton solutions and interaction aspects to the Estevez-Mansfield–Clarkson equation arising in diversity of fields*, Phys. Scr. **99** (2024), 115221.

[23] M. S. Osman and B. Ghanbari, *New optical solitary wave solutions of Fokas-Lenells equation in presence of perturbation terms by a novel approach*, Optik **175** (2018), 328–333.

[24] H. Qawaqneh and Y. Alrashedi, *Mathematical and physical analysis of fractional Estevez-Mansfield-Clarkson equation*, Fractal Fractional **8** (2024), 467, 19 pp.

[25] Z. Rahman, M. Z. Ali, and M. S. Ullah, *Analytical solutions of two space-time fractional nonlinear models using Jacobi elliptic function expansion method*, Contemp. Math. **2** (2021), 173–188.

[26] M. Roshid and H. Bashar, *Breather wave and Kinky periodic wave solutions of one-dimensional Oskolkov equation*, Math. Model. Eng. Probl. **6** (2019), 460–466.

[27] M. M. Roshid and H. O. Roshid, *Exact and explicit traveling wave solutions to two nonlinear evolution equations which describe incompressible viscoelastic Kelvin–Voigt fluid*, Heliyon **4** (2018), e00756, 21 pp.

[28] M. M. Roshid, M. Uddin, and G. Mostafa, *Dynamical structure of optical soliton solutions for M-fractional paraxial wave equation by using unified technique*, Results Phys. **51** (2023), 106632, 11 pp.

[29] A. R. Seadawy and S. Z. Alamri, *Mathematical methods via the nonlinear two-dimensional water waves of Olver dynamical equation and its exact solitary wave solutions*, Results Phys. **8** (2018), 286–291.

[30] M. S. Shehata, *Extended Jacobian elliptic function expansion method and its applications for solving some nonlinear evolution equations in mathematical physics*, Int. J. Comput. Appl. **109** (2015), 1–4.

[31] L. Shi-Kuo, F. Zun-Tao, L. Shi-Da, and Z. Qiang, *Expansion method about the Jacobi elliptic function and its applications to nonlinear wave equations*, Acta Phys. Sin. **50** (2001), 2068–2073.

[32] V. S. Shpakivskyi, *Conformable fractional derivative in commutative algebras.* J. Math. Sci. **274** (2023), 392–402.

[33] A. Sonmezoglu, *Exact solutions for some fractional differential equations*, Adv. Math. Phys. **2015** (2015), 567842, 10 pp.

[34] F. Tchier, A. Yusuf, A. I. Aliyu, and M. Inc, *Soliton solutions and conservation laws for lossy nonlinear transmission line equation*, Superlattic. Microstruct. **107** (2017), 320–336.

[35] A. Toplama, *A new approach to the directional derivative of fractional order*, Math. Meth. Appl. Sci. **48** (2025), 3926–3935.

[36] M. Vivas-Cortez, S. Arshed, M. Sadaf, Z. Perveen, and G. Akram, *Numerical simulations of the soliton dynamics for a nonlinear biological model: Modulation instability analysis*, PLoS One **18** (2023), e0281318, 12 pp.

[37] P. Xiao-Feng, *The lifetime of the soliton in the improved Davydov model at the biological temperature 300 K for protein molecules*, Eur. Phys. J. B **19** (2001), 297–316.

[38] B. Yasmeen, K. Ahmad, and N. Fatima, *Analyzing Approximate and Exact Wave Solutions of Optical Fibres in the Time-Fractional Estevez–Mansfield–Clarkson Equation*, Eur. J. Pure Appl. Math. **18** (2025), 5893, 17 pp.

[39] M. Younis, S. T. R. Rizvi, and S. Ali, *Analytical and soliton solutions: nonlinear model of nanobioelectronics transmission lines*, Appl. Math. Comput. **265** (2015), 994–1002.

[40] M. Younis and A. Zafar, *Exact solution to nonlinear differential equations of fractional order via (G'/G)-expansion method*, Appl. Math. **5** (2014), 1–6.

[41] H. Zhang, *Extended Jacobi elliptic function expansion method and its applications*, Commun. Nonlinear Sci. Numer. Simul. **12** (2007), 627–635.

[42] B. Zheng, *A new fractional Jacobi elliptic equation method for solving fractional partial differential equations*, Adv. Difference Equ. **2014** (2014), 228, 11 pp.

Department of Mathematics and Statistics, The University of Lahore, Lahore, Pakistan
E-mail address: `nauman.ahmed@math.uol.edu.pk`
E-mail address: `naveedpc75@gmail.com`
E-mail address: `umarkhanniazi426@gmail.com`

Department of Mathematics and Statistics, The University of Lahore, Sargodha Campus, Sargodha, Pakistan
E-mail address: `zafarullah8883@gmail.com`

Department of Exact Sciences and Technology, University Center of los Lagos, University of Guadalajara, Lagos de Moreno 47460, Mexico
E-mail address: `gallegos@culagos.udg.mx`

Department of Mathematics and Didactics of Mathematics, Tallinn University, 10120 Tallinn, Estonia; Departamento de Matemáticas y Física, Universidad Autónoma de Aguascalientes, Aguascalientes 20131, Mexico
E-mail address: `jorge.maciasdiaz@edu.uaa.mx`