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The connected niche graphs of split tournaments

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Abstract. The niche graph of a digraph D has $V(D)$ as the vertex set and an edge uv if and only if $(u,w) \in A(D)$ and $(v,w) \in A(D)$, or $(w,u) \in A(D)$ and $(w,v) \in A(D)$ for some $w \in V(D)$. In this paper, we find out the characteristics of the connected graph G and the split tournament $D = ST(I \cup K, A)$ when G is the niche graph of D.

1. Introduction

In this paper, we consider both undirected and directed graphs.

First some concepts in undirected graphs. All graphs considered in this paper are finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$ and $E(G)$ (or V and E in short) will denote its vertex-set and its edge-set, respectively. $|V(G)|$ and $|E(G)|$ (or $|V|$ and $|E|$ in short) are called the *order* and *size* of the graph G, respectively. The set of all neighbors of a subset $S \subseteq V(G)$ is denoted by $N_G(S)$ (or $N(S)$ in short). Further, for $W \subseteq V(G)$ the set $W \cap N_G(S)$ is denoted by $N_W(S)$. If $S = \{v\}$, then $N(S)$ and $N_W(S)$ are denoted shortly by $N(v)$ and $N_W(v)$, respectively. For a vertex $v \in V(G)$, the degree of v (resp., the degree of v with respect to W), denoted by $\deg(v)$ or $\mathrm{d}(v)$ (resp., $\deg_W(v)$ or $d_W(v)$), is $|N_G(v)|$ (resp., $|N_W(v)|$). Let G and H be two graphs. If $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then we say that H is a subgraph of G, and write $H \subseteq G$. The subgraph of G induced by $W \subseteq V(G)$ is denoted by $G[W]$. If all the vertices of G are pairwise connected by an edge then G is *complete*. A complete graph on n vertices is denoted by K_n, and K_3 is called a *triangle*. A set of vertices is *stable* if no two of its elements are adjacent. The stable set of order n is denoted by I_n. A *path* is a non-empty graph $P = (V, E)$ of

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form $V = \{x_0, x_1, \dots, x_k\}$, $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$, where the x_i are all distinct. The number of edges of a path is its *length*, and the path of length k is denoted by P_{k-1}. If $P = x_0 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C = P + x_{k-1}x_0$ is called a *cycle*. The *length* of a cycle is its number of edges (or vertices), the cycle of length k is denoted by C_k. Recall that if any two vertices of a graph can be joined by a path, then the graph is said to be *connected*, otherwise it is *disconnected*. A maximal connected subgraph of a graph G is a *component* of G. An *acyclic* graph, one not containing any cycles, is called a *forest*. A connected forest is called a *tree*. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs such that $V_1 \cap V_2 = \emptyset$. We call G_1 and G_2 *isomorphic*, and write $G_1 \cong G_2$, if there exists a bijection $f : V_1 \to V_2$ with $uv \in E_1$ if and only if $f(u)f(v) \in E_2$ for all $u, v \in V_1$. Unless otherwise indicated, our graph-theoretic terminology will follow [2].

Next we recall some concepts in directed graphs. Let D be a digraph. Then the vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ (or by V and A for short), respectively. A vertex $v \in V$ is called an *out-neighbor* of a vertex $u \in V$ if $(u, v) \in A$. We denote the set of all out-neighbors of u by $N_D^+(u)$. The *out-degree* of $u \in V$, denoted by $d_D^+(u)$, is $|N_D^+(u)|$. Similarly, a vertex $w \in V$ is called an *in-neighbor* of a vertex $u \in V$ if $(w, u) \in A$. We denote the set of all in-neighbors of u by $N_D^-(u)$. The *in-degree* of $u \in V$, denoted by $d_D^-(u)$, is $|N_D^-(u)|$. If $W \subseteq V$, then the subdigraph of D induced by W is denoted by $D[W]$. In addition, other concepts in directed graphs are fully presented in [1].

Cohen [5] introduced the notion of *competition graph* while studying predator prey concepts in ecological food webs. The competition graph of a digraph D is the graph having the vertex set $V(D)$ and an edge uv if and only if $(u, w) \in A(D)$ and $(v, w) \in A(D)$ for some $w \in V(D)$. Cohen's empirical observation that real-world competition graphs are usually interval graphs has led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. In the same vein, various variants of competition graphs have been introduced and studied, one of which is the notion of niche graph introduced by Cable et al. [4].

The *niche graph* of a digraph D, denoted by $\mathcal{N}(D)$, has $V(D)$ as the vertex set and an edge uv if and only if $(u, w) \in A(D)$ and $(v, w) \in A(D)$, or $(w, u) \in A(D)$ and $(w, v) \in A(D)$ for some $w \in V(D)$. Bowser, Cable, and Lundgren [3] studied the niche graphs of tournaments, Eoh, Choi, Kim, and Oh [6] studied the niche graphs of bipartite tournaments, Eoh, Choi, and Kim [7] studied the niche graphs of multipartite tournaments.

An *oriented graph* is a digraph with no cycle of length 2. A *split tournament* is an oriented graph $D = (V, A)$ with a partition $V = I \cup K$ such that $D[K]$ is a tournament, there are no arcs in the subdigraph $D[I]$ and for every two vertices $u \in I$, $v \in K$ exactly one of the arcs uv and vu is in A. We will denote such a digraph by $D = ST(I \cup K, A)$. This class of graphs can be considered an extension of the tournament graph. There have been many studies on the class of split tournaments, the most recent typical one is the study of the existence of disjoint cycles with different lengths in this class of graphs [8].

In this paper, we study the niche graphs of split tournaments $D = ST(I \cup K, A)$ for $|K| \geq 2$ which are connected. The niche graphs of split tournaments come in many different forms, which makes it hard to give a general characterization, if they are connected. Yet, we will determine the structure of a split tournament when its niche graph is the complete graph or the connected triangle-free graph. First, we determine the structure of a split tournament $D = ST(I \cup K, A)$ when its niche graph is the complete graph K_n for positive integers $n > |K| \geq 2$. The main result obtained in this paper is the following theorem.

Theorem 1. *Let $G = K_n$ be a complete graph. Then, there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is the niche of D if and only if $k \geq 3$.*

Finally, we determine the structure of a split tournament $D = ST(I \cup K, A)$ when its niche graph is a connected triangle-free graph. The main result about such graphs is the following theorem.

Theorem 2. *Let G be a connected triangle-free graph with $n = |V(G)| \geq 3$. Then there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is the niche of D if and only if $k \in \{2, 3, 4\}$ and*

(i) if $k = 2$, then G is isomorphic to a graph belonging to the set $\{P_3, P_4, C_5\}$;

(ii) if $k = 3$, then G is isomorphic to a graph belonging to the set $\{P_4, C_5\}$;

(iii) if $k = 4$, then G is isomorphic to the graph C_5.

2. Preliminary

For a digraph D, a digraph is said to be the *converse* of D and denoted by $\overleftarrow{D}$ if its vertex set is $V(D)$ and its arc set is $\{(u, v)|(v, u) \in A(D)\}$.

By the definition of niche graphs, the following observations are immediately true.

Observation 1. *For a digraph D, the niche graph of D and the niche graph of $\overleftarrow{D}$ are the same.*

Observation 2. *Let D be a digraph and D' be a subdigraph of D. Then the niche graph of D' is a subgraph of the niche graph of D.*

Observation 3. *For a digraph D, if the niche graph of D is K_m-free, then $d^+_D(u) \leq m - 1$ and $d^-_D(u) \leq m - 1$ for each vertex u in D.*

It is easy to check that the following lemma is true.

Lemma 1. *Let D be an orientation of K_3. Then the niche graph of D is isomorphic to*

$$\begin{cases} I_3, & \text{if } D \text{ is a directed cycle,} \\ P_3, & \text{otherwise.} \end{cases}$$

Bowser, Cable, and Lundgren [3] have shown that the complement of the niche graph of a tournament is one of the following: a cycle of odd order, a path of even order, a forest of odd order consisting of two paths, a forest of even order consisting of three paths, or a forest of four or more paths. By this result, we have the following lemma.

Lemma 2. *The niche graph of an orientation of K_4 is connected.*

Lemma 3. *For $|K| \geq 3$, the niche graph of a split tournament $D = ST(I \cup K, A)$ is connected.*

Proof. Let G be the niche graph of the split tournament $D = ST(I \cup K, A)$. Take two vertices x and y in G. It suffices to show that x and y are connected in G.

First, suppose that $x \in K$ and $y \in K$. Since $|K| \geq 3$, we may take $z \in I$ and $w \in K \setminus \{x, y\}$. Let D_1 be the subdigraph of D induced by $\{x, y, z, w\}$. Then D_1 is an orientation of K_4. Thus, by Lemma 2, the niche graph of D_1 is connected. By Observation 2, the niche graph of D_1 is a subgraph of G and so x and y are connected in G.

Now, suppose that among two vertices x and y there is one vertex belonging to K and one vertex belonging to I. Without loss of generality, we may assume that $x \in K$ and $y \in I$. Again, since $|K| \geq 3$, we may take $z, w \in K \setminus \{x\}$. Let D_2 be the subdigraph of D induced by $\{x, y, z, w\}$. Then D_1 is an orientation of K_4. Thus, by Lemma 2, the niche graph of D_1 is connected. By Observation 2, the niche graph of D_1 is a subgraph of G and so x and y are connected in G.

Finally, suppose that x and y belong to I. Take a vertex $z \in K$. Then x and z are connected in G, y and z are connected in G. Therefore x and y are connected in G. □

A *stable* set of a graph is a set of vertices no two of which are adjacent. A stable set in a graph is *maximum* if the graph contains no larger stable set. The cardinality of a maximum stable set in a graph G is called the *stability number* of G, denoted by $\alpha(G)$.

Lemma 4. *For $|K| \geq 2$, the niche graph of a split tournament $D = ST(I \cup K, A)$ has stability number at most 3.*

Proof. Let G be the niche graph of a split tournament $D = ST(I \cup K, A)$. Suppose, to the contrary, $\alpha(G) \geq 4$. Then we may take a stable set of size 4 in G. We denote it by $\{x_1, x_2, x_3, x_4\}$.

First, suppose that $|I \cap \{x_1, x_2, x_3, x_4\} \geq 3|$. Since $|K| \geq 2$, we may take a vertex $x_5 \in K, x_5 \notin \{x_1, x_2, x_3, x_4\}$. Since D is a split tournament, $\{x_1, x_2, x_3, x_4\} \subseteq N^+_D(x_5) \cup N^-_D(x_5)$. Therefore $|N^+_D(x_5) \cap \{x_1, x_2, x_3, x_4\}| \geq 2$ or $|N^-_D(x_5) \cap \{x_1, x_2, x_3, x_4\}| \geq 2$. Yet, each of $N^+_D(x_5) \cap \{x_1, x_2, x_3, x_4\}$ and $N^-_D(x_5) \cap \{x_1, x_2, x_3, x_4\}$ forms a clique in G, which is a contradiction to the assumption that $\{x_1, x_2, x_3, x_4\}$ is a stable set of G.

Now, suppose that $|I \cap \{x_1, x_2, x_3, x_4\} \leq 2|$. Hence there are two elements in $\{x_1, x_2, x_3, x_4\}$ belonging to K. Without loss of generality, we may assume that $x_4 \in K$. Then $\{x_1, x_2, x_3\} \subseteq N^+_D(x_4) \cup N^-_D(x_4)$ and so $|N^+_D(x_4) \cap \{x_1, x_2, x_3\}| \geq 2$ or $|N^-_D(x_4) \cap \{x_1, x_2, x_3\}| \geq 2$. Since each of $N^+_D(x_4) \cap \{x_1, x_2, x_3\}$ and $N^-_D(x_4) \cap \{x_1, x_2, x_3\}$ forms a clique in G, $\{x_1, x_2, x_3\}$ cannot be a stable set of G, which is a contradiction. This completes the proof. □

From the above lemma, the following corollary immediately follows.

Corollary 1. *For $|K| \geq 2$, the niche graph of a split tournament $D = ST(I \cup K, A)$ has at most three components.*

Lemma 3 tells us that, for a disconnected graph G and a split tournament $D = ST(I \cup K, A)$ with $|K| \geq 2$, if G is a niche graph of D, then $|K| = 2$. In addition, the niche graph of a split tournament $D = ST(I \cup K, A)$ has at most three components for $|K| \geq 2$ by Corollary 1.

3. Proof of Theorem 1

Now we prove Theorem 1.

Suppose that there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is the niche of D. Let $I = \{u_1, u_2, \ldots, u_m\}$, $K = \{v_1, v_2, \ldots, v_k\}$. Then $n = m + k$. By Lemma 1, it is not difficult to verify that $n \geq 4$. Suppose, to the contrary, that $k = 2$. Since $\mathcal{N}(D) \cong K_n$, u_1 and u_2 are adjacent in $\mathcal{N}(D)$, and so have a common out-neighbor or a common in-neighbor in D. By Observation 1, we may assume that they have a common out-neighbor and, by symmetry, we may assume that v_2 is a common out-neighbor of u_1 and u_2. Then, since v_1 and u_1 are adjacent in $\mathcal{N}(D)$, $(v_1, v_2) \in A$. Thus $N^-_D(v_2) \supseteq \{v_1, u_1, u_2\}$. On the other hand, since v_2 and u_1 (respectively, u_2) are adjacent in $\mathcal{N}(D)$, they have a common out-neighbor or a common in-neighbor in K. Yet, v_2 has no out-neighbor in K, so v_2 and u_1 (respectively, u_2) have a common in-neighbor that must be v_1. If $m = 2$, then $A = \{(v_1, u_1), (v_1, u_2), (v_1, v_2), (u_1, v_2), (u_2, v_2)\}$. Since v_1 has only out-neighbors and v_2 has only in-neighbors, they are not adjacent in $\mathcal{N}(D)$, which is a contradiction when $m = 2$. If $m \geq 3$, then it is not difficult to verify that $(v_1, u_i), (u_i, v_2) \in A$ for every $i = 3, \ldots m$. So, v_1 and

v_2 are not adjacent in $\mathcal{N}(D)$, which is a contradiction to the assumption that $\mathcal{N}(D) \cong K_n$. Thus, $k \geq 3$.

Conversely, suppose that $k \geq 3$. Then let $D = ST(I \cup K, A)$ be any split tournament with $I = \{u_1, u_2, \ldots, u_m\}$, $K = \{v_1, v_2, \ldots, v_k\}$ and arc set includes the following arc set (the remaining arcs have an arbitrary orientation):

$$A = A_1 \cup A_2 \cup A_3$$

with

$A_1 = \{(v_1, v_i) \mid 2 \leq i \leq k\} \cup \{(v_1, u_i) \mid 1 \leq i \leq m\}$,
$A_2 = \{(v_2, v_3), (v_3, v_4), \ldots, (v_{k-1}, v_k)\}$,
$A_3 = \{(v_k, u_1), (u_1, v_2), \ldots, (v_k, u_m), (u_m, v_2)\}$.

Since v_1 is a common in-neighbor of the remaining vertices, so the set $\{v_2, \ldots, v_k, u_1, u_2, \ldots, u_m\}$ forms a clique in $\mathcal{N}(D)$. Moreover, since v_i (respectively, u_j) has at least one out-neighbor in $\{v_2, \ldots, v_k, u_1, u_2, \ldots, u_m\}$ for each $2 \leq i \leq k$ (respectively, $1 \leq j \leq m$), v_1 and v_i (respectively, u_j) have a common out-neighbor in D, and so they are adjacent in $\mathcal{N}(D)$. Therefore $\mathcal{N}(D)$ is a complete graph K_n.

The proof of Theorem 1 is complete.

4. Proof of Theorem 2

First we prove some lemmas to serve the proof of Theorem 2.

Let G be a graph. Two vertices u and v of G are said to be *true twins* if they have the same closed neighborhood. We may introduce an analogous notion for a digraph. Let D be a digraph. Two vertices u and v of D are said to be true twins if they have the same open out-neighborhood and open in-neighborhood.

Lemma 5. *Let $D = ST(I \cup K, A)$ be a split tournament for $k = |K| \geq 1$ such that $\mathcal{N}(D)$ is connected. Then $\mathcal{N}(D)$ contains no induced path of length 5, that is, $\mathcal{N}(D)$ is P_6-free.*

Proof. Suppose, to the contrary, that $\mathcal{N}(D)$ contains an induced path P of length 5. Let $P = x_1x_2x_3x_4x_5x_6$. Take $v \in K$. Then $N^+_D(v) \cup N^-_D(v)$ contains at least five vertices in $V(P)$. Therefore $N^+_D(v)$ or $N^-_D(v)$ contains at least three vertices in $V(P)$. Since each of $N^+_D(v)$ and $N^-_D(v)$ forms a clique in $\mathcal{N}(D)$, the subgraph of $\mathcal{N}(D)$ induced by $V(P)$ contains a triangle, which contradicts the choice of P as an induced path of $\mathcal{N}(D)$. □

Corollary 2. *Let $D = ST(I \cup K, A)$ be a split tournament for $|K| \geq 2$. Then each component of $\mathcal{N}(D)$ has diameter at most 4.*

Using Theorem 1, we determine the structure of a split tournament $D = ST(I \cup K, A)$ when its niche graph is a path P_n or a cycle C_n for positive integers $n > |K| \geq 2$.

Lemma 6. *Let $G = P_n$ be a path with n vertices. Then there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is the niche of D if and only if $(n, k) \in \{(3, 2), (4, 2), (4, 3)\}$.*

Proof. Suppose that there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is the niche of D. Let $I = \{u_1, u_2, \ldots, u_m\}$, $K = \{v_1, v_2, \ldots, v_k\}$. By Lemma 5, $n = m + k \leq 5$. Thus we only need to show that (n, k) is neither $(5, 2)$, $(5, 3)$ nor $(5, 4)$. We denote P_5 by $x_1x_2x_3x_4x_5$. Since $\mathcal{N}(D) = P_5$, $\mathcal{N}(D)$ is triangle-free and so, by Observation 3, every vertex of D has in-degree at most two and out-degree at most two in D. First, assume that $x_2 \in K$. Then $N_D^+(x_2) \cup N_D^-(x_2) = V(D) \setminus \{x_2\}$, so $d_D^+(x_2) = 2$ and $d_D^-(x_2) = 2$. By Observation 1, we may assume that x_1 is an out-neighbor of x_2 in D. Since $N_D^+(x_2)$ forms an edge in $\mathcal{N}(D)$, x_1 is adjacent to a vertex in P_5 other than x_2 and we reach a contradiction. Therefore x_2 must belong to I. By symmetry, $x_4 \in I$. So $k \leq 3$. Now suppose that $k = 3$. Then $x_1, x_3, x_5 \in K$. Therefore $d_D^+(x_2) + d_D^-(x_2) = 3$ and so $d_D^+(x_2) = 2$ or $d_D^-(x_2) = 2$. By Observation 1, we may assume that $d_D^+(x_2) = 2$. Then the out-neighbors of x_2 in D are adjacent in $\mathcal{N}(D)$. However, the possible out-neighbors of x_2 in D are x_1, x_3, x_5, no two of which are consecutive on P_5. Hence we have reached a contradiction and so $k = 2$. Then it is not difficult to verify that $d_D^+(u) = d_D^-(u) = 1$ for all $u \in I$. By symmetry, we only need to consider two cases: $x_1, x_3 \in K$ or $x_1, x_5 \in K$. First let us consider $x_1, x_3 \in K$. Then $I = \{x_2, x_4, x_5\}$. By Observation 1, we may asumme that $(x_1, x_3) \in A$. Since x_4 and x_5 are adjacent on P_5, $(x_4, x_1), (x_5, x_1) \in A$ or $(x_1, x_4), (x_1, x_5) \in A$. If $(x_4, x_1), (x_5, x_1) \in A$ (respectively, $(x_1, x_4), (x_1, x_5) \in A$), then x_3 and x_4 are not adjacent on P_5 (respectively, x_3 and x_5 are adjacent on P_5), a contradiction. Now let us consider the case $x_1, x_5 \in K$. Then $I = \{x_2, x_3, x_4\}$. By Observation 1, we may asumme that $(x_1, x_5) \in A$. Since x_3 and x_4 are adjacent on P_5, $(x_3, x_1), (x_4, x_1) \in A$ or $(x_1, x_3), (x_1, x_4) \in A$. If $(x_3, x_1), (x_4, x_1) \in A$ (respectively, $(x_1, x_3), (x_1, x_4) \in A$), then x_4 and x_5 are not adjacent on P_5 (respectively, x_3 and x_5 are adjacent on P_5), a contradiction.

Conversely, suppose that $(n, k) \in \{(3, 2), (4, 2), (4, 3)\}$. Let D_1, D_2, and D_3 be the split tournaments $D = ST(I \cup K, A)$ in Table 1 corresponding to the pairs $(n, k) = (3, 2), (4, 2), (4, 3)$, respectively. It is easy to check that $\mathcal{N}(D_1) \cong P_3$, $\mathcal{N}(D_2) \cong P_4$, and $\mathcal{N}(D_3) \cong P_4$.

This completes the proof. □

TABLE 1. The graphs D_1, D_2 and D_3.

The graph $D = ST(I \cup K, A)$	The vertex-set $I \cup K$	The arc-set A
D_1 $((n,k) = (3,2))$	$I = \{u_1\}$, $K = \{v_1, v_2\}$.	$A = \{u_1v_1, u_1v_2, v_1v_2\}$,
D_2 $((n,k) = (4,2))$	$I = \{u_1, u_2\}$, $K = \{v_1, v_2\}$.	$A = \{u_1v_1, u_1v_2, v_1u_2, v_2u_2, v_1v_2\}$
D_3 $((n,k) = (4,3))$	$I = \{u_1\}$, $K = \{v_1, v_2, v_3\}$.	$A = \{u_1v_1, u_1v_2, v_3u_1, v_1v_2, v_1v_3,$ $v_2v_3\}$

Lemma 7. *Suppose that $\mathcal{N}(D)$ is a connected triangle-free niche graph of a split tournament $D = ST(I \cup K, A)$ with $|I| = m$ and $|K| = k \geq 2$. Then $k \in \{2, 3, 4\}$ and*
(i) if $k = 2$, then $1 \leq m \leq 3$;
(ii) if $k = 3$, then $1 \leq m \leq 2$;
(iii) if $k = 4$, then $m = 1$.

Proof. If $k \geq 5$, then $5 \leq d_D^+(v) + d_D^-(v)$ for each vertex v in D, which contradicts Observation 3. Thus $k \leq 4$. Let $I = \{u_1, u_2, \ldots, u_m\}$, $K = \{v_1, v_2, \ldots, v_k\}$.
(*i*) If $k = 2$, then $d_D^+(v_1) + d_D^-(v_1) = m + 1$. By Observation 3, $m + 1 \leq 4$. Thus $1 \leq m \leq 3$.
(*ii*) If $k = 3$, then $d_D^+(v_1) + d_D^-(v_1) = m + 2$. By Observation 3, $m + 2 \leq 4$. Thus $1 \leq m \leq 2$.
(*iii*) If $k = 4$, then $d_D^+(v_1) + d_D^-(v_1) = m + 3$. By Observation 3, $m + 3 \leq 4$. Thus $m = 1$. □

Lemma 8. *Let $G = C_n$ be a cycle with n vertices. Then there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that G is niche of D if and only if $(n, k) \in \{(5, 2), (5, 3), (5, 4)\}$.*

Proof. Suppose that there exists a split tournament $D = ST(I \cup K, A)$ with $n > k = |K| \geq 2$ such that $G = C_n$ is the niche of D. Let $I = \{u_1, u_2, \ldots, u_m\}$, $K = \{v_1, v_2, \ldots, v_k\}$. By Lemma 5, $n = m + k \leq 5$. Thus we only need to show that $(n, k) \notin \{(3, 2), (4, 2), (4, 3), (6, 2), (6, 3), (6, 4), (6, 5)\}$. By Lemma 7, $(n, k) \notin \{(6, 2), (6, 3), (6, 4), (6, 5)\}$ and by Lemma 1, $(n, k) \neq (3, 2)$. Suppose that $(n, k) \in \{(4, 2), (4, 3)\}$. Since $\mathcal{N}(D) \cong C_4$, $\mathcal{N}(D)$ is triangle-free and so, by Observation 3, every vertex of D has in-degree at most two and out-degree at most two in D. First let us consider the case

$(n,k) = (4,2)$. We will show that $\mathcal{N}(D)$ can have at most 3 edges. It will follow that it cannot be C_4. Since the in-degree and the out-degree in D of each vertex is at most 2 and their sum is at most 3, each vertex gives rice to at most one edge of $\mathcal{N}(D)$ between two of its neighbors. But u_1 and u_2 can only give rise to the same edge (v_1, v_2). The assertion follows.

Now let us consider the case $(n,k) = (4,3)$. In this case the split tournament is a tournament. The assertion follows by the result of [3] stated before Lemma 2.

Conversely, suppose that $(n,k) \in \{(5,2),(5,3),(5,4)\}$. Let D_4, D_5, and D_6 be the split tournaments $D = ST(I \cup K, A)$ in Table 2 corresponding to the pairs $(n,k) = (5,2),(5,3),(5,4)$, respectively. It is easy to check that $\mathcal{N}(D_i) \cong C_5$ for each $i = 4,5,6$.

This completes the proof. □

TABLE 2. The graphs D_4, D_5 and D_6.

The graph $D = ST(I \cup K, A)$	The vertex-set $I \cup K$	The arc-set A
D_4 $((n,k) = (5,2))$	$I = \{u_1, u_2, u_3\}$, $K = \{v_1, v_2\}$.	$A = \{u_1v_1, u_1v_2, v_1u_2, v_2u_2, u_3v_1,$ $v_2u_3, v_1v_2\}$
D_5 $((n,k) = (5,3))$	$I = \{u_1, u_2\}$, $K = \{v_1, v_2\, v_3\}$.	$A = \{u_1v_1, v_2u_1, u_1v_3, u_2v_1, v_2u_2,$ $v_3u_2, v_1v_2, v_1v_3, v_3v_2\}$
D_6 $((n,k) = (5,4))$	$I = \{u_1\}$, $K = \{v_1, v_2, v_3, v_4\}$.	$A = \{u_1v_1, v_2u_1, u_1v_3, v_4u_1, v_1v_3,$ $v_1v_4, v_2v_1, v_3v_2, v_3v_4, v_4v_2\}$

Let G_i with $i \in \{1,2,3,4\}$ be the graphs in Table 3.

Lemma 9. *Let G be a connected triangle-free graph with $3 \leq |V(G)| \leq 5$ and stability number at most 3. Then the following claims are true.*

1. *Each vertex in G has degree at most 3.*
2. *G is isomorphic to a path P_i for some $i \in \{3,4,5\}$ or a cycle C_j for some $j \in \{4,5\}$ or the graph G_k for some $k \in \{1,2,3,4\}$ given in Table 3.*

Proof. 1. We prove this statement in the contrapositive. Suppose that there exists a vertex x in G of degree at least 4. Then there exist four distinct vertices x_1, x_2, x_3, x_4 which are adjacent to x in G. Since G is triangle-free, x_i and x_j are not adjacent if $i \neq j$. Therefore $\{x_1, x_2, x_3, x_4\}$ is a stable set, which contradicts the hypothesis that G has stability number at most 3.

2. It is not difficult to see that the graph is either a tree other than $K_{1,4}$, a cycle C_4 or C_5, or $K_{2,3}$, or $K_{2,3}$ minus an edge. These possibilities other than paths or cycles are listed in Table 3 □

TABLE 3. The graphs G_1, G_2, G_3 and G_4.

The graph $G = (V(G), E(G))$	The vertex-set $V(G)$	The edge-set $E(G)$
G_1	$\{v_1, v_2, v_3, v_4\}$	$\{v_1v_2, v_2v_3, v_2v_4\}$
G_2	$\{v_1, v_2, \ldots, v_5\}$	$\{v_1v_2, v_2v_3, v_2v_4, v_4v_5\}$
G_3	$\{v_1, v_2, \ldots, v_5\}$	$\{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_2\}$
G_4	$\{v_1, v_2, \ldots, v_5\}$	$\{v_1v_2, v_1v_3, v_2v_4, v_2v_5, v_3v_4, v_3v_5\}$

Now we will continue to prove Theorem 2.

Suppose that there exists a split tournament $D = ST(I \cup K, A)$ with $k = |K| \geq 2$ such that G is the niche of D. By Lemma 7, $k \leq 4$ and $n \leq 5$. Thus, $k \in \{2, 3, 4\}$ and $n \leq 5$. First let us consider the case when G is a path or a cycle. Then, by Lemmas 6 and 8, G is isomorphic to P_3, P_4, or C_5 when $k = 2$; G is isomorphic to P_4 or C_5 when $k = 3$; G is isomorphic to C_5 when $k = 4$. Now let us consider the case G is neither a path nor a cycle. By Lemma 4 and Corollary 2, G has stability number at most 3. Therefore, by Lemma 9, G is isomorphic to the graph G_j given in Table 3 for some $j \in \{1, 2, 3, 4\}$. Since G is neither a path nor a cycle, there exists a vertex x_1 of degree at least 3 in G. If x_1 has degree at least 4, then $G \ncong G_i$ for each $1 \leq i \leq 4$. Therefore x_1 has degree 3. It follows that $n \geq 4$. Since each of x_1 and its neighbors has in-degree at most 2 and out-degree at most 2 by Observation 3, x_1 is adjacent to at most two vertices if $d_D^+(x_1) = \emptyset$ or $d_D^-(x_1) = \emptyset$, which is a contradiction. Therefore $d_D^+(x_1) \geq 1$ and $d_D^-(x_1) \geq 1$. If $d_D^+(x_1) = d_D^-(x_1) = 1$, then x_1 has degree at most 2 for the same reason as the previous one, which is a contradiction. Therefore $d_D^+(x_1) = 2$ or $d_D^-(x_1) = 2$ and $d_D^+(x_1) + d_D^-(x_1) \geq 3$. By Observation 1, we may assume that $d_D^+(x_1) = 2$. Let $N_D^+(x_1) = \{x_2, x_3\}$ and $x_4 \in N_D^-(x_1)$. If $n = 4$ then G is isomorphic to the graph G_1. But x_2 and x_3 are adjacent in G because $N_D^+(x_1) = \{x_2, x_3\}$, a contradiction. Thus, $n = 5$ and so $G \cong G_2$, $G \cong G_3$ or $G \cong G_4$. Let $x_5 \in V(G) \setminus \{x_1, x_2, x_3, x_4\}$. If $N_D^-(x_1) = \{x_4, x_5\}$, then $x_4x_5 \in E(G)$. It follows that $G - x_1$ has two separate edges x_2x_3 and x_4x_5, which cannot happen in any of G_2, G_3 and G_4. Therefore we may asume that $N_D^-(x_1) = \{x_4\}$. By a similar reasoning, either the in-neighborhood or the out-neighborhood of each vertex must have only one vertex. It follows that $x_i \in I$ for all i, a contradiction.

Conversely, suppose that $k \in \{2, 3, 4\}$ and the statements $(i), (ii), (iii)$ are true. From Lemmas 6 and 8 we can easily deduce what needs to be proven.

The proof of Theorem 2 is complete.

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