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Closed-form expressions for polylogarithmic integrals and related harmonic sums

Ali Olaikhan

Abstract. This paper derives closed-form expressions for a class of five-parameter polylogarithmic integrals, expressed in terms of the polylogarithm and Lerch transcendent, and reducible (under admissible parameter choices) to Riemann and Hurwitz zeta values. The paper further obtains closed forms for related linear Euler-type sums and BBP-type series. All results are established using purely real-analytic methods.

1. Introduction and preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{Z}_{\leqslant 0}$ the set of non-positive integers, $\mathbb{Z}_{\geqslant 1}$ the set of positive integers, and $\mathbb{Z}_{\geqslant 2}$ the set of integers greater than or equal to 2.

For $p \in \mathbb{Z}_{\geqslant 1}$ and $b = \pm 1$, we define

$$\mathbf{H}_{k,b}^{(p)} = \sum_{m=1}^{k} \frac{b^{m-1}}{m^p}, \quad \mathbf{H}_{k,b}^{(1)} := \mathbf{H}_{k,b}, \tag{1}$$

where $b = 1$ corresponds to the kth generalized harmonic numbers of order p, denoted by $H_k^{(p)}$, and $b = -1$ corresponds to the kth generalized skew-harmonic numbers of order p, denoted by $\overline{H}_k^{(p)}$. We also define

$$\mathbf{O}_{k,b}^{(p)} = \sum_{m=1}^{k} \frac{b^{m-1}}{(2m-1)^p}, \quad \mathbf{O}_{k,b}^{(1)} := \mathbf{O}_{k,b}, \tag{2}$$

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where $b = 1$ yields the kth generalized odd harmonic numbers of order p, denoted by $O_k^{(p)}$, and $b = -1$ yields the kth generalized odd skew-harmonic numbers of order p, denoted by $\overline{O}_k^{(p)}$.

A recent paper [14, Theorems 5.1 and 5.13] presented closed-form expressions for the *four-parameter* linear Euler-type sums

$$\sum_{k=1}^{\infty} \frac{a^k \, \mathbf{H}_{k,b}^{(p)}}{k^q} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{a^k \, \mathbf{O}_{k,b}^{(p)}}{k^q},$$

where $p, q \in \mathbb{Z}_{\geqslant 1}$, $a = \pm 1$, and $b = \pm 1$. Note that $q \neq 1$ when $a = b = 1$ to ensure the two sums converge. The first sum is restricted to the case where $p + q$ is odd, whereas the second sum requires $b = (-1)^{p+q-1}$.

In the present paper, we extend these results by giving closed-form expressions for the corresponding *five-parameter* sums (see Corollaries 1 and 4)

$$\sum_{k=1}^{\infty} \frac{a^k \, \mathbf{H}_{nk,b}^{(p)}}{k^q} \qquad \text{and} \qquad \sum_{k=1}^{\infty} \frac{a^k \, \mathbf{O}_{nk,b}^{(p)}}{k^q},$$

together with several BBP-type series (see Corollaries 2, 5, 3, and 6)

$$\sum_{k=1}^{\infty} (a^n)^k \, \mathbf{H}_{k,b}^{(p)} \sum_{j=1}^{n} \frac{a^{j-1}}{(nk+j)^q}, \quad \sum_{k=1}^{\infty} (a^n)^k \, \mathbf{H}_{k,b}^{(p)} \sum_{j=1}^{n} \frac{a^{j-1}}{(2nk+2j-1)^q},$$

$$\sum_{k=1}^{\infty} O_k^{(p)} \sum_{j=1}^{n} \frac{1}{(2nk-n+2j-1)^q}, \quad \text{and} \quad \sum_{k=1}^{\infty} O_k^{(p)} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{(4nk-2n+2j-1)^q},$$

where $n \in \mathbb{Z}_{\geqslant 1}$. A BBP-type series takes the general form

$$\sum_{k=0}^{\infty} \frac{1}{t^k} \sum_{j=1}^{n} \frac{h_j}{(nk+j)^r},$$

where $t, r, n \in \mathbb{Z}_{\geqslant 1}$ and $(h_1, \ldots, h_n) \in \mathbb{Z}^n$. This type of series is named after Bailey, Borwein, and Plouffe, who first discovered the BBP formula for the constant π [2],

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

Their discovery encouraged many researchers to explore new BBP-type formulas for other mathematical constants, see [1, 3, 4, 6, 17, 20, 22, 23].

The Euler-type sums and BBP-type series derived in this paper follow from establishing connections with the polylogarithmic integrals (see Theorems 3 and 4)

$$\int_0^1 \frac{\log^{q-1}(x) \, \mathrm{Li}_p(bx^n)}{1-ax} \, \mathrm{d}x \quad \text{and} \quad \int_0^1 \frac{\log^{q-1}(x) \, \mathrm{Li}_p(bx^{2n})}{1-ax^2} \, \mathrm{d}x,$$

which are respectively related to the integrals (see Theorems 1 and 2)

$$\int_0^\infty \frac{\log^{q-1}(x)\ \mathrm{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x \quad \text{and} \quad \int_0^\infty \frac{\log^{q-1}(x)\ \mathrm{Li}_p(bx^{2n})}{1-ax^2}\,\mathrm{d}x,$$

where Li_p denotes the polylogarithm function defined in (3).

The closed-form expressions for the integrals and sums discussed above are given in terms of the Lerch transcendent and some related special functions. The Lerch transcendent is defined by [18, p. 194]

$$\Phi(x,s,t) = \sum_{n=0}^{\infty} \frac{x^n}{(n+t)^s},$$

$$(t \notin \mathbb{Z}_{\leqslant 0};\ s \in \mathbb{R} \text{ when } |x|<1;\ s>1 \text{ when } |x|=1),$$

where only the case $|x| = 1$ is considered in this work, since x always appears in the forms a, b, $a^n b$, or ab^n, with $a = \pm 1$, $b = \pm 1$, and $n \in \mathbb{Z}_{\geqslant 1}$. This function generalizes several well-known special functions:

the generalized (or Hurwitz) zeta function

$$\zeta(s,t) = \sum_{n=0}^{\infty} \frac{1}{(n+t)^s} = \Phi(1,s,t),$$

the generalized eta function [15, Eq. (25.11.35)]

$$\eta(s,t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+t)^s} = \Phi(-1,s,t) = 2^{-s}\left(\zeta\left(s,\frac{t}{2}\right) - \zeta\left(s,\frac{t+1}{2}\right)\right),$$

the polylogarithm function

$$\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} = x\,\Phi(x,s,1), \tag{3}$$

the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \Phi(1,s,1),$$

the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \Phi(-1,s,1),$$

the Dirichlet lambda function

$$\lambda(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = 2^{-s}\,\Phi\left(1,s,\frac{1}{2}\right),$$

and the Dirichlet beta function

$$\beta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s} = 2^{-s}\,\Phi\left(-1,s,\frac{1}{2}\right).$$

In particular, $\beta(2)$ is Catalan's constant, denoted by G. The Dirichlet eta and Dirichlet lambda functions are related to the Riemann zeta function via

$$\eta(s) = \left(1 - 2^{1-s}\right)\zeta(s) \quad \text{and} \quad \lambda(s) = \left(1 - 2^{-s}\right)\zeta(s).$$

From the above relationships, we conclude that for any valid choice of the five parameters, the Lerch transcendent reduces to Riemann zeta and Hurwitz zeta values, as illustrated by the examples in Section 4.

All of the theorems and corollaries derived here appear to be new. However, several special cases have previously been established (see, for example [5, 6, 7, 8, 9, 11, 13, 16, 19, 21]).

2. Lemmas

Lemma 1. *Let $p \in \mathbb{Z}_{\geqslant 1}$, $c = \pm 1$, and $|x| < 1$. Then*

(*i*)

$$\sum_{k=1}^{\infty} \mathbf{H}_{k,c}^{(p)} x^k = \frac{c \operatorname{Li}_p(cx)}{1-x},$$

(*ii*)

$$\sum_{k=1}^{\infty} \mathbf{O}_{k,c}^{(p)} (cx^2)^k = \frac{cx\left(\operatorname{Li}_p(x) - \operatorname{Li}_p(-x)\right)}{2(1-cx^2)},$$

where $\mathbf{H}_{k,c}^{(p)}$ and $\mathbf{O}_{k,c}^{(p)}$ are respectively defined in (1) *and* (2), *and* Li_p *denotes the polylogarithm function.*

Proof. Rearranging the double sum terms according to

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k} a_k b_n = \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} a_k b_n,$$

we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbf{H}_{k,c}^{(p)} x^k &= \sum_{k=1}^{\infty}\sum_{n=1}^{k} \frac{c^{n-1}}{n^p} x^k = \sum_{n=1}^{\infty} \frac{c^{n-1}}{n^p} \sum_{k=n}^{\infty} x^k \\ &= \sum_{n=1}^{\infty} \frac{c^{n-1}}{n^p} \cdot \frac{x^n}{1-x} = \frac{c}{1-x} \sum_{n=1}^{\infty} \frac{(cx)^n}{n^p} = \frac{c \operatorname{Li}_p(cx)}{1-x}.\end{aligned}$$

This completes the proof of part (i). The proof of part (ii) follows similarly:

$$\begin{aligned}\sum_{k=1}^{\infty} \mathbf{O}_{k,c}^{(p)} (cx^2)^k &= \sum_{k=1}^{\infty}\sum_{m=1}^{k} \frac{c^{m-1}}{(2m-1)^p} \left(cx^2\right)^k = \sum_{m=1}^{\infty} \frac{c^{m-1}}{(2m-1)^p} \sum_{k=m}^{\infty} \left(cx^2\right)^k \\ &= \sum_{m=1}^{\infty} \frac{c^{m-1}}{(2m-1)^p} \cdot \frac{\left(cx^2\right)^m}{1-cx^2} = \frac{c}{1-cx^2} \sum_{m=1}^{\infty} \frac{x^{2m}}{(2m-1)^p}\end{aligned}$$

$$= \frac{cx}{2(1-cx^2)} \sum_{m=1}^{\infty} (1-(-1)^m) \frac{x^m}{m^p} = \frac{cx\,(\mathrm{Li}_p(x) - \mathrm{Li}_p(-x))}{2(1-cx^2)}.$$

□

Lemma 2. *Let $c = \pm 1$ and $s \in (0,1)$. Then*

$$\int_0^{\infty} \frac{x^{s-1}}{1-cx}\,\mathrm{d}x = -2\sum_{r=0}^{\infty} \mathrm{Li}_{2r}(c) s^{2r-1},$$

where Li_p is the polylogarithm function. This integral must be viewed as the Cauchy principal value when $c = 1$.

Proof. The proof follows by taking $c = \pm 1$ in [10, Entry 3.222.2],

$$\int_0^{\infty} \frac{x^{s-1}}{x+c}\,\mathrm{d}x = \begin{cases} -(-c)^{s-1}\pi\cot(\pi s) & c < 0, \\ c^{s-1}\pi\csc(\pi s) & c > 0, \end{cases} \tag{4}$$

and then expanding the cotangent and cosecant functions into their series:

$$\pi\cot(\pi s) = -2\sum_{r=0}^{\infty} \mathrm{Li}_{2r}(1) s^{2r-1}, \qquad |s| < 1, \tag{5}$$

$$\pi\csc(\pi s) = -2\sum_{r=0}^{\infty} \mathrm{Li}_{2r}(-1) s^{2r-1}, \qquad |s| < 1, \tag{6}$$

where (5) is given in [18, p. 271, Eq. (18)], and (6) follows from employing (5) in the trigonometric identity $\csc(x) = \cot(x/2) - \cot(x)$. □

Remark 1. In Lemma 2, the $r = 0$ term of the summation $\sum_{r=0}^{\infty}$ produces $\mathrm{Li}_0(c)$. In the case $c = 1$, this becomes $\mathrm{Li}_0(1)$, which is interpreted as $-1/2$ via the analytic continuation of the Riemann zeta function,

$$\mathrm{Li}_0(1) = \zeta(0) = -\frac{1}{2}.$$

We note that *Mathematica* also returns $-1/2$ when evaluating $\mathrm{Li}_r(1)$ in the limit $r \to 0$. This interpretation is used throughout this work for every occurrence of $\mathrm{Li}_0(1)$.

Lemma 3. *Let $q \in \mathbb{Z}_{\geqslant 1}$, $s, r > 0$, $c = \pm 1$, and $q \neq 1$ when $c = 1$. Then*

$$\int_0^1 \frac{x^{s-1}\log^{q-1}(x)}{1-cx^r}\,\mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{r^q}\,\Phi\Big(c, q, \frac{s}{r}\Big),$$

where Φ is the Lerch transcendent.

Proof. The proof proceeds by expanding $\frac{1}{1-cx^r}$ into its Maclaurin series, interchanging the order of summation and integration (justified by uniform convergence) and then applying the integral identity [13, Eq. (1.36)]

$$\int_0^1 x^{r-1}\log^{q-1}(x)\,\mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{r^q}, \qquad (r>0,\ q\in\mathbb{Z}_{\geqslant 1}). \tag{7}$$

□

Lemma 4. *Let $q\in\mathbb{Z}_{\geqslant 1}$, $c=\pm 1$, and $s/r\in(0,1)$. Then*

$$\int_0^\infty \frac{x^{s-1}\log^{q-1}(x)}{1-cx^r}\,\mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{r^q}\,\Theta(c,q,s,r),$$

where

$$\Theta(c,q,s,r) = \Phi\Big(c,q,\frac{s}{r}\Big) + c(-1)^q\,\Phi\Big(c,q,\frac{r-s}{r}\Big), \tag{8}$$

and Φ denotes the Lerch transcendent. This integral must be viewed as the Cauchy principal value for the case $(q,c)=(1,1)$.

Proof. Using $\int_0^\infty f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x + \int_1^\infty f(x)\,\mathrm{d}x$, and then making the substitution $x\mapsto 1/x$ in the second integral, we find

$$\int_0^\infty \frac{x^{s-1}\log^{q-1}(x)}{1-cx^r}\,\mathrm{d}x = \int_0^1 \frac{x^{s-1}\log^{q-1}(x)}{1-cx^r}\,\mathrm{d}x + c(-1)^q\int_0^1 \frac{x^{r-s-1}\log^{q-1}(x)}{1-cx^r}\,\mathrm{d}x.$$

The proof follows from Lemma 3. □

Remark 2. By substituting $(q,c,x^r)=(1,1,y)$ into the integral in Lemma 4 and then applying the case $c<0$ from (4), we obtain

$$\Theta(1,1,s,r) = \pi\cot\Big(\frac{\pi s}{r}\Big), \qquad \frac{s}{r}\in(0,1).$$

This validates the case $(q,c)=(1,1)$ in the identity (8), noting that its right-hand side must be considered as the limit as $c\to 1$ when $q=1$.

Lemma 5. *Let $n\in\mathbb{Z}_{\geqslant 1}$, $q\in\mathbb{Z}_{\geqslant 2}$, $a=\pm 1$, $b=\pm 1$, and $y\in(0,1)$. Then*

$$\int_0^\infty \frac{\log^{q-1}(x)}{x}\left(\frac{1}{1-ax}-\frac{1}{1-byx^n}\right)\mathrm{d}x = -(q-1)!\,(1+(-1)^q)\,\mathrm{Li}_q(a) + \frac{2(-1)^q(q-1)!}{n^q}\sum_{k=0}^{\lfloor \frac{q}{2}\rfloor}\frac{\log^{q-2k}(y)}{(q-2k)!}\,\mathrm{Li}_{2k}(b),$$

where Li_p is the polylogarithm function, and $\lfloor\cdot\rfloor$ is the floor function. This integral must be understood as the Cauchy principal value when $b=1$.

Proof. Differentiating under the integral sign yields

$$\int_0^\infty \frac{\log^{q-1}(x)}{x}\left(\frac{1}{1-ax}-\frac{1}{1-byx^n}\right)\mathrm{d}x$$
$$=\lim_{s\to 0}\frac{\partial^{q-1}}{\partial s^{q-1}}\left(\int_0^\infty \frac{x^{s-1}}{1-ax}\,\mathrm{d}x-\int_0^\infty \frac{x^{s-1}}{1-byx^n}\,\mathrm{d}x\right). \quad (9)$$

This differentiation is justified by the dominated convergence theorem for $0<\Re(s)<1$ and $a\neq 1$ when the first integral, and for $0<\Re(s)<n$ when the second integral. Applying Lemma 2 in the first integral, it follows that

$$\int_0^\infty \frac{x^{s-1}}{1-ax}\,\mathrm{d}x=-2\sum_{r=0}^{\infty}\mathrm{Li}_{2r}(a)s^{2r-1}=-\sum_{r=0}^{\infty}\left(1+(-1)^r\right)\mathrm{Li}_r(a)s^{r-1}. \quad (10)$$

Making the change of variable $x=(\frac{t}{y})^{1/n}$ in the second integral and then using Lemma 2, we get

$$\int_0^\infty \frac{x^{s-1}}{1-byx^n}\,\mathrm{d}x=-2y^{-\frac{s}{n}}\sum_{r=0}^{\infty}n^{-2r}\,\mathrm{Li}_{2r}(b)s^{2r-1}.$$

Expanding $y^{-s/n}$ into its Maclaurin series as $\sum_{r=0}^{\infty}\frac{(-s/n)^r\log^r(y)}{r!}$, whose radius of convergence is ∞, and so there is no restriction on $\log(y)$ or s, and then applying the special case of the Cauchy product for two convergent series [13, Eq. (2.67)]

$$\left(\sum_{r=0}^{\infty}f(r)s^r\right)\left(\sum_{r=0}^{\infty}g(2r)s^{2r}\right)=\sum_{r=0}^{\infty}\left(\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}f(r-2k)g(2k)\right)s^r,$$

we find

$$\int_0^\infty \frac{x^{s-1}}{1-byx^n}\,\mathrm{d}x=2\sum_{r=0}^{\infty}\frac{(-1)^{r-1}}{n^r}\left(\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\log^{r-2k}(y)}{(r-2k)!}\,\mathrm{Li}_{2k}(b)\right)s^{r-1}. \quad (11)$$

Plugging (10) and (11) into (9), we have

$$\int_0^\infty \frac{\log^{q-1}(x)}{x}\left(\frac{1}{1-ax}-\frac{1}{1-byx^n}\right)\mathrm{d}x=\lim_{s\to 0}\frac{\partial^{q-1}}{\partial s^{q-1}}$$
$$\times\sum_{r=0}^{\infty}\underbrace{\left(-\left(1+(-1)^r\right)\mathrm{Li}_r(a)+\frac{2(-1)^r}{n^r}\sum_{k=0}^{\lfloor\frac{r}{2}\rfloor}\frac{\log^{r-2k}(y)}{(r-2k)!}\,\mathrm{Li}_{2k}(b)\right)}_{f(r)}s^{r-1}$$
$$=\lim_{s\to 0}\frac{\partial^{q-1}}{\partial s^{q-1}}\sum_{r=0}^{\infty}f(r)s^{r-1}=(q-1)!f(q).$$

The last step follows from differentiating term by term $(q-1)$ times with respect to s and then evaluating the limit when s approaches zero. $\square$

Lemma 6. *Let $p \in \mathbb{Z}_{\geqslant 2}$, $c = \pm 1$ and $x \in (0,1)$. Then*

$$\operatorname{Li}_p(cx) + (-1)^p \operatorname{Li}_p\left(\frac{c}{x}\right) = 2\sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \frac{\log^{p-2j}(x)}{(p-2j)!} \operatorname{Li}_{2j}(c),$$

where Li_p denotes the polylogarithm function, and $\lfloor \cdot \rfloor$ represents the floor function. Only the real part of $\operatorname{Li}_p(c/x)$ is considered when $c = 1$.

Proof. The case $c = -1$ follows from Lewin [12, Eq. (7.20)] by adding and subtracting the $j = 0$ term. For the case $c = 1$, the result follows by dividing both sides of the dilogarithm identity

$$\operatorname{Li}_2(t) + \Re \operatorname{Li}_2\left(\frac{1}{t}\right) = -\frac{1}{2}\log^2(t) + 2\operatorname{Li}_2(1), \qquad t \in (0,1),$$

(see [13, Eq. (1.106)]) by t, and then repeatedly integrating the resulting relation over the interval $t \in (x, 1)$. $\square$

3. Main results

Theorem 1. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $a = \pm 1$, and $b = \pm 1$. Then*

$$\begin{aligned}
\int_0^\infty \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x &= -\left(1+(-1)^q\right)(q-1)!\operatorname{Li}_q(a)\operatorname{Li}_p(a^n b) \\
&+ \frac{2(q-1)!}{n^q}\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor}\binom{p+q-2j-1}{p-1}\operatorname{Li}_{2j}(b)\operatorname{Li}_{p+q-2j}(a^n b) \\
&+ \frac{a^n b(q-1)!}{n^q}\sum_{j=1}^{n-1}\sum_{k=1}^{q}\binom{p+q-k-1}{p-1}a^j(-1)^k\,\Theta(b,k,j,n) \\
&\times \Phi\left(a^n b, p+q-k, \frac{n-j}{n}\right),
\end{aligned}$$

where Li_p is defined in (13), Φ denotes the Lerch transcendent, and Θ is given in (8). Considering the Cauchy principal value (PV) of the integral when $a = 1$ is unnecessary, since the singularity at $x = 1$ is suppressed by the logarithm. Indeed,

$$\lim_{x\to 1}\frac{\log^{q-1}(x)}{1-x} = \begin{cases} -1, & q = 2, \\ 0, & q > 2. \end{cases} \tag{12}$$

Proof. Employing the integral form of the polylogarithm [13, p. 35]

$$\mathrm{Li}_p(z) = \frac{(-1)^{p-1}}{(p-1)!}\begin{cases} \displaystyle\int_0^1 \frac{z\,\log^{p-1}(y)}{1-zy}\,\mathrm{d}y, & |z|\leqslant 1,\\ \mathrm{PV}\displaystyle\int_0^1 \frac{z\,\log^{p-1}(y)}{1-zy}\,\mathrm{d}y, & z>1,\end{cases} \qquad p\in\mathbb{Z}_{\geqslant 2}, \tag{13}$$

which guarantees that the polylogarithm yields a real value when $z > 1$, we may write

$$\int_0^\infty \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x = \frac{(-1)^{p-1}b}{(p-1)!}\int_0^\infty\int_0^1 \frac{x^{n-1}\log^{q-1}(x)\log^{p-1}(y)}{(1-ax)(1-byx^n)}\,\mathrm{d}y\,\mathrm{d}x.$$

Swapping the order of integration (justified by Fubini's theorem) we have

$$\int_0^\infty \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x = \frac{(-1)^{p-1}}{(p-1)!}\int_0^1 \log^{p-1}(y)\left(\int_0^\infty \frac{bx^{n-1}\log^{q-1}(x)}{(1-ax)(1-byx^n)}\,\mathrm{d}x\right)\mathrm{d}y. \tag{14}$$

By applying partial fraction decomposition for several values of $n\in\mathbb{Z}_{\geqslant 1}$, we observe a recurring structure that holds for $a=\pm1$ and $b=\pm1$,

$$\frac{x^n}{(1-ax)(1-byx^n)} = \frac{a^n}{1-a^nby}\left(\frac{1}{1-ax} - \frac{\sum_{j=0}^{n-1}(ax)^j}{1-byx^n}\right), \tag{15}$$

from which it follows that

$$\int_0^\infty \frac{x^{n-1}\log^{q-1}(x)}{(1-ax)(1-byx^n)}\,\mathrm{d}x = \frac{a^n}{1-a^nby}\int_0^\infty \frac{\log^{q-1}(x)}{x}\left(\frac{1}{1-ax} - \frac{\sum_{j=0}^{n-1}(ax)^j}{1-byx^n}\right)\mathrm{d}x.$$

To avoid divergence, we separate the $j = 0$ term and then change the order of integration and summation. This yields

$$\begin{aligned}\int_0^\infty \frac{x^{n-1}\log^{q-1}(x)}{(1-ax)(1-byx^n)}\,\mathrm{d}x &= \frac{a^n}{1-a^nby}\int_0^\infty \frac{\log^{q-1}(x)}{x}\left(\frac{1}{1-ax} - \frac{1}{1-byx^n}\right)\mathrm{d}x\\ &\quad - \frac{a^n}{1-a^nby}\sum_{j=1}^{n-1} a^j\int_0^\infty \frac{x^{j-1}\log^{q-1}(x)}{1-byx^n}\,\mathrm{d}x. \end{aligned}\tag{16}$$

Performing the substitution $x = ty^{-1/n}$ in the rightmost integral, applying the binomial expansion to $\log^{q-1}(ty^{-1/n})$,

$$\begin{aligned}\log^{q-1}\left(ty^{-\frac{1}{n}}\right) &= \left(\log(t) - \frac{\log(y)}{n}\right)^{q-1} \\ &= \sum_{k=1}^{q} \binom{q-1}{k-1} \log^{k-1}(t) \left(-\frac{\log(y)}{n}\right)^{q-k},\end{aligned}$$

and then employing Lemma 4, we find

$$\int_0^\infty \frac{x^{j-1}\log^{q-1}(x)}{1-byx^n}\,\mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{n^q} \sum_{k=1}^{q} \frac{\Theta(b,k,j,n)}{(q-k)!} y^{-\frac{j}{n}} \log^{q-k}(y). \tag{17}$$

Substituting Lemma 5 and (17) into (16), we conclude

$$\begin{aligned}\int_0^\infty \frac{x^{n-1}\log^{q-1}(x)}{(1-ax)(1-byx^n)}\,\mathrm{d}x &= -\frac{a^n(q-1)!}{1-a^n by}\left(1+(-1)^q\right)\mathrm{Li}_q(a) \\ &- \frac{2a^n(-1)^{q-1}(q-1)!}{n^q(1-a^n by)} \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} \frac{\mathrm{Li}_{2j}(b)}{(q-2j)!} \log^{q-2j}(y) \\ &- \frac{a^n(-1)^{q-1}(q-1)!}{n^q(1-a^n by)} \sum_{j=1}^{n-1}\sum_{k=1}^{q} \frac{a^j\,\Theta(b,k,j,n)}{(q-k)!} y^{-\frac{j}{n}} \log^{q-k}(y).\end{aligned} \tag{18}$$

Plugging (18) into (14) and then using Lemma 3, the conclusion follows. $\square$

Theorem 2. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $a = \pm 1$, and $b = \pm 1$. Then*

$$\begin{aligned}\int_0^\infty \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^{2n})}{1-ax^2}\,\mathrm{d}x &= -\frac{a(q-1)!}{2^q}\left(1+a(-1)^q\right)\mathrm{Li}_p(a^n b)\,\Phi\left(a, q, \frac{1}{2}\right) \\ &+ \frac{a^{n-1}b(q-1)!}{(2n)^q} \sum_{j=1}^{n}\sum_{k=1}^{q} \binom{p+q-k-1}{p-1} a^j (-1)^k\,\Theta(b,k,2j-1,2n) \\ &\times \Phi\left(a^n b, p+q-k, \frac{2n-2j+1}{2n}\right),\end{aligned}$$

where Li_p is defined in (13), Φ denotes the Lerch transcendent, and Θ is given in (8). Considering the Cauchy principal value (PV) of the integral when $a = 1$ is unnecessary, since the singularity at $x = 1$ is suppressed by the logarithm, see (12).

Proof. Following the previous proof, we write

$$\int_0^\infty \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^{2n})}{1-ax^2}\,\mathrm{d}x$$

$$= \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \log^{p-1}(y) \left(\int_0^\infty \frac{bx^{2n}\log^{q-1}(x)}{(1-ax^2)(1-byx^{2n})}\, \mathrm{d}x \right) \mathrm{d}y. \quad (19)$$

Replacing x with x^2 in (15) and (j,n) with $(2j-1, 2n)$ in (17), we obtain, respectively,

$$\frac{x^{2n}}{(1-ax^2)(1-byx^{2n})} = \frac{a^n}{1-a^n by} \left(\frac{1}{1-ax^2} - \frac{\sum_{j=1}^n \left(ax^2\right)^{j-1}}{1-byx^{2n}} \right)$$

and

$$\int_0^\infty \frac{x^{2j-2}\log^{q-1}(x)}{1-byx^{2n}}\, \mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{(2n)^q} \sum_{k=1}^q \frac{\Theta(b,k,2j-1,2n)}{(q-k)!} y^{-\frac{2j-1}{2n}} \log^{q-k}(y).$$

Employing the latter two identities, and then Lemma 4, it follows that

$$\int_0^\infty \frac{x^{2n}\log^{q-1}(x)}{(1-ax^2)(1-byx^{2n})}\, \mathrm{d}x = -\frac{a^{n-1}(q-1)!\,(1+a(-1)^q)}{2^q(1-a^n by)} \Phi\left(a, q, \frac{1}{2}\right)$$
$$-\frac{a^n(-1)^{q-1}(q-1)!}{(2n)^q(1-a^n by)} \sum_{j=1}^n a^{j-1} \sum_{k=1}^q \frac{\Theta(b,k,2j-1,2n)}{(q-k)!} y^{-\frac{2j-1}{2n}} \log^{q-k}(y). \quad (20)$$

The proof is finalized by plugging (20) into (19) and then using Lemma 3. □

Theorem 3. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $p+q$ odd, $a = \pm 1$, and $b = \pm 1$. Then*

$$\int_0^1 \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^n)}{1-ax}\, \mathrm{d}x$$
$$= -\frac{a}{2}\left(1+(-1)^q\right)(q-1)!\,\mathrm{Li}_q(a)\,\mathrm{Li}_p(a^n b) + \frac{a(q-1)!}{2(-n)^q}\,\mathrm{Li}_{p+q}(b)$$
$$+ \frac{a(q-1)!}{n^q} \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} \binom{p+q-2j-1}{p-1} \mathrm{Li}_{2j}(b)\,\mathrm{Li}_{p+q-2j}(a^n b)$$
$$+ a(q-1)!n^p \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} n^{-2j}\,\mathrm{Li}_{2j}(b)\,\mathrm{Li}_{p+q-2j}(a)$$
$$+ \frac{a^{n-1}b(q-1)!}{2n^q} \sum_{j=1}^{n-1} \sum_{k=1}^q \binom{p+q-k-1}{p-1} a^j (-1)^k\,\Theta(b,k,j,n)$$
$$\times \Phi\left(a^n b, p+q-k, \frac{n-j}{n}\right),$$

where Li_p *denotes the polylogarithm function,* Φ *represents the Lerch transcendent,* Θ *is described in* (8)*, and* $\lfloor\cdot\rfloor$ *is the floor function.*

Proof. We begin with employing $\int_0^\infty f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x + \int_1^\infty f(x)\,\mathrm{d}x$,

$$\int_0^\infty \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x = \int_0^1 \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x + \int_1^\infty \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x.$$

Write $\frac{1}{x(1-ax)} = \frac{1}{x} + \frac{a}{1-ax}$ in the first integral, substitute $x \mapsto 1/x$ in the second, and then set $(-1)^q = (-1)^{p-1}$, since $p+q$ is odd, we obtain

$$\int_0^\infty \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x(1-ax)}\,\mathrm{d}x = \int_0^1 \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x}\,\mathrm{d}x + a\int_0^1 \frac{\log^{q-1}(x)}{1-ax}\left(\operatorname{Li}_p(bx^n) - (-1)^p \operatorname{Li}_p\left(\frac{b}{x^n}\right)\right)\mathrm{d}x. \tag{21}$$

For the first integral on the right-hand side, expand $\operatorname{Li}_p(bx^n)$ in series as given in (3), revert the order of integration and summation, and then use the integral (7) to find

$$\int_0^1 \frac{\log^{q-1}(x)\operatorname{Li}_p(bx^n)}{x}\,\mathrm{d}x = \frac{(-1)^{q-1}(q-1)!}{n^q}\operatorname{Li}_{p+q}(b). \tag{22}$$

Substituting the results from Theorem 1 and (22) into (21), we reach

$$\begin{aligned}
&\int_0^1 \frac{\log^{q-1}(x)}{1-ax}\left(\operatorname{Li}_p(bx^n) - (-1)^p \operatorname{Li}_p\left(\frac{b}{x^n}\right)\right)\mathrm{d}x \\
&= -a\left(1+(-1)^q\right)(q-1)!\operatorname{Li}_p(a^n b)\operatorname{Li}_q(a) + \frac{a(q-1)!}{(-n)^q}\operatorname{Li}_{p+q}(b) \\
&\quad + \frac{2a(q-1)!}{n^q}\sum_{j=0}^{\lfloor \frac{q}{2}\rfloor}\binom{p+q-2j-1}{p-1}\operatorname{Li}_{2j}(b)\operatorname{Li}_{p+q-2j}(a^n b) \\
&\quad + \frac{a^{n-1}b(q-1)!}{n^q}\sum_{j=1}^{n-1}\sum_{k=1}^{q}\binom{p+q-k-1}{p-1}a^j(-1)^k\,\Theta(b,k,j,n) \\
&\quad \times \Phi\left(a^n b, p+q-k, \frac{n-j}{n}\right).
\end{aligned} \tag{23}$$

From Lemmas 6 and 3, with $p+q$ odd, it follows that

$$\int_0^1 \frac{\log^{q-1}(x)}{1-ax}\left(\operatorname{Li}_p(bx^n) + (-1)^p \operatorname{Li}_p\left(\frac{b}{x^n}\right)\right)\mathrm{d}x$$

$$= 2a(q-1)!n^p \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} n^{-2j} \operatorname{Li}_{2j}(b) \operatorname{Li}_{p+q-2j}(a). \quad (24)$$

The result then readily follows on combining (23) with (24). This completes the proof. □

Corollary 1. *Let $p, q, n \in \mathbb{Z}_{\geqslant 1}$, $a = \pm 1$, $b = \pm 1$, $q \neq 1$ when $a = 1$, and $p + q$ odd. Then*

$$\sum_{k=1}^{\infty} \frac{a^k \mathbf{H}_{nk,b}^{(p)}}{k^q} = \frac{b}{2}(1 + (-1)^q) \operatorname{Li}_q(a) \operatorname{Li}_p(b) + \frac{b}{2n^p} \operatorname{Li}_{p+q}(ab^n)$$
$$- b(-n)^q \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} \binom{p+q-2j-1}{p-1} n^{-2j} \operatorname{Li}_{2j}(ab^n) \operatorname{Li}_{p+q-2j}(b)$$
$$+ \frac{(-1)^p b}{n^p} \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} \operatorname{Li}_{2j}(ab^n) \operatorname{Li}_{p+q-2j}(a)$$
$$+ \frac{(-1)^p ab}{2n^p} \sum_{j=1}^{n-1} \sum_{k=1}^{p} \binom{p+q-k-1}{q-1} b^j (-1)^k \Theta(ab^n, k, j, n)$$
$$\times \Phi\left(a, p+q-k, \frac{n-j}{n}\right),$$

where $\mathbf{H}_{nk,b}^{(p)}$ is defined in (1), Li_p denotes the polylogarithm function, Φ represents the Lerch transcendent, Θ is described in (8), and $\lfloor \cdot \rfloor$ is the floor function.

Proof. Using the integral form of $1/m^p$ given in (7), we have

$$\mathbf{H}_{nk,b}^{(p)} = \sum_{m=1}^{nk} \frac{b^{m-1}}{m^p} = \sum_{m=1}^{nk} b^{m-1} \cdot \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 x^{m-1} \log^{p-1}(x)\, dx$$
$$= \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \log^{p-1}(x) \sum_{m=1}^{nk} (bx)^{m-1}\, dx$$
$$= \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \log^{p-1}(x) \cdot \frac{1-(bx)^{nk}}{1-bx}\, dx. \quad (25)$$

Multiplying this by a^k/k^q, considering the summation over integers $k \geqslant 1$, and then changing the order of integration and summation, we obtain

$$\sum_{k=1}^{\infty} \frac{a^k \mathbf{H}_{nk,b}^{(p)}}{k^q} = \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \frac{\log^{p-1}(x)}{1-bx} \sum_{k=1}^{\infty} \frac{a^k - (ab^n x^n)^k}{k^q}\, dx$$

$$= \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \frac{\log^{p-1}(x)}{1-bx} \left(\operatorname{Li}_q(a) - \operatorname{Li}_q(ab^n x^n)\right) \mathrm{d}x$$
$$= b \operatorname{Li}_q(a) \operatorname{Li}_p(b) - \frac{(-1)^{p-1}}{(p-1)!} \int_0^1 \frac{\log^{p-1}(x) \operatorname{Li}_q(ab^n x^n)}{1-bx} \mathrm{d}x. \quad (26)$$

By substituting $(q, p, a, b) = (p, q, b, ab^n)$ in Theorem 3, we obtain the result of the integral appearing in (26) and the proof is complete. □

Remark 3. It is valid to take $(p, q, b) = (1, 2q, 1)$ in Corollary 1, since in this case the first term containing the factor $\operatorname{Li}_p(b)$ is canceled by the term arising from $\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor}$. This cancellation follows from applying the summation identity

$$\sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} f(2j) = \frac{1}{2}\left(1 + (-1)^q\right) f(q) + \sum_{j=0}^{\lfloor \frac{q-1}{2} \rfloor} f(2j),$$

which, upon noting that $\mathbf{H}_{nk,1}^{(1)} = H_{nk}$, yields

$$\sum_{k=1}^{\infty} \frac{a^k H_{nk}}{k^{2q}} = \frac{1 - 4q \operatorname{Li}_0(a)}{2n} \operatorname{Li}_{2q+1}(a) - n^{2q} \sum_{j=0}^{q-1} n^{-2j} \operatorname{Li}_{2j}(a)\, \zeta(2q - 2j + 1)$$
$$+ \frac{a}{2n} \sum_{j=1}^{n-1} \Theta(a, 1, j, n)\, \Phi\left(a, 2q, \frac{n-j}{n}\right).$$

Corollary 2. *Let $p, q, n \in \mathbb{Z}_{\geqslant 1}$, $a = \pm 1$, $b = \pm 1$, $p \neq 1$ when $b = 1$, $q \neq 1$ when $a = 1$, and $p + q$ odd. Then*

$$\sum_{k=1}^{\infty} (a^n)^k \mathbf{H}_{k,b}^{(p)} \sum_{j=1}^{n} \frac{a^{j-1}}{(nk+j)^q} = \frac{ab}{2}\left(1 + (-1)^q\right) \operatorname{Li}_q(a) \operatorname{Li}_p(b)$$
$$- \frac{ab}{2n^q} \operatorname{Li}_{p+q}(a^n b) - \frac{(-1)^q ab}{n^q} \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} \binom{p+q-2j-1}{p-1} \operatorname{Li}_{2j}(a^n b) \operatorname{Li}_{p+q-2j}(b)$$
$$+ ab(-n)^p \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} n^{-2j} \operatorname{Li}_{2j}(a^n b) \operatorname{Li}_{p+q-2j}(a)$$
$$- \frac{(-1)^q a}{2n^q} \sum_{j=1}^{n-1} \sum_{k=1}^{q} \binom{p+q-k-1}{p-1} a^j (-1)^k\, \Theta(a^n b, k, j, n)$$
$$\times \Phi\left(b, p+q-k, \frac{n-j}{n}\right),$$

where $\mathbf{H}_{k,b}^{(p)}$ *is defined in* (1), Li_p *denotes the polylogarithm function,* Φ *represents the Lerch transcendent,* Θ *is described in* (8), *and* $\lfloor\cdot\rfloor$ *is the floor function.*

Proof. Replacing (c,x) with $(b,a^n x^n)$ in part (i) of Lemma 1, we get

$$\sum_{k=1}^{\infty}(a^n)^k\,\mathbf{H}_{k,b}^{(p)}\,x^{nk}=\frac{b\,\mathrm{Li}_p(a^n b x^n)}{1-a^n x^n},$$

from which it follows that

$$\begin{aligned}\frac{\mathrm{Li}_p(a^n b x^n)}{1-ax}&=\frac{\mathrm{Li}_p(a^n b x^n)}{1-a^n x^n}\cdot\frac{1-a^n x^n}{1-ax}\\&=b\sum_{k=1}^{\infty}(a^n)^k\,\mathbf{H}_{k,b}^{(p)}\,x^{nk}\cdot\sum_{j=1}^{n}(ax)^{j-1}\\&=b\sum_{k=1}^{\infty}(a^n)^k\,\mathbf{H}_{k,b}^{(p)}\sum_{j=1}^{n}a^{j-1}x^{nk+j-1}.\end{aligned}\tag{27}$$

Multiplying this by $\log^{q-1}(x)$, and then integrating for $x\in(0,1)$, we obtain

$$\sum_{k=1}^{\infty}(a^n)^k\,\mathbf{H}_{k,b}^{(p)}\sum_{j=1}^{n}\frac{a^{j-1}}{(nk+j)^q}=\frac{b(-1)^{q-1}}{(q-1)!}\int_0^1\frac{\log^{q-1}(x)\,\mathrm{Li}_p(a^n b x^n)}{1-ax}\,\mathrm{d}x.$$

Taking $b=a^n b$ in Theorem 3 delivers the result of the integral appearing in the latter identity and completes the proof. □

Corollary 3. *Let* $n\in\mathbb{Z}_{\geqslant 1}$, $p,q\in\mathbb{Z}_{\geqslant 2}$, *and* $p+q$ *odd. Then*

$$\begin{aligned}&\sum_{k=1}^{\infty}O_k^{(p)}\sum_{j=1}^{n}\frac{1}{(2nk+2j-n-1)^q}\\&=\frac{1}{8}\left(1+(-1)^q\right)\Big\{2\zeta(q)\lambda(p)+\eta(q)\Big[\mathrm{Li}_p((-1)^n)-\mathrm{Li}_p((-1)^{n-1})\Big]\Big\}\\&\quad-\frac{(-1)^q}{4n^q}\sum_{j=0}^{\lfloor\frac{q}{2}\rfloor}\binom{p+q-2j-1}{p-1}\Big\{\zeta(2j)\Big[\zeta(p+q-2j)-\mathrm{Li}_{p+q-2j}((-1)^n)\Big]\\&\quad-\eta(2j)\Big[\eta(p+q-2j)+\mathrm{Li}_{p+q-2j}((-1)^{n-1})\Big]\Big\}\\&\quad+(-n)^p\sum_{j=0}^{\lfloor\frac{p}{2}\rfloor}\binom{p+q-2j-1}{q-1}n^{-2j}\lambda(2j)\lambda(p+q-2j)-\frac{(-1)^q}{8n^q}\\&\quad\times\sum_{j=1}^{n-1}\sum_{k=1}^{q}\binom{p+q-k-1}{p-1}(-1)^k\Big\{\Theta(1,k,j,n)\Big[\Phi\Big(1,p+q-k,\frac{n-j}{n}\Big)\end{aligned}$$

$$- (-1)^{j+n}\,\Phi\left((-1)^n, p+q-k, \frac{n-j}{n}\right)\Big] + \Theta(-1,k,j,n)\times$$
$$\left[\Phi\left(-1, p+q-k, \frac{n-j}{n}\right) - (-1)^{j+n}\Phi\left((-1)^{n-1}, p+q-k, \frac{n-j}{n}\right)\right]\Bigg\},$$

where $O_k^{(p)} = \sum_{m=1}^k \frac{1}{(2m-1)^p}$, Li_p *denotes the polylogarithm function,* ζ *is the Riemann zeta function,* η *is the Dirichlet eta function,* λ *is the Dirichlet lambda function,* Φ *represents the Lerch transcendent,* Θ *is described in* (8), *and* $\lfloor\cdot\rfloor$ *is the floor function.*

Proof. Setting $(c,x) = (a^n, x^n)$ in part (ii) of Lemma 1, we have

$$2\sum_{k=1}^{\infty} (a^n)^{k-1}\, \mathbf{O}_{k,a^n}^{(p)}\, x^{2nk-n} = \frac{\mathrm{Li}_p(x^n) - \mathrm{Li}_p(-x^n)}{1 - a^n x^{2n}},$$

from which it follows that

$$\frac{\mathrm{Li}_p(x^n) - \mathrm{Li}_p(-x^n)}{1 - ax^2} = \frac{\mathrm{Li}_p(x^n) - \mathrm{Li}_p(-x^n)}{1 - a^n x^{2n}} \cdot \frac{1 - a^n x^{2n}}{1 - ax^2}$$
$$= 2\sum_{k=1}^{\infty} (a^n)^{k-1}\, \mathbf{O}_{k,a^n}^{(p)}\, x^{2nk-n} \cdot \sum_{j=1}^{n} \left(ax^2\right)^{j-1}.$$

Multiplying this by $\log^{q-1}(x)$, then integrating for $x \in (0,1)$, we find

$$\sum_{k=1}^{\infty} (a^n)^{k-1}\, \mathbf{O}_{k,a^n}^{(p)} \sum_{j=1}^{n} \frac{a^{j-1}}{(2nk+2j-n-1)^q}$$
$$= \frac{(-1)^{q-1}}{2(q-1)!} \int_0^1 \frac{\log^{q-1}(x)\,(\mathrm{Li}_p(x^n) - \mathrm{Li}_p(-x^n))}{1 - ax^2}\,\mathrm{d}x. \quad (28)$$

From (28), setting $a = 1$ yields

$$\sum_{k=1}^{\infty} O_k^{(p)} \sum_{j=1}^{n} \frac{1}{(2nk+2j-n-1)^q}$$
$$= \frac{(-1)^{q-1}}{2(q-1)!} \int_0^1 \frac{\log^{q-1}(x)\,(\mathrm{Li}_p(x^n) - \mathrm{Li}_p(-x^n))}{1 - x^2}\,\mathrm{d}x. \quad (29)$$

Combining the cases $a = \pm 1$ in Theorem 3 leads to

$$\int_0^1 \frac{\log^{q-1}(x)\,\mathrm{Li}_p(bx^n)}{1 - x^2}\,\mathrm{d}x$$
$$= -\frac{1}{4}(1 + (-1)^q)(q-1)!\,[\zeta(q)\,\mathrm{Li}_p(b) + \eta(q)\,\mathrm{Li}_p((-1)^n b)]$$
$$+ \frac{(q-1)!}{2n^q} \sum_{j=0}^{\lfloor \frac{q}{2} \rfloor} \binom{p+q-2j-1}{p-1} \mathrm{Li}_{2j}(b)\Big[\mathrm{Li}_{p+q-2j}(b) - \mathrm{Li}_{p+q-2j}((-1)^n b)\Big]$$

$$+ (q-1)!n^p \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} n^{-2j} \operatorname{Li}_{2j}(b) \lambda(p+q-2j)$$
$$+ \frac{b(q-1)!}{4n^q} \sum_{j=1}^{n-1} \sum_{k=1}^{q} \binom{p+q-k-1}{p-1} (-1)^k \Theta(b,k,j,n)$$
$$\times \left\{ \Phi\left(b, p+q-k, \frac{n-j}{n}\right) - (-1)^{j+n} \Phi\left((-1)^n b, p+q-k, \frac{n-j}{n}\right) \right\}.$$

By substituting $b = \pm 1$ in the latter identity, we obtain the results of the two integrals appearing in (29) and the proof is complete. □

Theorem 4. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $b = \pm 1$, and $a = (-1)^{p+q-1}$. Then*

$$\int_0^1 \frac{\log^{q-1}(x) \operatorname{Li}_p(bx^{2n})}{1-ax^2} \, dx = -\frac{a(q-1)!}{2^{q+1}} (1-(-1)^p) \operatorname{Li}_p(a^n b) \Phi\left(a, q, \frac{1}{2}\right)$$
$$+ \frac{(q-1)!n^p}{a2^q} \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} n^{-2j} \operatorname{Li}_{2j}(b) \Phi\left(a, p+q-2j, \frac{1}{2}\right)$$
$$+ \frac{a^{n-1}b(q-1)!}{2(2n)^q} \sum_{j=1}^{n} \sum_{k=1}^{q} \binom{p+q-k-1}{p-1} a^j (-1)^k \Theta(b,k,2j-1,2n)$$
$$\times \Phi\left(a^n b, p+q-k, \frac{2n-2j+1}{2n}\right),$$

where Li_p denotes the polylogarithm function, Φ represents the Lerch transcendent, Θ is described in (8), and $\lfloor \cdot \rfloor$ is the floor function.

Proof. Employing $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$, and then letting $x \mapsto 1/x$ in the second integral, we obtain

$$\int_0^\infty \frac{\log^{q-1}(x) \operatorname{Li}_p(bx^{2n})}{1-ax^2} \, dx$$
$$= \int_0^1 \frac{\log^{q-1}(x)}{1-ax^2} \left(\operatorname{Li}_p(bx^{2n}) + a(-1)^q \operatorname{Li}_p\left(\frac{b}{x^{2n}}\right) \right) dx.$$

Recalling the result from Theorem 2, then substituting $a(-1)^q = (-1)^{p-1}$, we get

$$\int_0^1 \frac{\log^{q-1}(x)}{1-ax^2} \left(\operatorname{Li}_p(bx^{2n}) - (-1)^p \operatorname{Li}_p\left(\frac{b}{x^{2n}}\right) \right) dx$$
$$= -\frac{a(q-1)!}{2^q} (1-(-1)^p) \operatorname{Li}_p(a^n b) \Phi\left(a, q, \frac{1}{2}\right)$$

$$+\frac{a^{n-1}b(q-1)!}{(2n)^q}\sum_{j=1}^{n}\sum_{k=1}^{q}\binom{p+q-k-1}{p-1}a^j(-1)^k\,\Theta(b,k,2j-1,2n)$$
$$\times\,\Phi\left(a^nb,p+q-k,\frac{2n-2j+1}{2n}\right). \tag{30}$$

From Lemmas 6 and 3, with $a=(-1)^{p+q-1}$, one has

$$\int_0^1\frac{\log^{q-1}(x)}{1-ax^2}\left(\operatorname{Li}_p(bx^{2n})+(-1)^p\operatorname{Li}_p\left(\frac{b}{x^{2n}}\right)\right)\mathrm{d}x$$
$$=\frac{a(q-1)!n^p}{2^{q-1}}\sum_{j=0}^{\lfloor\frac{p}{2}\rfloor}\binom{p+q-2j-1}{q-1}n^{-2j}\operatorname{Li}_{2j}(b)\,\Phi\left(a,p+q-2j,\frac{1}{2}\right). \tag{31}$$

Summing (30) and (31) delivers the stated result and completes the proof. □

Corollary 4. *Let $p,q,n\in\mathbb{Z}_{\geqslant1}$, $a=\pm1$, $b=(-1)^{p+q-1}$, and $q\neq1$ when $a=1$. Then*

$$\sum_{k=1}^{\infty}\frac{a^k\,\mathbf{O}_{nk,b}^{(p)}}{k^q}=\frac{1}{2^{p+1}}\left(1+(-1)^q\right)\operatorname{Li}_q(a)\,\Phi\left(b,p,\frac{1}{2}\right)$$
$$-\frac{(-n)^q}{2^p}\sum_{j=0}^{\lfloor\frac{q}{2}\rfloor}\binom{p+q-2j-1}{p-1}n^{-2j}\operatorname{Li}_{2j}(ab^n)\,\Phi\left(b,p+q-2j,\frac{1}{2}\right)$$
$$-\frac{(-1)^qa}{2(2n)^p}\sum_{j=1}^{n}\sum_{k=1}^{p}\binom{p+q-k-1}{q-1}b^j(-1)^k\,\Theta(ab^n,k,2j-1,2n)$$
$$\times\,\Phi\left(a,p+q-k,\frac{2n-2j+1}{2n}\right),$$

where $\mathbf{O}_{nk,b}^{(p)}$ is defined in (2), Li_p denotes the polylogarithm function, Φ represents the Lerch transcendent, Θ is described in (8), and $\lfloor\cdot\rfloor$ is the floor function.

Proof. Following the same technique as used in (25), we find

$$\mathbf{O}_{nk,b}^{(p)}=\sum_{m=1}^{nk}\frac{b^{m-1}}{(2m-1)^p}=\frac{(-1)^{p-1}}{(p-1)!}\int_0^1\log^{p-1}(x)\cdot\frac{1-\left(bx^2\right)^{nk}}{1-bx^2}\,\mathrm{d}x,$$

and consequently

$$\sum_{k=1}^{\infty}\frac{a^k\,\mathbf{O}_{nk,b}^{(p)}}{k^q}=\frac{1}{2^p}\operatorname{Li}_q(a)\,\Phi\left(b,p,\frac{1}{2}\right)-\frac{(-1)^{p-1}}{(p-1)!}\int_0^1\frac{\log^{p-1}(x)\operatorname{Li}_q(ab^nx^{2n})}{1-bx^2}\,\mathrm{d}x.$$

Setting $b = ab^n$ in Theorem 4 gives the result of the integral appearing in the latter identity and completes the proof. □

Remark 4. The case $(p, q, b) = (1, 2q, 1)$ in Corollary 4 is valid for the same reason as explained in Remark 3. In this situation, noting that $\mathbf{O}^{(1)}_{nk,1} = O_{nk}$, we obtain

$$\sum_{k=1}^{\infty} \frac{a^k\, O_{nk}}{k^{2q}} = -\frac{n^{2q}}{2} \sum_{j=0}^{q-1} n^{-2j}\, \mathrm{Li}_{2j}(a)\, \Phi\left(1, 2q - 2j + 1, \frac{1}{2}\right) + \frac{a}{4n} \sum_{j=1}^{n} \Theta(a, 1, 2j-1, 2n)\, \Phi\left(a, 2q, \frac{2n - 2j + 1}{2n}\right).$$

Corollary 5. *Let $p, q, n \in \mathbb{Z}_{\geqslant 1}$, $b = \pm 1$, $a = (-1)^{p+q-1}$, $p \neq 1$ when $b = 1$, and $q \neq 1$ when $a = 1$. Then*

$$\begin{aligned}\sum_{k=1}^{\infty} (a^n)^k\, \mathbf{H}^{(p)}_{k,b} \sum_{j=1}^{n} \frac{a^{j-1}}{(2nk + 2j - 1)^q} &= \frac{b}{2^{q+1}} \left(1 - (-1)^p\right) \mathrm{Li}_p(b)\, \Phi\left(a, q, \frac{1}{2}\right) \\ &+ \frac{b(-n)^p}{2^q} \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p + q - 2j - 1}{q - 1} n^{-2j}\, \mathrm{Li}_{2j}(a^n b)\, \Phi\left(a, p + q - 2j, \frac{1}{2}\right) \\ &+ \frac{(-1)^p}{2(2n)^q} \sum_{j=1}^{n} \sum_{k=1}^{q} \binom{p + q - k - 1}{p - 1} a^j (-1)^k\, \Theta(a^n b, k, 2j-1, 2n) \\ &\times \Phi\left(b, p + q - k, \frac{2n - 2j + 1}{2n}\right),\end{aligned}$$

where $\mathbf{H}^{(p)}_{k,b}$ is defined in (1), Li_p denotes the polylogarithm function, Φ represents the Lerch transcendent, Θ is described in (8), and $\lfloor \cdot \rfloor$ is the floor function.

Proof. Replacing x with x^2 in (27) yields

$$\frac{\mathrm{Li}_p(a^n b x^{2n})}{1 - ax^2} = b \sum_{k=1}^{\infty} (a^n)^k\, \mathbf{H}^{(p)}_{k,b} \sum_{j=1}^{n} a^{j-1} x^{2nk+2j-2}.$$

Multiplying this by $\log^{q-1}(x)$, then integrating for $x \in (0, 1)$, we obtain

$$\sum_{k=1}^{\infty} (a^n)^k\, \mathbf{H}^{(p)}_{k,b} \sum_{j=1}^{n} \frac{a^{j-1}}{(2nk + 2j - 1)^q} = \frac{(-1)^{q-1} b}{(q-1)!} \int_0^1 \frac{\log^{q-1}(x)\, \mathrm{Li}_p(a^n b x^{2n})}{1 - ax^2}\, \mathrm{d}x.$$

Substituting $b = a^n b$ in Theorem 4 produces the result of the integral appearing in the latter identity and completes the proof. □

Remark 5. In Corollary 5 we stated that $p \neq 1$ when $b = 1$. However, this restriction can be relaxed when $n = 1$, since in this case the first term containing the factor $\mathrm{Li}_p(b)$ is canceled by the $k = q$ term in the double sum $\sum_{j=1}^{n} \sum_{k=1}^{q}$. This cancellation, noting that $\mathbf{H}_{k,1}^{(1)} = H_k$, yields

$$\sum_{k=1}^{\infty} \frac{((-1)^q)^k H_k}{(2k+1)^q} = -\frac{\log(2)}{2^{q-1}} \Phi\left((-1)^q, q, \frac{1}{2}\right) - \frac{q}{2^q} \mathrm{Li}_0((-1)^q) \Phi\left((-1)^q, q+1, \frac{1}{2}\right) - \frac{(-1)^q}{2^{q+1}} \sum_{k=1}^{q-1} (-1)^k \Theta((-1)^q, k, 1, 2) \zeta\left(q-k+1, \frac{1}{2}\right).$$

Corollary 6. *Let $n \in \mathbb{Z}_{\geqslant 1}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, and $p+q$ even. Then*

$$\begin{aligned}
&\sum_{k=1}^{\infty} O_k^{(p)} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{(4nk-2n+2j-1)^q} \\
&= \frac{1}{4}(1-(-1)^p)\beta(q)\Big[\mathrm{Li}_p((-1)^n) - \mathrm{Li}_p((-1)^{n-1})\Big] \\
&\quad + (-2n)^p \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p+q-2j-1}{q-1} (2n)^{-2j} \beta(p+q-2j)\lambda(2j) \\
&\quad + \frac{(-1)^{n+q}}{4(2n)^q} \sum_{j=1}^{n} \sum_{k=1}^{q} \binom{p+q-k-1}{p-1} (-1)^{k+j} \Big\{\Theta(1, k, 2j-1, 2n) \\
&\quad \times \Phi\left((-1)^n, p+q-k, \frac{2n-2j+1}{2n}\right) + \Theta(-1, k, 2j-1, 2n) \\
&\quad \times \Phi\left((-1)^{n-1}, p+q-k, \frac{2n-2j+1}{2n}\right)\Big\},
\end{aligned}$$

where $O_k^{(p)} = \sum_{m=1}^{k} \frac{1}{(2m-1)^p}$, Li_p denotes the polylogarithm function, ζ is the Riemann zeta function, η is the Dirichlet eta function, β is the Dirichlet beta function, λ is the Dirichlet lambda function, Φ represents the Lerch transcendent, Θ is described in (8)*, and $\lfloor \cdot \rfloor$ is the floor function.*

Proof. Taking $(a, n) = (-1, 2n)$ in (28) results in

$$\sum_{k=1}^{\infty} O_k^{(p)} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{(4nk+2j-2n-1)^q} = \frac{(-1)^{q-1}}{2(q-1)!} \int_0^1 \frac{\log^{q-1}(x)\left(\mathrm{Li}_p(x^{2n}) - \mathrm{Li}_p(-x^{2n})\right)}{1+x^2} \, \mathrm{d}x.$$

Taking $(a,b)=(-1,\pm 1)$ in Theorem 4 yields the results of the two integrals appearing in the latter identity and completes the proof. □

4. Examples

To obtain results from all formulas in this work without encountering divergence issues, it is important to use the following *Mathematica* code:

```
Unprotect[PolyLog];
PolyLog[0, 1] = -1/2;
Protect[PolyLog];
theta[1, 1, s_, r_] := Pi Cot[Pi s/r];
theta[c_, q_, s_, r_] := LerchPhi[c, q, s/r] +
c (-1)^q LerchPhi[c, q, (r - s)/r];
formulaExpression // FullSimplify
```

Example 1. Take $(n,p,q,a,b)=(3,2,3,-1,-1)$ in Corollary 1. Then

$$\sum_{k=1}^{\infty}\frac{(-1)^k\overline{H}_{3k}^{(2)}}{k^3}=\frac{53\pi^2\zeta(3)}{24}-\frac{7343\zeta(5)}{144}+\frac{\pi\zeta(4,\frac{1}{3})}{96\sqrt{3}}+\frac{\pi\zeta(4,\frac{1}{6})}{96\sqrt{3}}-\frac{\pi\zeta(4,\frac{2}{3})}{96\sqrt{3}}-\frac{\pi\zeta(4,\frac{5}{6})}{96\sqrt{3}}.$$

Example 2. Take $(n,p,q,a,b)=(3,2,3,-1,-1)$ in Corollary 2. Then

$$\sum_{k=1}^{\infty}(-1)^k\overline{H}_k^{(2)}\left(\frac{1}{(3k+1)^3}-\frac{1}{(3k+2)^3}+\frac{1}{(3k+3)^3}\right)=-\frac{79\pi^2\zeta(3)}{72}+\frac{10957\zeta(5)}{432}+\frac{\pi^3\zeta(2,\frac{2}{3})}{162\sqrt{3}}+\frac{\pi^3\zeta(2,\frac{5}{6})}{162\sqrt{3}}+\frac{\pi\zeta(4,\frac{2}{3})}{288\sqrt{3}}+\frac{\pi\zeta(4,\frac{5}{6})}{288\sqrt{3}}-\frac{\pi^3\zeta(2,\frac{1}{3})}{162\sqrt{3}}-\frac{\pi^3\zeta(2,\frac{1}{6})}{162\sqrt{3}}-\frac{\pi\zeta(4,\frac{1}{3})}{288\sqrt{3}}-\frac{\pi\zeta(4,\frac{1}{6})}{288\sqrt{3}}.$$

Example 3. Take $(n,p,q)=(3,2,3)$ in Corollary 3. Then

$$\sum_{k=1}^{\infty}O_k^{(2)}\left(\frac{1}{(6k)^3}+\frac{1}{(6k-2)^3}+\frac{1}{(6k+2)^3}\right)=\frac{679\pi^2\zeta(3)}{1728}-\frac{31\zeta(5)}{432}+\frac{5\pi^3\zeta(2,\frac{2}{3})}{2592\sqrt{3}}-\frac{5\pi^3\zeta(2,\frac{1}{3})}{2592\sqrt{3}}+\frac{17\pi\zeta(4,\frac{2}{3})}{1152\sqrt{3}}-\frac{17\pi\zeta(4,\frac{1}{3})}{1152\sqrt{3}}-\frac{\pi\zeta(4,\frac{1}{6})}{1152\sqrt{3}}+\frac{\pi\zeta(4,\frac{5}{6})}{1152\sqrt{3}}-\frac{\pi^3\zeta(2,\frac{1}{6})}{2592\sqrt{3}}+\frac{\pi^3\zeta(2,\frac{5}{6})}{2592\sqrt{3}}.$$

Example 4. Take $(n,p,q,a,b)=(3,2,4,-1,-1)$ in Corollary 4. Then

$$\sum_{k=1}^{\infty}\frac{(-1)^k\overline{O}_{3k}^{(2)}}{k^4}=-\frac{\pi^4G}{48}-\frac{17\pi^2\zeta(4,\frac{1}{4})}{48}+\frac{17\pi^2\zeta(4,\frac{3}{4})}{48}+\frac{\pi\zeta(5,\frac{5}{12})}{192\sqrt{3}}$$

$$+\frac{\pi\zeta(5,\frac{7}{12})}{192\sqrt{3}}+\frac{405\zeta(6,\frac{1}{4})}{512}-\frac{405\zeta(6,\frac{3}{4})}{512}-\frac{\pi\zeta(5,\frac{1}{12})}{192\sqrt{3}}-\frac{\pi\zeta(5,\frac{11}{12})}{192\sqrt{3}}.$$

Example 5. Take $(n,p,q,a,b)=(3,2,4,-1,-1)$ in Corollary 5. Then

$$\sum_{k=1}^{\infty}(-1)^k\overline{H}_k^{(2)}\left(\frac{1}{(6k+1)^4}-\frac{1}{(6k+3)^4}+\frac{1}{(6k+5)^4}\right)=-\frac{133\pi^4G}{648}$$
$$-\frac{\pi^3\zeta(3,\frac{1}{12})}{864\sqrt{3}}+\frac{\pi^3\zeta(3,\frac{5}{12})}{864\sqrt{3}}-\frac{\pi^3\zeta(3,\frac{11}{12})}{864\sqrt{3}}+\frac{\pi^3\zeta(3,\frac{7}{12})}{864\sqrt{3}}+\frac{\pi\zeta(5,\frac{5}{12})}{6912\sqrt{3}}$$
$$+\frac{\pi\zeta(5,\frac{7}{12})}{6912\sqrt{3}}+\frac{45\zeta(6,\frac{1}{4})}{1024}-\frac{45\zeta(6,\frac{3}{4})}{1024}-\frac{\pi\zeta(5,\frac{1}{12})}{6912\sqrt{3}}-\frac{\pi\zeta(5,\frac{11}{12})}{6912\sqrt{3}}.$$

Example 6. Take $(n,p,q)=(2,3,3)$ in Corollary 6. Then

$$\sum_{k=1}^{\infty}O_k^{(3)}\left(\frac{1}{(8k-3)^3}-\frac{1}{(8k-1)^3}+\frac{1}{(8k+1)^3}-\frac{1}{(8k+3)^3}\right)=-\frac{105\pi^3\zeta(3)}{256}$$
$$-\frac{93\pi\zeta(5)}{4}+\frac{3\pi^3\zeta(3,\frac{1}{8})}{2048\sqrt{2}}+\frac{3\pi^3\zeta(3,\frac{7}{8})}{2048\sqrt{2}}-\frac{3\pi^3\zeta(3,\frac{3}{8})}{2048\sqrt{2}}-\frac{3\pi^3\zeta(3,\frac{5}{8})}{2048\sqrt{2}}$$
$$-\frac{15\pi^2\zeta(4,\frac{1}{4})}{512}+\frac{15\pi^2\zeta(4,\frac{3}{4})}{512}+\frac{3\pi^2\zeta(4,\frac{1}{8})}{2048\sqrt{2}}+\frac{3\pi^2\zeta(4,\frac{3}{8})}{2048\sqrt{2}}-\frac{3\pi^2\zeta(4,\frac{5}{8})}{2048\sqrt{2}}$$
$$-\frac{3\pi^2\zeta(4,\frac{7}{8})}{2048\sqrt{2}}+\frac{3\pi\zeta(5,\frac{1}{8})}{2048\sqrt{2}}+\frac{3\pi\zeta(5,\frac{7}{8})}{2048\sqrt{2}}-\frac{3\pi\zeta(5,\frac{3}{8})}{2048\sqrt{2}}-\frac{3\pi\zeta(5,\frac{5}{8})}{2048\sqrt{2}}.$$

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College of Humanities and Social Sciences, Grand Canyon University, Phoenix, Arizona, United States
E-mail address: alishathri@yahoo.com