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Lacunary strong Riesz summability

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Abstract. In this paper, we introduce a new class of sequence spaces $N_{\theta,R}^{(k)}$ by combining lacunary block structures with Riesz-type weighted means of order k. This construction extends the classical notion of lacunary strong convergence to a higher-order weighted setting. We establish several inclusion relations between the classical strong Riesz summability method of order k and the corresponding lacunary space $N_{\theta,R}^{(k)}$ in the spirit of Freedman-type comparisons. The sharpness of the obtained conditions is illustrated by appropriate counterexamples. Basic topological properties of the new spaces are also discussed.

1. Introduction and preliminaries

Lacunary summability methods have played an important role in the theory of strong convergence since the pioneering work of Freedman et al. [3], who introduced the space N_θ of lacunary strongly convergent sequences. On the other hand, Riesz means [11] provide a flexible weighted summability method that generalizes Cesàro summability and admits higher-order extensions. For a comprehensive treatment of summability methods we refer to [1], [5] and [2].

The purpose of this paper is to localize Riesz means of order k within lacunary intervals. To the best of our knowledge, no previous study has combined lacunary block structures with higher-order Riesz kernels. The resulting method yields a new summability process that simultaneously reflects localization, weighting and order effects. Related studies on strong summability can be found in [6] and [7], while lacunary statistical convergence aspects are explored in [9], [13], and [10].

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Let $\theta = (k_r)$ be a sequence of positive integers with $k_0 = 0$. It is called *lacunary* if

$$h_r := k_r - k_{r-1} \longrightarrow \infty \quad (r \to \infty).$$

The lacunary intervals determined by θ are denoted by

$$I_r = (k_{r-1}, k_r].$$

Throughout the paper, given positive sequences (A_r) and (B_r) the notation $A_r \asymp B_r$ means that there exist constants $C_1, C_2 > 0$ such that

$$C_1 B_r \leq A_r \leq C_2 B_r \quad \text{for all sufficiently large } r.$$

Beyond their intrinsic interest, lacunary summability methods provide a natural framework for studying localized averaging processes, where global information may fail to detect oscillatory or sparse phenomena. Such localization effects arise naturally in Tauberian theory, matrix domain investigations of sequence spaces, and in the study of convergence of operator sequences where non-uniform behavior is present.

On the other hand, Riesz-type means allow flexible weighting and higher-order regularization, which play an important role in summability theory and Fourier analysis [15]. By combining lacunary block structures with higher-order Riesz kernels, the spaces $N_{\theta,R}^{(k)}$ capture simultaneously localization, weighting, and order effects. This interaction leads to new phenomena that are invisible in both the purely lacunary and purely classical settings and motivates a detailed comparison between classical and lacunary strong Riesz summability.

The present work goes beyond a routine lacunary modification of classical strong Riesz summability. Unlike the classical strong Riesz method, which relies on a single global normalization over the entire initial segment, we introduce a genuinely blockwise framework in which the Riesz regularization is reinitialized on each lacunary interval. This localized construction yields the new spaces $N_{\theta,R}^{(k)}$, where convergence is tested independently on successive lacunary blocks via weighted kernels of order k. As a consequence, the proposed method captures convergence phenomena arising from temporally heterogeneous or intermittently irregular behavior that are invisible to global Riesz averages. The sharp distinction between the classical and lacunary settings is further highlighted through precise inclusion theorems and counterexamples of Freedman–Sember–Raphael [3] type.

2. The space $N_{\theta,R}^{(k)}$

Let $\lambda = (\lambda_n)$ be a strictly increasing sequence of real numbers such that $\lambda_n \to \infty$. Define the Riesz increments

$$p_i := \lambda_i - \lambda_{i-1} > 0 \qquad (i \geq 1).$$

Definition 1. Let $k \geq 0$ be a fixed integer. For $r \geq 1$ and $i \in I_r$ define the *lacunary Riesz kernel of order* k by

$$\varkappa_{i,r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}} \right)^k .$$

The associated block weights are

$$w_{i,r}^{(k)} := \varkappa_{i,r}^{(k)} \, p_i,$$

and the total weight of the r-th block is

$$\Lambda_r^{(k)} = \sum_{i \in I_r} w_{i,r}^{(k)} = \sum_{i \in I_r} \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}} \right)^k (\lambda_i - \lambda_{i-1}).$$

Throughout the paper, we assume that $\Lambda_r^{(k)} > 0$ for all r, which is automatically satisfied under the above assumptions.

Definition 2 (Strong lacunary Riesz convergence of order k)**.** A sequence $x = (x_i)$ is said to be *strongly lacunary Riesz convergent of order* k *to* L if

$$\lim_{r \to \infty} \frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} w_{i,r}^{(k)} \, |x_i - L| = 0.$$

The class of all such sequences is denoted by $N_{\theta,R}^{(k)}$.

3. Structural analysis and special cases

The construction of the space $N_{\theta,R}^{(k)}$ allows for a flexible recovery of several classical summability methods. In the base case where $k = 0$, the lacunary Riesz kernel simplifies to a constant factor $\varkappa_{i,r}^{(0)} \equiv 1$. Consequently, the total block weight becomes the simple difference of the Riesz weights:

$$\Lambda_r^{(0)} = \sum_{i \in I_r} p_i = \lambda_{k_r} - \lambda_{k_{r-1}}.$$

In this setting, the space $N_{\theta,R}^{(0)}$ reduces to the classical space of strong lacunary Riesz convergence. If we further assume the arithmetic weight sequence $\lambda_n = n$, then $p_i \equiv 1$ and the space $N_{\theta,R}^{(0)}$ coincides with the original space N_θ introduced by Freedman et al. [3].

For higher orders $k \geq 1$, the kernel $\varkappa_{i,r}^{(k)}$ introduces a localized weighting mechanism. It acts as a polynomial decay factor within each lacunary block I_r, assigning larger weights to the initial indices and vanishing at the right endpoint $i = k_r$. This specific structure facilitates a higher-order regularization of the sequence.

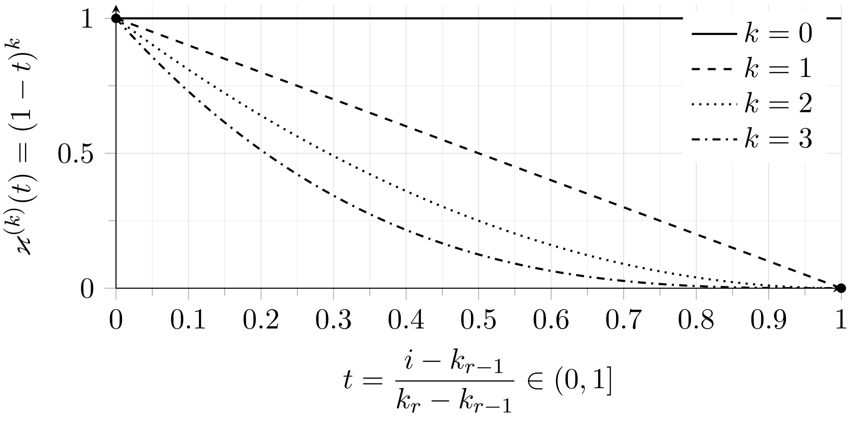

FIGURE 1. Normalized decay of the lacunary Riesz kernel within a block. Using the normalized position $t = (i - k_{r-1})/(k_r - k_{r-1})$, the kernel becomes $\varkappa^{(k)}(t) = (1-t)^k$, independent of the specific lacunary sequence and the weights λ. Larger k increases the localization effect toward the beginning of each lacunary interval.

4. Topological properties

Theorem 1. *The space* $N_{\theta,R}^{(k)}$ *is a linear space over* $\mathbb{C}$.

The proof follows directly from the linearity of the weighted block averages and is therefore omitted.

Theorem 2. *The functional*

$$g(x) = \sup_r \left(\frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} w_{i,r}^{(k)} |x_i| \right)$$

defines a paranorm on $N_{\theta,R}^{(k)}$.

Proof. Non-negativity and absolute homogeneity follow immediately from the definition: $g(x) \geq 0$, $g(0) = 0$, and $g(\alpha x) = |\alpha| g(x)$ for all $\alpha \in \mathbb{C}$.

For subadditivity, since $w_{i,r}^{(k)} \geq 0$,

$$\frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} w_{i,r}^{(k)} |x_i + y_i| \leq \frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} w_{i,r}^{(k)} (|x_i| + |y_i|),$$

and taking $\sup_r$ yields $g(x+y) \leq g(x) + g(y)$.

To show that $g(x) = 0$ implies $x = 0$, fix an arbitrary index $i \in \mathbb{N}$. Because $k_r \to \infty$ as $r \to \infty$, there exists an index r such that $i < k_r$. For this r, we have $i \in I_r$ and, moreover,

$$\varkappa_{i,r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}} \right)^k > 0,$$

since λ is strictly increasing and $i < k_r$ implies $\lambda_i < \lambda_{k_r}$. Consequently, $w_{i,r}^{(k)} = \varkappa_{i,r}^{(k)} p_i > 0$. If $g(x) = 0$, then, in particular,

$$\frac{1}{\Lambda_r^{(k)}} \sum_{j \in I_r} w_{j,r}^{(k)} |x_j| = 0,$$

and since all terms in the sum are nonnegative and $w_{i,r}^{(k)} > 0$, we must have $x_i = 0$. As i was arbitrary, $x = 0$.

Finally, continuity of scalar multiplication follows from

$$g(\alpha_n x) = |\alpha_n| g(x) \to 0$$

whenever $\alpha_n \to 0$. Thus g is a paranorm on $N_{\theta,R}^{(k)}$. □

5. Inclusion results

Lemma 1. *Let* $0 \leq k < h$. *Assume that the block normalizations satisfy the uniform ratio bound*

$$\sup_{r \geq 1} \frac{\Lambda_r^{(k)}}{\Lambda_r^{(h)}} < \infty. \tag{1}$$

Then

$$N_{\theta,R}^{(k)} \subset N_{\theta,R}^{(h)}.$$

Proof. Fix $x \in N_{\theta,R}^{(k)}$ with $N_{\theta,R}^{(k)} - \lim x_i = L$. For $i \in I_r$, define

$$a_{i,r} := \frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}}.$$

Because λ is increasing and $i \leq k_r$, we have $0 \leq a_{i,r} \leq 1$. Since the function $t \mapsto t^\alpha$ is decreasing on $[0, 1]$ for $\alpha > 0$, the inequality $0 \leq k < h$ implies $a_{i,r}^h \leq a_{i,r}^k$ for all $i \in I_r$. Hence

$$\sum_{i \in I_r} a_{i,r}^h \, p_i \, |x_i - L| \leq \sum_{i \in I_r} a_{i,r}^k \, p_i \, |x_i - L|.$$

Dividing by $\Lambda_r^{(h)}$ and using (1) yields

$$\frac{1}{\Lambda_r^{(h)}} \sum_{i \in I_r} a_{i,r}^h \, p_i \, |x_i - L| \leq \frac{\Lambda_r^{(k)}}{\Lambda_r^{(h)}} \cdot \frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} a_{i,r}^k \, p_i \, |x_i - L|.$$

Taking $\lim_{r \to \infty}$ and using $x \in N_{\theta,R}^{(k)}$, the right-hand side tends to 0 because the first factor is uniformly bounded. Therefore $x \in N_{\theta,R}^{(h)}$. □

Remark 1. Lemma 1 shows that the expected order inclusion $N_{\theta,R}^{(k)} \subset N_{\theta,R}^{(h)}$ $(k < h)$ holds once the block normalizations are comparable in the sense of (1). Without such a comparability assumption, an inclusion in either direction cannot be concluded from the pointwise inequality $a_{i,r}^h \le a_{i,r}^k$ alone, since both the numerator and the denominator depend on k.

Corollary 1. *Let* $0 \le k < h$ *and assume there exists* $\delta \in (0,1)$ *such that*

$$\inf_{r\ge 1} \frac{1}{\Lambda_r^{(k)}} \sum_{\substack{i\in I_r \\ a_{i,r}\ge\delta}} a_{i,r}^k\, p_i \;>\; 0. \tag{2}$$

Then (1) *holds, and hence* $N_{\theta,R}^{(k)} \subset N_{\theta,R}^{(h)}$.

Proof. For indices with $a_{i,r} \ge \delta$, we have $a_{i,r}^h \ge \delta^{\,h-k}\, a_{i,r}^k$. Therefore,

$$\Lambda_r^{(h)} = \sum_{i\in I_r} a_{i,r}^h\, p_i \ge \sum_{\substack{i\in I_r \\ a_{i,r}\ge\delta}} a_{i,r}^h\, p_i \ge \delta^{\,h-k} \sum_{\substack{i\in I_r \\ a_{i,r}\ge\delta}} a_{i,r}^k\, p_i.$$

Divide by $\Lambda_r^{(k)}$ to get

$$\frac{\Lambda_r^{(h)}}{\Lambda_r^{(k)}} \ge \delta^{\,h-k} \cdot \frac{1}{\Lambda_r^{(k)}} \sum_{\substack{i\in I_r \\ a_{i,r}\ge\delta}} a_{i,r}^k\, p_i,$$

and by (2) the right-hand side is bounded below by a positive constant. Hence $\sup_r \Lambda_r^{(k)}/\Lambda_r^{(h)} < \infty$, i.e. (1) holds. Now apply Lemma 1. ☐

5.1. A Freedman-type comparison between classical strong Riesz means and $N_{\theta,R}^{(k)}$. We establish inclusion relations between the lacunary space $N_{\theta,R}^{(k)}$ and a natural *classical* strong Riesz space of order k, in the spirit of the classical comparison results of Freedman–Sember–Raphael [3] for strong Cesàro and N_θ.

Throughout, let $\lambda = (\lambda_n)$ be strictly increasing with $\lambda_n \to \infty$, and $p_i := \lambda_i - \lambda_{i-1} > 0$. We also assume $\lambda_0 = 0$ for convenience.

Definition 3 (Classical strong Riesz method of order k)**.** Fix an integer $k \ge 0$. For $n \ge 1$ and $1 \le i \le n$, define

$$a_{i,n}^{(k)} := \left(\frac{\lambda_n - \lambda_i}{\lambda_n - \lambda_0}\right)^k = \left(\frac{\lambda_n - \lambda_i}{\lambda_n}\right)^k \in [0,1].$$

Set

$$W_n^{(k)} := \sum_{i=1}^{n} a_{i,n}^{(k)}\, p_i.$$

A sequence $x = (x_i)$ is said to be *strongly Riesz convergent of order k to L*, and we write $x \xrightarrow{\mathcal{R}^{(k)}} L$, if

$$\lim_{n\to\infty} \frac{1}{W_n^{(k)}} \sum_{i=1}^{n} a_{i,n}^{(k)}\, p_i\, |x_i - L| = 0.$$

Denote by $\mathcal{R}^{(k)}$ the class of all sequences possessing this property.

This definition is classical and goes back to Riesz [11] (see also Hardy [5]).

Remark 2. For $n = k_r$ and $i \in I_r = (k_{r-1}, k_r]$, we have the exact relation

$$a_{i,k_r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_{k_{r-1}}}{\lambda_{k_r}}\right)^k \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}}\right)^k = c_r^{(k)}\, \varkappa_{i,r}^{(k)},$$

where

$$c_r^{(k)} := \left(\frac{\lambda_{k_r} - \lambda_{k_{r-1}}}{\lambda_{k_r}}\right)^k.$$

Consequently,

$$\sum_{i\in I_r} a_{i,k_r}^{(k)}\, p_i = c_r^{(k)} \sum_{i\in I_r} \varkappa_{i,r}^{(k)}\, p_i = c_r^{(k)}\, \Lambda_r^{(k)}.$$

5.2. From classical strong Riesz to lacunary strong Riesz.

Theorem 3 (Classical-to-lacunary inclusion). *Fix $k \geq 0$ and assume there exists $C > 0$ such that, for all $r \geq 1$,*

$$W_{k_r}^{(k)} \leq C\, c_r^{(k)}\, \Lambda_r^{(k)}. \tag{3}$$

Then

$$\mathcal{R}^{(k)} \subset N_{\theta,R}^{(k)}.$$

Proof. Let $x \xrightarrow{\mathcal{R}^{(k)}} L$. Using Remark 2 and positivity of the weights, for each r we obtain

$$\frac{1}{\Lambda_r^{(k)}} \sum_{i\in I_r} \varkappa_{i,r}^{(k)} p_i |x_i - L| = \frac{1}{c_r^{(k)} \Lambda_r^{(k)}} \sum_{i\in I_r} a_{i,k_r}^{(k)} p_i |x_i - L| \leq \frac{1}{c_r^{(k)} \Lambda_r^{(k)}} \sum_{i=1}^{k_r} a_{i,k_r}^{(k)} p_i |x_i - L|.$$

Divide and multiply by $W_{k_r}^{(k)}$ to get

$$\frac{1}{\Lambda_r^{(k)}} \sum_{i\in I_r} \varkappa_{i,r}^{(k)} p_i |x_i - L| \leq \frac{W_{k_r}^{(k)}}{c_r^{(k)} \Lambda_r^{(k)}} \cdot \frac{1}{W_{k_r}^{(k)}} \sum_{i=1}^{k_r} a_{i,k_r}^{(k)} p_i |x_i - L|.$$

By (3), the first factor is uniformly bounded, and the second factor tends to 0 along $n = k_r$ since $x \xrightarrow{\mathcal{R}^{(k)}} L$. Hence $x \xrightarrow{N_{\theta,R}^{(k)}} L$. $\square$

Remark 3. Condition (3) is a weighted analogue of the classical boundedness condition on lacunary ratios in the Cesàro case. It prevents the classical normalization $W_{k_r}^{(k)}$ from being excessively large compared to the effective contribution of the last lacunary block.

5.3. Counterexamples and sharpness (general k and general λ). In this subsection, we show that the additional assumptions imposed in the Freedman-type comparison theorems cannot, in general, be removed. All examples are given for arbitrary $k \geq 0$ and a general strictly increasing sequence $\lambda = (\lambda_n)$ with $\lambda_n \to \infty$.

Recall that $p_i = \lambda_i - \lambda_{i-1}$ and

$$\varkappa_{i,r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r} - \lambda_{k_{r-1}}} \right)^k .$$

Proposition 1 (Classical strong Riesz but not $N_{\theta,R}^{(k)}$)**.** *Let $\theta = (k_r)$ be lacunary and assume that λ satisfies (Δ_2). Suppose that there exists a subsequence (r_j) such that*

$$\frac{\lambda_{k_{r_j}} - \lambda_{k_{r_j-1}}}{\lambda_{k_{r_j}}} \longrightarrow 0 \qquad (j \to \infty). \tag{4}$$

Fix any $\delta \in (0,1)$ and, for each j, choose an index $i_j \in I_{r_j} = (k_{r_j-1}, k_{r_j}]$ satisfying

$$\frac{\lambda_{k_{r_j}} - \lambda_{i_j}}{\lambda_{k_{r_j}} - \lambda_{k_{r_j-1}}} \geq \delta. \tag{5}$$

Define $x = (x_i)$ by

$$x_{i_j} := \frac{\lambda_{k_{r_j}} - \lambda_{k_{r_j-1}}}{p_{i_j}}, \qquad x_i := 0 \textit{ for all other } i.$$

Then $x \xrightarrow{\mathcal{R}^{(k)}} 0$, but $x \notin N_{\theta,R}^{(k)}$.

Proof. Step 1: failure of lacunary strong Riesz convergence. For $r = r_j$, the block I_r contains exactly one nonzero term, located at $i = i_j$. By (5),

$$\varkappa_{i_j,r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_{i_j}}{\lambda_{k_r} - \lambda_{k_{r-1}}} \right)^k \geq \delta^k.$$

Therefore,

$$\frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} \varkappa_{i,r}^{(k)} p_i |x_i| = \frac{1}{\Lambda_r^{(k)}} \varkappa_{i_j,r}^{(k)} p_{i_j} |x_{i_j}| \geq \delta^k \, \frac{\lambda_{k_r} - \lambda_{k_{r-1}}}{\Lambda_r^{(k)}}.$$

Since $0 \le \varkappa_{i,r}^{(k)} \le 1$ for all $i \in I_r$, we have

$$\Lambda_r^{(k)} = \sum_{i \in I_r} \varkappa_{i,r}^{(k)} p_i \le \sum_{i \in I_r} p_i = \lambda_{k_r} - \lambda_{k_{r-1}}.$$

Hence

$$\frac{1}{\Lambda_r^{(k)}} \sum_{i \in I_r} \varkappa_{i,r}^{(k)} p_i |x_i| \ge \delta^k > 0,$$

so the lacunary strong Riesz means cannot converge to 0. Thus $x \notin N_{\theta,R}^{(k)}$.

Step 2: classical strong Riesz convergence to 0. Fix j and set $n = k_{r_j}$. The numerator of the classical mean has exactly one nonzero term:

$$\sum_{i=1}^{n} a_{i,n}^{(k)} p_i |x_i| = a_{i_j,n}^{(k)} p_{i_j} |x_{i_j}|.$$

Using $a_{i_j,n}^{(k)} = \big(\frac{\lambda_n - \lambda_{i_j}}{\lambda_n}\big)^k$ and $p_{i_j}|x_{i_j}| = \lambda_n - \lambda_{k_{r_j-1}}$, we get

$$\sum_{i=1}^{n} a_{i,n}^{(k)} p_i |x_i| = \left(\frac{\lambda_n - \lambda_{i_j}}{\lambda_n}\right)^k \big(\lambda_n - \lambda_{k_{r_j-1}}\big) \le \big(\lambda_n - \lambda_{k_{r_j-1}}\big),$$

since $0 \le a_{i_j,n}^{(k)} \le 1$.

On the other hand, by Lemma 2 (which holds under (Δ_2)), there exists $c_k > 0$ such that $W_n^{(k)} \ge c_k \lambda_n$ for all n. Therefore

$$\frac{1}{W_n^{(k)}} \sum_{i=1}^{n} a_{i,n}^{(k)} p_i |x_i| \le \frac{1}{c_k} \cdot \frac{\lambda_n - \lambda_{k_{r_j-1}}}{\lambda_n} \longrightarrow 0$$

by (4). Hence $x \xrightarrow{\mathcal{R}^{(k)}} 0$. ☐

Remark 4. Proposition 1 shows that, in general, the implication

$$\mathcal{R}^{(k)} \subset N_{\theta,R}^{(k)}$$

fails without an additional hypothesis linking the *global* classical normalization $W_{k_r}^{(k)}$ to the *local* weight concentrated on the last lacunary block.

Indeed, the construction places a single spike inside I_{r_j} so that the lacunary mean captures it with a uniformly positive contribution (because $\varkappa_{i,r}^{(k)}$ stays bounded away from 0 on a fixed interior portion of each block), whereas the classical strong Riesz mean at $n = k_{r_j}$ normalizes by a quantity of order $\lambda_{k_{r_j}}$ and therefore washes out the spike precisely under (4). Consequently, a comparability condition of the form (3) (or any equivalent control preventing the last block from being negligible relative to the full initial segment) is not merely technical but genuinely necessary. This is the natural weighted analogue of the classical lacunary–Cesàro phenomenon in the sense of Freedman–Sember–Raphael [3].

5.4. From lacunary strong Riesz to classical strong Riesz.

Definition 4 (A mild regularity assumption on λ)**.** We say that $\lambda = (\lambda_n)$ satisfies the *doubling condition* (Δ_2) if there exists a constant $D \geq 1$ such that

$$\lambda_{2n} \leq D\,\lambda_n \text{ for all } n \in \mathbb{N}. \tag{6}$$

Remark 5. Condition (Δ_2) is a standard doubling-type regularity assumption. Iterating (6) yields $\lambda_{2^m} \leq D^m \lambda_1$, and hence $\lambda_n = O\big(n^{\log_2 D}\big)$ as $n \to \infty$. In particular, (Δ_2) rules out exponential growth and faster, while it is satisfied by many classical choices of weights in summability theory (e.g. $\lambda_n = n^\alpha$, regularly varying sequences, and other polynomial-type regimes).

Lemma 2 (A uniform lower bound for $W_n^{(k)}$ under (Δ_2))**.** *Assume* (Δ_2)*. Fix* $k \geq 0$ *and define*

$$a_{i,n}^{(k)} = \Big(1 - \frac{\lambda_i}{\lambda_n}\Big)^k, \qquad W_n^{(k)} = \sum_{i=1}^{n} a_{i,n}^{(k)} p_i, \quad p_i = \lambda_i - \lambda_{i-1}.$$

Then there exists a constant $c_k > 0$ *(depending only on* k *and* D*) such that*

$$W_n^{(k)} \geq c_k\,\lambda_n \qquad \textit{for all } n \in \mathbb{N}. \tag{7}$$

Proof. Fix $n \geq 2$ and let $m = \lfloor n/2 \rfloor$. Since λ is increasing,

$$W_n^{(k)} \geq \sum_{i=1}^{m} a_{i,n}^{(k)} p_i.$$

For $1 \leq i \leq m$, we have $\lambda_i \leq \lambda_m$. Since $n \leq 2m + 1 \leq 4m$ for all $m \geq 1$, the doubling condition applied twice gives

$$\lambda_n \leq \lambda_{4m} \leq D^2 \lambda_m.$$

Hence

$$\frac{\lambda_m}{\lambda_n} \leq 1 \qquad \text{and} \qquad \frac{\lambda_m}{\lambda_n} \geq \frac{1}{D^2}.$$

Therefore,

$$1 - \frac{\lambda_i}{\lambda_n} \geq 1 - \frac{\lambda_m}{\lambda_n} \geq 1 - \frac{1}{D^2}.$$

Consequently,

$$W_n^{(k)} \geq \Big(1 - \frac{1}{D^2}\Big)^k \sum_{i=1}^{m} p_i = \Big(1 - \frac{1}{D^2}\Big)^k \lambda_m.$$

Using again $\lambda_n \leq D^2 \lambda_m$, we obtain

$$W_n^{(k)} \geq D^{-2} \Big(1 - \frac{1}{D^2}\Big)^k \lambda_n.$$

Thus (7) holds with

$$c_k = D^{-2}\left(1-\frac{1}{D^2}\right)^k.$$

□

Theorem 4 (Lacunary-to-classical inclusion along (k_r))**.** *Fix $k \geq 0$ and assume that λ satisfies (Δ_2). Let $\theta = (k_r)$ be lacunary and let $x = (x_i)$ be bounded. If $x \xrightarrow{N_{\theta,R}^{(k)}} L$, then*

$$\lim_{r\to\infty} \frac{1}{W_{k_r}^{(k)}} \sum_{i=1}^{k_r} a_{i,k_r}^{(k)}\, p_i\, |x_i - L| = 0.$$

Remark 6. The conclusion above is obtained only along the lacunary indices $n = k_r$. In general, $N_{\theta,R}^{(k)}$-convergence does *not* force $\mathcal{R}^{(k)}$-convergence for all n.

Proof of the Theorem 4. Let $M := \sup_i |x_i - L| < \infty$. Fix $\varepsilon > 0$. Since $x \in N_{\theta,R}^{(k)}$, there exists r_0 such that for all $j \geq r_0$,

$$\frac{1}{\Lambda_j^{(k)}} \sum_{i\in I_j} \varkappa_{i,j}^{(k)} p_i |x_i - L| < \varepsilon. \tag{8}$$

Fix $r > r_0$ and consider the classical Riesz mean at $n = k_r$. We decompose the numerator into lacunary blocks:

$$\sum_{i=1}^{k_r} a_{i,k_r}^{(k)} p_i |x_i - L| = \sum_{j=1}^{r} \sum_{i\in I_j} a_{i,k_r}^{(k)} p_i |x_i - L|.$$

For $i \in I_j$, we have $\lambda_i \geq \lambda_{k_{j-1}}$ and $\lambda_i \leq \lambda_{k_j}$, hence

$$a_{i,k_r}^{(k)} = \left(\frac{\lambda_{k_r} - \lambda_i}{\lambda_{k_r}}\right)^k \leq \left(\frac{\lambda_{k_r} - \lambda_{k_{j-1}}}{\lambda_{k_r}}\right)^k =: A_{j,r}.$$

Therefore,

$$\sum_{i\in I_j} a_{i,k_r}^{(k)} p_i |x_i - L| \leq A_{j,r} \sum_{i\in I_j} p_i |x_i - L|.$$

Now split each block I_j into a “good” part where $\varkappa_{i,j}^{(k)}$ is bounded below and a small endpoint strip. Fix $\delta \in (0, 1/2)$ and define

$$I_j(\delta) := \left\{ i \in I_j :\ \frac{\lambda_{k_j} - \lambda_i}{\lambda_{k_j} - \lambda_{k_{j-1}}} \geq \delta \right\}, \qquad J_j(\delta) := I_j \setminus I_j(\delta).$$

For $i \in I_j(\delta)$ we have $\varkappa_{i,j}^{(k)} \geq \delta^k$, hence

$$\sum_{i\in I_j(\delta)} p_i |x_i - L| \leq \delta^{-k} \sum_{i\in I_j} \varkappa_{i,j}^{(k)} p_i |x_i - L| \leq \delta^{-k} \varepsilon\, \Lambda_j^{(k)} \quad (j \geq r_0),$$

by (8). For the endpoint strip we only use boundedness:

$$\sum_{i\in J_j(\delta)} p_i|x_i-L| \le M \sum_{i\in J_j(\delta)} p_i.$$

But by definition of $J_j(\delta)$, every $i\in J_j(\delta)$ satisfies $\lambda_i > \lambda_{k_j}-\delta(\lambda_{k_j}-\lambda_{k_{j-1}})$. Hence

$$\sum_{i\in J_j(\delta)} p_i = \lambda_{k_j} - \lambda_{\min J_j(\delta)-1} \le \delta(\lambda_{k_j}-\lambda_{k_{j-1}}).$$

Combining, for $j \ge r_0$ we obtain

$$\sum_{i\in I_j} p_i|x_i-L| \le \delta^{-k}\varepsilon\,\Lambda_j^{(k)} + M\delta(\lambda_{k_j}-\lambda_{k_{j-1}}).$$

Thus

$$\sum_{i\in I_j} a_{i,k_r}^{(k)} p_i|x_i-L| \le A_{j,r}\Big(\delta^{-k}\varepsilon\,\Lambda_j^{(k)} + M\delta(\lambda_{k_j}-\lambda_{k_{j-1}})\Big) \qquad (j\ge r_0).$$

Summing over $j=r_0,\dots,r$ and adding the finitely many initial blocks $j<r_0$, we get

$$\sum_{i=1}^{k_r} a_{i,k_r}^{(k)} p_i|x_i-L| \le C_0 + \delta^{-k}\varepsilon\sum_{j=r_0}^{r} A_{j,r}\Lambda_j^{(k)} + M\delta\sum_{j=r_0}^{r} A_{j,r}(\lambda_{k_j}-\lambda_{k_{j-1}}),$$

where $C_0 := M\sum_{i=1}^{k_{r_0}-1} p_i = M\lambda_{k_{r_0-1}}$ is independent of r.

Now divide by $W_{k_r}^{(k)}$. By Lemma 2 (under (Δ_2)) we have $W_{k_r}^{(k)} \ge c_k\lambda_{k_r}$. Since $0\le \varkappa_{i,j}^{(k)}\le 1$ for all $i\in I_j$, we have

$$\Lambda_j^{(k)} = \sum_{i\in I_j}\varkappa_{i,j}^{(k)} p_i \le \sum_{i\in I_j} p_i = \lambda_{k_j}-\lambda_{k_{j-1}}.$$

Also, $0\le A_{j,r}\le 1$, so

$$\sum_{j=r_0}^{r} A_{j,r}\Lambda_j^{(k)} \le \sum_{j=r_0}^{r}(\lambda_{k_j}-\lambda_{k_{j-1}}) = \lambda_{k_r}-\lambda_{k_{r_0-1}} \le \lambda_{k_r}.$$

Similarly,

$$\sum_{j=r_0}^{r} A_{j,r}(\lambda_{k_j}-\lambda_{k_{j-1}}) \le \lambda_{k_r}.$$

Hence

$$\frac{1}{W_{k_r}^{(k)}}\sum_{i=1}^{k_r} a_{i,k_r}^{(k)} p_i|x_i-L| \le \frac{C_0}{c_k\lambda_{k_r}} + \frac{\delta^{-k}}{c_k}\varepsilon + \frac{M}{c_k}\delta.$$

Letting $r\to\infty$ gives $C_0/(c_k\lambda_{k_r})\to 0$. Now choose $\delta>0$ so small that $\frac{M}{c_k}\delta<\varepsilon$. As $\varepsilon>0$ was arbitrary, we conclude that the right-hand side tends to 0, completing the proof. □

6. Conclusion

We introduced the lacunary strong Riesz spaces $N_{\theta,R}^{(k)}$ and investigated their fundamental structural properties. Precise inclusion relations between classical and lacunary strong Riesz summability were established under natural comparability and growth conditions. The necessity of these assumptions was demonstrated by explicit counterexamples, showing that the obtained results are essentially sharp.

The present approach provides a flexible setting for further investigations, including Tauberian theorems for weighted lacunary methods and applications to convergence problems in matrix domains and operator theory. Our present construction could also be extended to more general settings, including double sequences [14], sequences of sets [4], and Fourier series [12], where strong Riesz-type summability has recently attracted attention. These directions will be addressed in future work.

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