

A note on attaining diameter two properties in Lipschitz-free spaces

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ABSTRACT. We prove that in Lipschitz-free spaces the strong diameter two property, the diameter two property, and the local diameter two property coincide with their corresponding attaining variants.

1. Introduction

Let M be a metric space with a distinguished point $0 \in M$. We denote by $\text{Lip}_0(M)$ the Banach space of all real-valued Lipschitz functions vanishing at 0, equipped with the norm

$$\|f\| = \sup \left\{ \frac{|f(p) - f(q)|}{d(p, q)} : p, q \in M, p \neq q \right\}.$$

Let $\delta : M \rightarrow \text{Lip}_0(M)^*$ denote the canonical embedding, defined by

$$\langle f, \delta(p) \rangle = f(p), \quad p \in M, f \in \text{Lip}_0(M).$$

It is well known that $\text{Lip}_0(M)$ is a dual space whose canonical predual is *the Lipschitz-free space* $\mathcal{F}(M)$ defined by

$$\mathcal{F}(M) = \overline{\text{span}} \delta(M) \subset \text{Lip}_0(M)^*.$$

For $p, q \in M$ with $p \neq q$, we denote by

$$m_{p,q} = \frac{\delta(p) - \delta(q)}{d(p, q)}$$

a norm one elementary molecule in $\mathcal{F}(M)$. For background, we refer to [6] and [15].

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In these spaces we are interested in the following properties. We say that a Banach space X has the *Daugavet property*, if for every $x \in S_X$, every slice S of B_X , and every $\varepsilon > 0$, there exists $y \in S$ such that

$$\|x - y\| \geq 2 - \varepsilon.$$

Following [1], a Banach space X has the *local diameter two property (LD2P)*, the *diameter two property (D2P)*, or the *strong diameter two property (SD2P)* if every slice of B_X , every nonempty weakly open subset of B_X , or every convex combination of slices of B_X , has diameter two, respectively. In general,

$$\text{Daugavet property} \Rightarrow \text{SD2P} \Rightarrow \text{D2P} \Rightarrow \text{LD2P},$$

and the implications are strict.

One may also consider the corresponding *attaining* variants (ALD2P, AD2P, and ASD2P), where the diameter two is required to be attained. Analogous implications hold for these properties and they are strictly stronger than their non-attaining counterparts (see [4] and [12]).

In Lipschitz-free space, diameter two properties are remarkably rigid. It follows from [2, 5, 7, 10] that for a complete metric space M , the following assertions are equivalent:

- M is a length space;
- $\mathcal{F}(M)$ has the Daugavet property;
- $\mathcal{F}(M)$ has the SD2P;
- $\mathcal{F}(M)$ has the D2P;
- $\mathcal{F}(M)$ has the LD2P;
- $\mathcal{F}(M)$ is locally almost square;
- the unit ball of $\mathcal{F}(M)$ does not have strongly exposed points.

It is therefore natural to ask whether the attaining variants also fall under the same characterisation. We answer this affirmatively.

Theorem 1. *Let M be a complete metric space. The following assertions are equivalent:*

- (1) M is a length space;
- (2) $\mathcal{F}(M)$ has the Daugavet property;
- (3) $\mathcal{F}(M)$ has the SD2P;
- (4) $\mathcal{F}(M)$ has the D2P;
- (5) $\mathcal{F}(M)$ has the LD2P;
- (6) $\mathcal{F}(M)$ has the ASD2P;
- (7) $\mathcal{F}(M)$ has the AD2P;
- (8) $\mathcal{F}(M)$ has the ALD2P.

The equivalence of (1)–(5) is known from previously cited results. Thus, it remains to show that if M is a length space, then $\mathcal{F}(M)$ has the ASD2P. We prove this in Theorem 2.

It is an open question whether every Banach space with the Daugavet property also has the ASD2P (see [4, Question 6.10]). Theorem 1 provides a partial positive answer to this question for Lipschitz-free spaces.

One may also consider other attaining variants of related properties, in which the diameter is required to be attained. One such property is the *perfect Daugavet property*, requiring that the distance two in the definition of the Daugavet property is attained (see [11, page 212]). It would be interesting to know whether there exists a Lipschitz-free space with this property.

We note that a separable example cannot exist, since separable Lipschitz-free spaces already lack the attaining diametral local diameter two property (see [3] for background on diametral diameter two properties). Indeed, if M is separable and $\{p_n : n \in \mathbb{N}\} \subset M \setminus \{0\}$ is dense in M , then the element

$$\mu = \sum_{n=1}^{\infty} 2^{-n} m_{p_n,0} \in S_{\mathcal{F}(M)}$$

admits a unique norming functional $d(\cdot, 0)$. Consider the slice S determined by the functional $d(\cdot, 0)$ and 1. If $\mathcal{F}(M)$ would have the perfect Daugavet property, then there exist $\nu \in S$ and $f \in S_{\text{Lip}_0(M)}$ so that $\langle f, \mu \rangle = 1$ and $\langle f, \nu \rangle = -1$. But then $f(\cdot) = d(\cdot, 0)$, which contradicts $\nu \in S$.

Analogous questions arise for spaces of Lipschitz functions. As these are dual Banach spaces, one can also consider weak*-slices and weak*-open sets, and define the corresponding weak*-diameter two properties. Previous work in spaces of Lipschitz functions has provided characterisations of weak*-diameter two properties (see [9], [13], [14]) and established necessary conditions for diameter two properties (see [8]). To the best of our knowledge, the attaining variants in these spaces have not been investigated.

Notation. We only consider real Banach spaces. For a Banach space X we denote the closed unit ball by B_X , the unit sphere by S_X , and the dual space by X^* . Given $x^* \in S_{X^*}$ and $\alpha > 0$, the slice of B_X , determined by x^* and α , is

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\}.$$

We only consider complete metric spaces. Let M be a metric space. A continuous mapping $\gamma : [0, 1] \rightarrow M$ is called a *path* joining $\gamma(0)$ and $\gamma(1)$. Its length is defined by

$$L(\gamma) = \sup \left\{ \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})) : n \in \mathbb{N}, 0 = t_0 \leq \dots \leq t_n = 1 \right\}.$$

The metric space M is a *length space* if for every $p, q \in M$,

$$d(p, q) = \inf \{L(\gamma) : \gamma \text{ is a path joining } p \text{ and } q\}.$$

Given p in M and $r > 0$, we denote by $B(p, r)$ the open ball in M centred at p of radius r .

2. Proof of the main theorem

We begin with an auxiliary lemma.

Lemma 1. *Let M be a length space, $f \in S_{\text{Lip}_0(M)}$, $\varepsilon > 0$, $r > 0$, and let $x, y \in M$ with $x \neq y$ satisfy $\langle f, m_{x,y} \rangle > 1 - \varepsilon$ and $r < d(x, y)$. For every $\gamma : [0, 1] \rightarrow M$ with $\gamma(0) = x$ and $\gamma(1) = y$, we have*

$$\langle f, m_{u,v} \rangle > 1 - \frac{L(\gamma) - (1 - \varepsilon)d(x, y)}{r}$$

for any $u, v \in \text{Im } \gamma$ satisfying $d(u, v) \geq r$ and $d(x, u) < d(x, v)$.

Proof. The proof follows from a decomposition argument of $\delta(x) - \delta(y)$ along the path γ . Fix $u, v \in \text{Im } \gamma$ satisfying $d(u, v) \geq r$ and $d(x, u) < d(x, v)$. Since

$$\delta(x) - \delta(y) = (\delta(x) - \delta(u)) + (\delta(u) - \delta(v)) + (\delta(v) - \delta(y)),$$

we have

$$\begin{aligned} \langle f, m_{x,y} \rangle &= \frac{\langle f, \delta(x) - \delta(y) \rangle}{d(x, y)} \\ &= \frac{\langle f, \delta(x) - \delta(u) \rangle}{d(x, y)} + \frac{\langle f, \delta(u) - \delta(v) \rangle}{d(x, y)} + \frac{\langle f, \delta(v) - \delta(y) \rangle}{d(x, y)} \\ &= \frac{d(x, u)}{d(x, y)} \langle f, m_{x,u} \rangle + \frac{d(u, v)}{d(x, y)} \langle f, m_{u,v} \rangle + \frac{d(v, y)}{d(x, y)} \langle f, m_{v,y} \rangle. \end{aligned}$$

Since $\|f\| = 1$, we have $\langle f, m_{x,u} \rangle \leq 1$ and $\langle f, m_{v,y} \rangle \leq 1$. Therefore,

$$1 - \varepsilon < \langle f, m_{x,y} \rangle \leq \frac{d(x, u)}{d(x, y)} + \frac{d(u, v)}{d(x, y)} \langle f, m_{u,v} \rangle + \frac{d(v, y)}{d(x, y)}.$$

Rearranging gives

$$\langle f, m_{u,v} \rangle > (1 - \varepsilon) \frac{d(x, y)}{d(u, v)} - \frac{d(x, u) + d(v, y)}{d(u, v)}.$$

Finally, because $d(u, v) \geq r$ and

$$d(x, u) + d(v, y) \leq L(\gamma) - d(u, v),$$

we obtain

$$\langle f, m_{u,v} \rangle > 1 - \frac{L(\gamma) - (1 - \varepsilon)d(x, y)}{d(u, v)} \geq 1 - \frac{L(\gamma) - (1 - \varepsilon)d(x, y)}{r}.$$

□

We are now ready to prove the main result of this note.

Theorem 2. *Let M be a length space. Then $\mathcal{F}(M)$ has the ASD2P.*

Proof. Assume M is a length space. Let $n \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n > 0$ with $\sum_i \lambda_i = 1$, let $f_1, \dots, f_n \in S_{\text{Lip}_0(M)}$, and let $\alpha > 0$. For each $i \in \{1, \dots, n\}$ consider the slice $S_i = S(f_i, \alpha)$. It suffices to find $\mu, \nu \in \sum_{i=1}^n \lambda_i S_i$ so that $\|\mu - \nu\| = 2$.

Fix $i \in \{1, \dots, n\}$ and let

$$\varepsilon = \frac{\alpha}{18n}.$$

Find $u_i, v_i \in M$ with $u_i \neq v_i$ such that $\langle f_i, m_{u_i, v_i} \rangle > 1 - \varepsilon$. Denote

$$r_i = \frac{d(u_i, v_i)}{9n}$$

and assume without loss of generality that $r_1 \leq \dots \leq r_n$. Since M is length, we choose a path $\gamma_i : [0, 1] \rightarrow M$ joining u_i to v_i such that

$$L(\gamma_i) < (1 + \varepsilon)d(u_i, v_i) = \alpha r_i + (1 - \varepsilon)d(u_i, v_i),$$

implying

$$\frac{L(\gamma_i) - (1 - \varepsilon)d(u_i, v_i)}{r_i} < \alpha. \quad (1)$$

Next, we inductively construct points $x_i, y_i \in \text{Im } \gamma_i$ such that

$$\min\{d(x_i, x_j), d(x_i, y_j), d(y_i, y_j) : i > j\} \geq 2r_i \quad (2)$$

and

$$\min\{d(x_i, v_i), d(y_i, u_i), d(x_i, y_i) : i = 1, \dots, n\} \geq 2r. \quad (3)$$

For $i = 1$ we set $x_1 = u_1$ and $y_1 = v_1$. Let $i \in \{2, \dots, n\}$ and assume that we have defined points $x_j, y_j \in M$ for every $j < i$. To choose x_i , we consider the set

$$F_i = \{x_1, y_1, \dots, x_{i-1}, y_{i-1}, v_i\}.$$

We claim that

$$\text{Im } \gamma_i \setminus \bigcup_{p \in F_i} B(p, 2r_i) \neq \emptyset.$$

Indeed, if $\text{Im } \gamma_i \subset \bigcup_{p \in F_i} B(p, 2r_i)$, then

$$L(\gamma_i) \leq \sum_{p \in F_i} \text{diam}(B(p, 2r_i)) \leq |F_i| \cdot 4r_i \leq 8nr_i = \frac{8}{9}d(u_i, v_i),$$

contradicting $L(\gamma_i) \geq d(u_i, v_i)$. Thus we may choose $x_i \in \text{Im } \gamma_i$ with $d(x_i, F_i) \geq 2r_i$.

Next we define

$$G_i = \{x_1, y_1, \dots, x_{i-1}, y_{i-1}, x_i, u_i\}.$$

The same argument shows that

$$\text{Im } \gamma_i \setminus \bigcup_{p \in G_i} B(p, 2r_i) \neq \emptyset,$$

so we may choose $y_i \in \text{Im } \gamma_i$ with $d(y_i, G_i) \geq 2r_i$. This completes the induction and establishes (2) and (3).

Fix $i \in \{1, \dots, n\}$. By (3), we can choose points $z_i, w_i \in \text{Im } \gamma_i$ such that

$$d(x_i, z_i) = r_i, \quad d(u_i, x_i) < d(u_i, z_i),$$

and

$$d(y_i, w_i) = r_i, \quad d(u_i, w_i) < d(u_i, y_i).$$

Applying Lemma 1 to the path γ_i (with $x = u_i$, $y = v_i$, and r_i as above), and using (2.1), we obtain

$$\langle f_i, m_{x_i, z_i} \rangle > 1 - \alpha \quad \text{and} \quad \langle f_i, m_{w_i, y_i} \rangle > 1 - \alpha.$$

Hence $m_{x_i, z_i}, m_{w_i, y_i} \in S_i$.

Now we are ready to define

$$\mu = \sum_{i=1}^n \lambda_i m_{x_i, z_i} \quad \text{and} \quad \nu = \sum_{i=1}^n \lambda_i m_{w_i, y_i}.$$

Then $\mu, \nu \in \sum_{i=1}^n \lambda_i S_i$.

Define $g : M \rightarrow \mathbb{R}$ by

$$g(p) = \max\{r_i - d(x_i, p), r_i - d(y_i, p) : i = 1, \dots, n\}.$$

Each function $p \mapsto r_i - d(x_i, p)$ and $p \mapsto r_i - d(y_i, p)$ is 1-Lipschitz, hence so is g . Set $\tilde{g}(p) = g(p) - g(0)$. Then $\tilde{g} \in S_{\text{Lip}_0(M)}$.

By (2) and (3), all of the balls $B(x_i, r)$ and $B(y_i, r)$ are pairwise disjoint. Moreover,

$$g(x_i) = r_i, \quad g(z_i) = 0, \quad g(y_i) = r_i, \quad g(w_i) = 0.$$

Therefore,

$$\langle \tilde{g}, m_{x_i, z_i} \rangle = 1, \quad \langle \tilde{g}, m_{w_i, y_i} \rangle = -1,$$

and consequently,

$$\|\mu - \nu\| \geq \langle \tilde{g}, \mu - \nu \rangle = \sum_{i=1}^n \lambda_i (1 - (-1)) = 2.$$

□

Remark 1. Small modifications to the previous proof imply that every convex combination of slices of $B_{\mathcal{F}(M)}$ contains a square point (see [12, Definition 3.1]).

We can require that (2) and (3) hold with $3r_i$. It then follows that we define $\mu, \nu \in \sum \lambda_i S_i$ such that $\|\mu \pm \nu\| = 2$ and thus

$$\left\| \frac{\mu + \nu}{2} + \frac{\mu - \nu}{2} \right\| = \left\| \frac{\mu + \nu}{2} - \frac{\mu - \nu}{2} \right\| = 1.$$

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