ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 30, Number 1, June 2026
Available online at https://ojs.utlib.ee/index.php/ACUTM

Stability of properties α and β under absolute sums of Banach spaces

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Abstract. We study the stability of properties α and β under absolute sums of two real Banach spaces. We prove that if N is an absolute normalised norm on $\mathbb{R}^2$ whose unit sphere is polygonal, then $X \oplus_N Y$ has property α whenever both X and Y have property α, and similarly for property β.

We also obtain partial converse results. For property α, if $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$ and $X \oplus_N Y$ has property α, then X has property α. For property β, if $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$ and $X \oplus_N Y$ has property β, then X has property β. As corollaries, we recover the finite ℓ_1- and ℓ_∞-cases and obtain corresponding equivalence results.

1. Introduction

Bishop and Phelps proved in [1] that norm-attaining functionals are dense in the dual of every Banach space. Motivated by the corresponding question for operators, Lindenstrauss introduced properties A and B in [2]. Later, Schachermayer defined the stronger properties α and β in [4].

In the present paper we study the behaviour of properties α and β under absolute sums of two real Banach spaces. Our main interest is in determining when these properties pass from the summands to the sum, and when one may recover the property of a summand from the corresponding property of the sum.

Our first main result shows that polygonal absolute norms provide a common framework for positive stability results. More precisely, if N is an absolute normalised norm on $\mathbb{R}^2$ whose unit sphere is polygonal, then $X \oplus_N Y$ has property α whenever both X and Y have property α, and likewise $X \oplus_N Y$ has property β whenever both X and Y have property β. In particular, this

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Received April 14, 2026.
2020 *Mathematics Subject Classification.* 46B20, 47A05.
Key words and phrases. Absolute sums, norm-attaining operators, properties α and β.
https://doi.org/10.12697/ACUTM.2026.30.12

recovers the finite ℓ_1- and ℓ_∞-cases. Geometrically, the ℓ_1-case corresponds to the situation where the coordinate points are extreme, whereas the ℓ_∞-case corresponds to the complementary geometry in which they fail to be extreme.

Our second main theme is the converse direction. Here the geometry of the coordinate points $(1,0)$ and $(0,1)$ in $B_{(\mathbb{R}^2,N)}$ becomes important. On the α-side, if $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$ and $X \oplus_N Y$ has property α, then X has property α. On the β-side, if $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$ and $X \oplus_N Y$ has property β, then X has property β. Combining these converse results with the forward stability theorems, we obtain equivalence results in the corresponding geometric settings. It would be interesting to determine to what extent these results extend beyond the polygonal case and to more general absolute sums.

The paper is organised as follows. Section 2 contains the relevant definitions and the basic geometric facts on absolute norms and absolute sums that will be used later. In Section 3, we deal with property α, proving hereditary results for ℓ_1-sums, forward stability under polygonal absolute sums, and partial converse results. Section 4 treats property β in a parallel way.

2. Preliminaries

Throughout the paper, all Banach spaces are real and non-trivial. For a Banach space X, we denote by B_X and S_X its closed unit ball and unit sphere, respectively. When discussing the quadrants of $\mathbb{R}^2$, we consider the axes to be part of the adjacent quadrants, i.e. we define quadrants nonstrictly.

For a subset A of a vector space, we denote by $\operatorname{aconv}(A)$ its absolute convex hull and by $\overline{\operatorname{aconv}}(A)$ its closed absolute convex hull. If I is an index set, then $-I$ denotes a disjoint copy of I, with its elements written as $-i$ for $i \in I$.

We next recall the two geometric properties studied in this paper.

Definition 2.1. A Banach space X is said to have *property* α if there exist a family

$$\{(x_i, x_i^*) : i \in I\} \subset S_X \times S_{X^*}$$

and a constant $\lambda \in [0,1)$ such that:

(i) $x_i^*(x_i) = 1$ for every $i \in I$;
(ii) $|x_i^*(x_j)| \le \lambda$ whenever $i, j \in I$ and $i \neq j$;
(iii) $\overline{\operatorname{aconv}}(\{x_i : i \in I\}) = B_X$.

Definition 2.2. A Banach space Y is said to have *property* β if there exist a family

$$\{(y_i, y_i^*) : i \in I\} \subset S_Y \times S_{Y^*}$$

and a constant $\lambda \in [0,1)$ such that:

(i) $y_i^*(y_i) = 1$ for every $i \in I$;
(ii) $|y_i^*(y_j)| \leq \lambda$ whenever $i, j \in I$ and $i \neq j$;
(iii) for every $y \in Y$, $||y|| = \sup_{i\in I} |y_i^*(y)|$.

We now recall the notion of an absolute norm on $\mathbb{R}^2$.

Definition 2.3. A norm N on $\mathbb{R}^2$ is said to be *absolute* if

$$N(a, b) = N(|a|, |b|) \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

It is said to be *normalised* if

$$N(1, 0) = N(0, 1) = 1.$$

A norm N on $\mathbb{R}^2$ is *polygonal* if its unit sphere $S_{(\mathbb{R}^2,N)}$ is a polygon.

If N is an absolute normalised norm on $\mathbb{R}^2$ and X, Y are Banach spaces, then the absolute sum $X \oplus_N Y$ is the vector space $X \times Y$ equipped with the norm

$$||(x, y)||_N = N(||x||, ||y||).$$

For every absolute normalised norm N on $\mathbb{R}^2$, one has

$$||(a, b)||_\infty \leq N(a, b) \leq ||(a, b)||_1 \quad \text{for all } (a, b) \in \mathbb{R}^2.$$

In particular,

$$N(a, 0) = |a| \quad \text{and} \quad N(0, b) = |b| \quad \text{for all } a, b \in \mathbb{R}.$$

The dual norm of N is the absolute normalised norm N^* on $\mathbb{R}^2$ defined by

$$N^*(c, d) = \sup \{|ac| + |bd| : N(a, b) \leq 1\}.$$

With this notation, one has the canonical isometric identification

$$(X \oplus_N Y)^* = X^* \oplus_{N^*} Y^*.$$

Thus, if $(x^*, y^*) \in (X \oplus_N Y)^*$, then

$$||(x^*, y^*)|| = N^*(||x^*||, ||y^*||).$$

We shall repeatedly use the following pruning lemma.

Lemma 2.4. *Let X be a Banach space and let $B \subset B_X$ satisfy*

$$\overline{\text{aconv}}(B) = B_X.$$

Then for every $\delta > 0$ one has

$$\overline{\text{aconv}}(\{b \in B : ||b|| > 1 - \delta\}) = B_X.$$

Proof. It is enough to prove that every $x \in S_X$ belongs to the closure of the absolute convex hull on the left-hand side. Fix $x \in S_X$ and $\varepsilon > 0$, and choose $\gamma > 0$ so small that

$$\gamma + \frac{\gamma}{\delta} < \varepsilon.$$

Since $x \in \overline{\mathrm{aconv}}(B)$, there exist $b_1, \dots, b_n \in B$ and scalars $\alpha_1, \dots, \alpha_n$ such that

$$\sum_{k=1}^{n} |\alpha_k| \leq 1 \qquad \text{and} \qquad ||x - \sum_{k=1}^{n} \alpha_k b_k|| < \gamma.$$

Set

$$J = \{k \in \{1, \dots, n\} : ||b_k|| \leq 1 - \delta\} \qquad \text{and} \qquad \tau = \sum_{k \in J} |\alpha_k| .$$

Then

$$\begin{aligned} 1 - \gamma < ||\sum_{k=1}^{n} \alpha_k b_k|| &\leq \sum_{k=1}^{n} |\alpha_k| \; ||b_k|| \\ &= \sum_{k \notin J} |\alpha_k| \; ||b_k|| + \sum_{k \in J} |\alpha_k| \; ||b_k|| \\ &\leq (1 - \tau) + \tau(1 - \delta) = 1 - \delta\tau, \end{aligned}$$

hence

$$\tau < \frac{\gamma}{\delta}.$$

Therefore,

$$\begin{aligned} ||x - \sum_{k \notin J} \alpha_k b_k|| &\leq ||x - \sum_{k=1}^{n} \alpha_k b_k|| + ||\sum_{k \in J} \alpha_k b_k|| \\ &< \gamma + \sum_{k \in J} |\alpha_k| \; ||b_k|| \leq \gamma + \tau < \gamma + \frac{\gamma}{\delta} < \varepsilon. \end{aligned}$$

Since each b_k with $k \notin J$ satisfies $||b_k|| > 1 - \delta$, the proof is complete. □

For polygonal absolute norms, we shall also use the following elementary facts about supporting functionals.

Lemma 2.5. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal. Then the first-quadrant part of $S_{(\mathbb{R}^2,N)}$ is a polygonal chain, and each edge of this chain admits a supporting functional $(c, d) \in S_{(\mathbb{R}^2,N)^*}$ with $c, d \geq 0$ which takes the value 1 precisely on that edge and strictly smaller values on all the remaining vertices of the chain.*

Proof. The first-quadrant part of $S_{(\mathbb{R}^2,N)}$ is a compact polygonal chain. Each edge lies on a supporting line of the polygon, and by absoluteness, we may choose a supporting functional with non-negative coordinates. Since the chain has only finitely many vertices, that functional can be chosen so that it attains the value 1 exactly on the given edge and is strictly smaller than 1 at all the remaining vertices of the chain. □

Lemma 2.6. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal, and let*

$$(a_1, b_1), \ldots, (a_R, b_R)$$

be those vertices of $S_{(\mathbb{R}^2,N)}$ which lie in the first quadrant, listed in their natural order along the boundary. Then for every $r \in \{1, \ldots, R\}$, one may choose a functional $(c_r, d_r) \in S_{(\mathbb{R}^2,N)^*}$ with $c_r, d_r \geq 0$ such that

$$c_r a_r + d_r b_r = 1$$

and

$$c_r a_s + d_r b_s < 1 \quad \text{for every } s \in \{1, \ldots, R\} \setminus \{r\}\,.$$

In other words, each first-quadrant vertex of the polygonal chain of $S_{(\mathbb{R}^2,N)}$ may be exposed by a supporting functional which is strictly smaller than 1 at all the other first-quadrant vertices of that chain.

Proof. If (a_r, b_r) is a vertex, then it is the unique intersection point of the two adjacent supporting lines. Any strictly positive convex combination of the corresponding supporting functionals still supports the unit ball at (a_r, b_r), exposes that vertex, and is strictly smaller than 1 at all remaining vertices of the chain. ☐

3. Property α

We begin with a hereditary fact for ℓ_1-sums which clarifies the place of the finite absolute-sum results in the article. Although Propositions 3.1 and 4.1 are essentially contained in the later equivalence results for ℓ_1- and ℓ_∞-sums (Corollaries 3.8 and 4.10) derived from the main theorems, we state them separately in order to emphasize that their proofs are direct and substantially simpler than the proofs of the main theorems.

Proposition 3.1. *Let*

$$Z = \Big(\bigoplus_{\gamma \in \Gamma} X_\gamma \Big)_{\ell_1}.$$

If Z has property α, then X_γ has property α for every $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$ and set

$$Y_\gamma = \Big(\bigoplus_{\eta \in \Gamma \setminus \{\gamma\}} X_\eta \Big)_{\ell_1}.$$

Then Z is canonically isometric to

$$X_\gamma \oplus_1 Y_\gamma.$$

Now fix the witness family

$$\{(z_i, z_i^*) : i \in I\} \subset S_Z \times S_{Z^*}$$

and constant $\lambda \in [0,1)$ for property α of Z. Write

$$z_i = (x_i, y_i) \quad \text{and} \quad z_i^* = (x_i^*, y_i^*)$$

with $x_i \in X_\gamma$, $y_i \in Y_\gamma$ and $x_i^* \in X_\gamma^*, y_i^* \in Y_\gamma^*$. Since

$$Z^* = X_\gamma^* \oplus_\infty Y_\gamma^*,$$

we have $\|z_i^*\| = \max\{\|x_i^*\|, \|y_i^*\|\} = 1$. Moreover,

$$1 = z_i^*(z_i) = x_i^*(x_i) + y_i^*(y_i) \le \|x_i^*\|\|x_i\| + \|y_i^*\|\|y_i\| \le \|x_i\| + \|y_i\| = \|z_i\| = 1.$$

Equality hence holds throughout and, therefore, for every $i \in I$ with $\|x_i\| > 0$, one has

$$\|x_i^*\| = 1 \quad \text{and} \quad x_i^*(x_i) = \|x_i\|.$$

Let $P_\gamma : Z \to X_\gamma$ be the coordinate projection. Since $\|P_\gamma\| = 1$ and $P_\gamma(B_Z) = B_{X_\gamma}$, condition (iii) for property α in Z gives

$$B_{X_\gamma} = P_\gamma(B_Z) = P_\gamma\big(\overline{\text{aconv}}(\{z_i : i \in I\})\big) \subset \overline{\text{aconv}}(\{x_i : i \in I\}).$$

The reverse inclusion is obvious because $x_i \in B_{X_\gamma}$ for all i, so in fact

$$\overline{\text{aconv}}(\{x_i : i \in I\}) = B_{X_\gamma}.$$

Choose $\nu \in \big((1+\lambda)/2, 1\big)$ and define

$$I_\gamma = \{i \in I : \|x_i\| > \nu\}.$$

Applying Lemma 2.4 to the set $\{x_i : i \in I\} \subset B_{X_\gamma}$ with $\delta = 1 - \nu$, we obtain

$$\overline{\text{aconv}}(\{x_i : i \in I_\gamma\}) = B_{X_\gamma}.$$

In particular, $I_\gamma \neq \varnothing$. Set

$$A_\gamma = \left\{\frac{x_i}{\|x_i\|} : i \in I_\gamma\right\} \subset S_{X_\gamma}$$

and

$$A_\gamma^* = \{x_i^* : i \in I_\gamma\} \subset S_{X_\gamma^*}.$$

We claim that these sets witness property α of X_γ.

For condition (iii), every element x_i with $i \in I_\gamma$ belongs to $\text{aconv}(A_\gamma)$ because

$$x_i = \|x_i\| \frac{x_i}{\|x_i\|} \quad \text{and} \quad 0 < \|x_i\| \le 1.$$

Hence

$$B_{X_\gamma} = \overline{\text{aconv}}(\{x_i : i \in I_\gamma\}) \subset \overline{\text{aconv}}(A_\gamma) \subset B_{X_\gamma},$$

so $\overline{\text{aconv}}(A_\gamma) = B_{X_\gamma}$.

Condition (i) holds because for $i \in I_\gamma$, we have

$$x_i^*\Big(\frac{x_i}{\|x_i\|}\Big) = 1.$$

Finally, for condition (ii), let $i, j \in I_\gamma$ with $i \neq j$. Then

$$\lambda \geq |z_i^*(z_j)| = \left| x_i^*(x_j) + y_i^*(y_j) \right| \geq |x_i^*(x_j)| - |y_i^*(y_j)|$$
$$\geq |x_i^*(x_j)| - \|y_j\| = |x_i^*(x_j)| - (1 - \|x_j\|).$$

Since $\|x_j\| > \nu > (1+\lambda)/2$, it follows that

$$|x_i^*(x_j)| \leq \lambda + 1 - \|x_j\| < \lambda + 1 - \nu < \frac{1+\lambda}{2}.$$

Therefore,

$$\left| x_i^*\left(\frac{x_j}{\|x_j\|}\right) \right| = \frac{|x_i^*(x_j)|}{\|x_j\|} < \frac{(1+\lambda)/2}{\nu} =: \mu < 1.$$

Condition (ii) thus holds with constant μ. Therefore X_γ has property α. □

We now turn to the finite absolute-sum results.

Theorem 3.2. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal. If X and Y have property α, then $X \oplus_N Y$ has property α.*

Proof. Fix witness families

$$\{(x_i, x_i^*) : i \in I\} \subset S_X \times S_{X^*} \quad \text{and} \quad \big\{(y_j, y_j^*) : j \in J\big\} \subset S_Y \times S_{Y^*}$$

for property α of X and Y with respective constants $\lambda_X, \lambda_Y \in [0,1)$.

We now use the polygonal structure of the first-quadrant part of $S_{(\mathbb{R}^2,N)}$ to construct a witness family for $X \oplus_N Y$. Let

$$E = \{(a_r, b_r) : r = 1, \ldots, R\}$$

be the set of such vertices of $S_{(\mathbb{R}^2,N)}$ which lie in the first quadrant, numbered in the natural order along the boundary from $(1,0)$ to $(0,1)$, so that

$$a_1 > \cdots > a_R \quad \text{and} \quad b_1 < \cdots < b_R.$$

Note that $(1,0)$ and $(0,1)$ need not themselves be in this set. Thus, if $b_1 > 0$, then the first boundary edge is the line segment joining $(1,0)$ to (a_1, b_1), whereas if $b_1 = 0$, then necessarily $(a_1, b_1) = (1,0)$. Similarly, if $a_R > 0$, then the last boundary edge is the line segment joining (a_R, b_R) to $(0,1)$, whereas if $a_R = 0$, then necessarily $(a_R, b_R) = (0,1)$.

For each $r \in \{1, \ldots, R\}$, choose a supporting functional $(c_r, d_r) \in S_{(\mathbb{R}^2,N)^*}$ with $c_r, d_r \geq 0$ such that

$$c_r a_r + d_r b_r = 1$$

and

$$c_r a_s + d_r b_s < 1 \quad \text{for every } s \in \{1, \ldots, R\} \setminus \{r\}.$$

Such a choice is possible thanks to Lemma 2.6. Moreover, we choose these functionals so that

$$a_r > 0 \quad \text{implies} \quad c_r a_r > 0$$

and

$$b_r > 0 \quad \text{implies} \quad d_r b_r > 0.$$

This is possible by taking at (a_r, b_r) a strictly positive convex combination of the two adjacent supporting functionals.

Define a signed copy of the witness family of X by setting

$$I^{\pm} = I \cup (-I).$$

For $i \in I^{\pm}$, put

$$z_i = \begin{cases} x_i, & \text{if } i \in I, \\ -x_{-i}, & \text{if } i \in -I, \end{cases} \qquad \text{and} \qquad z_i^* = \begin{cases} x_i^*, & \text{if } i \in I, \\ -x_{-i}^*, & \text{if } i \in -I. \end{cases}$$

Fix $i_0 \in I$, $j_0 \in J$. For each $r \in \{1, \ldots, R\}$, define

$$\Xi_r = \begin{cases} I^{\pm} \times J, & \text{if } a_r b_r > 0, \\ I \times \{j_0\}, & \text{if } a_r > 0,\ b_r = 0, \\ \{i_0\} \times J, & \text{if } a_r = 0,\ b_r > 0. \end{cases}$$

Note that $a_r = b_r = 0$ is impossible due to their selection.

Now define

$$\Xi = \{\xi = (r, i, j) : 1 \leq r \leq R,\ (i, j) \in \Xi_r\}$$

and for each $r \in \{1, \ldots, R\}$, define

$$A_r = \{(a_r z_i, b_r y_j) : (i, j) \in \Xi_r\} \subset S_{X \oplus_N Y},$$

$$A_r^* = \left\{(c_r z_i^*, d_r y_j^*) : (i, j) \in \Xi_r\right\} \subset S_{(X \oplus_N Y)^*}.$$

For $\xi = (r, i, j) \in \Xi$, let

$$u_\xi = (a_r z_i, b_r y_j) \qquad \text{and} \qquad u_\xi^* = (c_r z_i^*, d_r y_j^*).$$

We claim that the family

$$\left\{(u_\xi, u_\xi^*) : \xi \in \Xi\right\} \subset S_{X \oplus_N Y} \times S_{(X \oplus_N Y)^*}$$

witnesses property α of $X \oplus_N Y$. Let

$$A = \{u_\xi : \xi \in \Xi\} \qquad \text{and} \qquad A^* = \left\{u_\xi^* : \xi \in \Xi\right\}.$$

Condition (i) is immediate from the definitions. Indeed, for every $\xi \in \Xi$ one has

$$u_\xi^*(u_\xi) = 1.$$

We next verify condition (ii). Let $P = \{r : 1 \leq r \leq R,\ a_r b_r > 0\}$. In the following definitions, the maximum over the empty set is understood to be 0. Define

$$\gamma_0 = \max\{c_r a_s + d_r b_s : 1 \leq r, s \leq R,\ r \neq s\},$$

$$\gamma_1 = \max\{c_r a_r \lambda_X + d_r b_r : r \in P\},$$

$$\gamma_2 = \max\{c_r a_r + d_r b_r \lambda_Y : r \in P\},$$

$$\gamma_3 = \max\{|-c_r a_r + d_r b_r| : r \in P\}.$$

All these constants are strictly smaller than 1:

- $\gamma_0 < 1$ by the choice of the supporting functionals;
- $\gamma_1 < 1$ and $\gamma_2 < 1$ because
$$c_r a_r + d_r b_r = 1, \quad \lambda_X, \lambda_Y < 1, \quad \text{and} \quad c_r a_r, d_r b_r > 0;$$
- if $a_r b_r > 0$, then $c_r a_r > 0$ and $d_r b_r > 0$, so
$$|-c_r a_r + d_r b_r| < c_r a_r + d_r b_r = 1,$$
hence $\gamma_3 < 1$.

Set
$$\lambda = \max\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \lambda_X, \lambda_Y\} < 1.$$

Take two distinct elements of A, say
$$u = (a_r z_i, b_r y_j) \in A_r \quad \text{and} \quad v = (a_{r'} z_{i'}, b_{r'} y_{j'}) \in A_{r'}.$$
We estimate the value of the functional from A_r^* corresponding to u on the point v.

If $r \neq r'$, then
$$\left|(c_r z_i^*, d_r y_j^*)(a_{r'} z_{i'}, b_{r'} y_{j'})\right| \leq c_r a_{r'} + d_r b_{r'} \leq \gamma_0 < 1.$$

Assume now that $r = r'$. Consider first the case $a_r b_r > 0$. If $j \neq j'$, then
$$\left|(c_r z_i^*, d_r y_j^*)(a_r z_{i'}, b_r y_{j'})\right| \leq c_r a_r + d_r b_r \lambda_Y \leq \gamma_2 < 1.$$
If $j = j'$ and $i' \neq i$, then there are two subcases. If $i' \neq -i$, then
$$\left|(c_r z_i^*, d_r y_j^*)(a_r z_{i'}, b_r y_j)\right| \leq c_r a_r \lambda_X + d_r b_r \leq \gamma_1 < 1.$$
If $i' = -i$, then
$$\left|(c_r z_i^*, d_r y_j^*)(a_r z_{-i}, b_r y_j)\right| = |-c_r a_r + d_r b_r| \leq \gamma_3 < 1.$$

If $a_r > 0$ and $b_r = 0$, then $j = j_0$ is fixed and $i, i' \in I$. For $i \neq i'$,
$$|(c_r z_i^*, d_r y_j^*)(a_r z_{i'}, 0)| = c_r a_r |x_i^*(x_{i'})| \leq \lambda_X < 1.$$
If $a_r = 0$ and $b_r > 0$, then similarly $i = i_0$ is fixed and $j, j' \in J$. For $j \neq j'$,
$$|(c_r x_i^*, d_r y_j^*)(0, b_r y_{j'})| = d_r b_r |y_j^*(y_{j'})| \leq \lambda_Y < 1.$$

Condition (ii) thus holds with constant λ.

It remains to verify condition (iii). Let
$$C = \{x_i : i \in I\} \subset S_X \quad \text{and} \quad D = \{y_j : j \in J\} \subset S_Y.$$
Since X and Y have property α,
$$\overline{\text{aconv}}(C) = B_X \quad \text{and} \quad \overline{\text{aconv}}(D) = B_Y.$$
Because $\overline{\text{aconv}}(A)$ is a closed absolutely convex subset of $B_{X \oplus_N Y}$, it is enough to show that
$$S_{X \oplus_N Y} \subset \overline{\text{aconv}}(A).$$

We will now verify condition (iii). For this, we first establish two elementary claims.

Claim 1. If $r \in \{1, \dots, R\}$ is such that $a_r b_r > 0$, then for every $u \in \operatorname{aconv}(C)$ and every $v \in \operatorname{aconv}(D)$, one has

$$(a_r u, b_r v) \in \operatorname{aconv}(A_r).$$

Proof of Claim 1. Write

$$u = \sum_{k=1}^{n} \alpha_k x_{i_k} \quad \text{and} \quad v = \sum_{l=1}^{m} \beta_l y_{j_l},$$

where

$$\sum_{k=1}^{n} |\alpha_k| \leq 1 \quad \text{and} \quad \sum_{l=1}^{m} |\beta_l| \leq 1.$$

Since C and D are non-empty, we may enlarge these representations, if necessary, by adding cancelling pairs to arrange that

$$\sum_{k=1}^{n} |\alpha_k| = 1 \quad \text{and} \quad \sum_{l=1}^{m} |\beta_l| = 1.$$

Indeed, if $\sum_{k=1}^{n} |\alpha_k| < 1$, choose any $i_0 \in I$ and replace the representation of u by

$$u = \sum_{k=1}^{n} \alpha_k x_{i_k} + \frac{1 - \sum_{k=1}^{n} |\alpha_k|}{2} x_{i_0} - \frac{1 - \sum_{k=1}^{n} |\alpha_k|}{2} x_{i_0},$$

and similarly for v.

Discarding zero-valued coefficients if necessary, we may assume that $\alpha_k \neq 0$ and $\beta_l \neq 0$ for all k, l. Then

$$(a_r u, b_r v) = \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_k \beta_l \operatorname{sign}(\alpha_k) \left(\frac{\operatorname{sign}(\beta_l)}{\operatorname{sign}(\alpha_k)} a_r x_{i_k},\, b_r y_{j_l} \right).$$

Indeed, the first coordinate of the right-hand side is

$$\begin{aligned} \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_k \beta_l \operatorname{sign}(\alpha_k) \frac{\operatorname{sign}(\beta_l)}{\operatorname{sign}(\alpha_k)} a_r x_{i_k} &= \sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_k \beta_l \operatorname{sign}(\beta_l)\, a_r x_{i_k} \\ &= \sum_{k=1}^{n} \alpha_k \left(\sum_{l=1}^{m} |\beta_l| \right) a_r x_{i_k} = a_r u, \end{aligned}$$

and similarly, the second coordinate is

$$\sum_{k=1}^{n} \sum_{l=1}^{m} \alpha_k \beta_l \operatorname{sign}(\alpha_k)\, b_r y_{j_l} = \sum_{l=1}^{m} \beta_l \left(\sum_{k=1}^{n} |\alpha_k| \right) b_r y_{j_l} = b_r v.$$

Moreover,

$$\sum_{k=1}^{n}\sum_{l=1}^{m} |\alpha_k \beta_l \operatorname{sign}(\alpha_k)| = \sum_{k=1}^{n} |\alpha_k| \sum_{l=1}^{m} |\beta_l| = 1.$$

Since $a_r b_r > 0$, we have $\Xi_r = I^{\pm} \times J$, and therefore

$$\left(\frac{\operatorname{sign}(\beta_l)}{\operatorname{sign}(\alpha_k)}\, a_r x_{i_k},\, b_r y_{j_l} \right) \in A_r$$

for every k, l. Hence $(a_r u, b_r v) \in \operatorname{aconv}(A_r)$. □

Claim 2. For every $u \in \operatorname{aconv}(C)$ one has $(u, 0) \in \operatorname{aconv}(A_1)$, and for every $v \in \operatorname{aconv}(D)$ one has $(0, v) \in \operatorname{aconv}(A_R)$.

Proof of Claim 2. We only prove the first statement; the second is analogous.

If $b_1 = 0$, then $(a_1, b_1) = (1, 0)$ and

$$A_1 = \{(x_i, 0) : i \in I\},$$

so $(u, 0) \in \operatorname{aconv}(A_1)$ is immediate.

Assume now that $b_1 > 0$. Then $a_1 = 1$ and $\Xi_1 = I^{\pm} \times J$. Write

$$u = \sum_{k=1}^{n} \alpha_k x_{i_k}, \qquad \sum_{k=1}^{n} |\alpha_k| \leq 1.$$

Fix any $j_0 \in J$. Then

$$(u, 0) = \sum_{k=1}^{n} \frac{\alpha_k}{2} (x_{i_k}, b_1 y_{j_0}) + \sum_{k=1}^{n} \frac{-\alpha_k}{2} (-x_{i_k}, b_1 y_{j_0}),$$

and the sum of the absolute values of the coefficients is

$$\sum_{k=1}^{n} \frac{|\alpha_k|}{2} + \sum_{k=1}^{n} \frac{|\alpha_k|}{2} \leq 1.$$

Hence $(u, 0) \in \operatorname{aconv}(A_1)$. □

We now prove that every point of $S_{X \oplus_N Y}$ belongs to $\overline{\operatorname{aconv}}(A)$. Let $(x, y) \in S_{X \oplus_N Y}$, meaning

$$N(\|x\|, \|y\|) = 1.$$

Since N is absolute, we may work in the first quadrant of $(\mathbb{R}^2, N)$. We treat the cases $y = 0$, $x = 0$, the boundary cases $\|x\| = 1$ or $\|y\| = 1$, and finally the genuinely interior case $0 < \|x\|, \|y\| < 1$ separately.

If $y = 0$, then $\|x\| = 1$. Choose $u \in \operatorname{aconv}(C)$ arbitrarily close to x. By Claim 2, $(u, 0) \in \operatorname{aconv}(A_1)$, and hence

$$(x, 0) \in \overline{\operatorname{aconv}}(A).$$

The case $x = 0$ is analogous.

Assume next that $x \neq 0$, $y \neq 0$, and $||x|| = 1$. Then the point $(1, ||y||)$ lies on the first edge of the first-quadrant polygonal chain, that is, on the line segment joining $(1,0)$ and $(a_1, b_1) = (1, b_1)$. Thus there exists $\theta \in (0,1]$ such that

$$(1, ||y||) = (1-\theta)(1,0) + \theta(1, b_1),$$

so $||y|| = \theta b_1$. Hence

$$(x,y) = (1-\theta)(x,0) + \theta\left(x, b_1 \frac{y}{||y||}\right).$$

Choose $u \in \operatorname{aconv}(C)$ arbitrarily close to x and $v \in \operatorname{aconv}(D)$ arbitrarily close to $y/||y||$. By Claim 2, $(u,0) \in \operatorname{aconv}(A_1)$. Since $b_1 > 0$, Claim 1 also gives

$$(u, b_1 v) \in \operatorname{aconv}(A_1).$$

Therefore

$$(1-\theta)(u,0) + \theta(u, b_1 v) \in \operatorname{aconv}(A).$$

Moreover, one may choose this point to be arbitrarily close to (x,y). Thus

$$(x,y) \in \overline{\operatorname{aconv}}(A).$$

The case $||y|| = 1$ is analogous, using the last vertex (a_R, b_R) and Claim 2.

Finally, assume that

$$0 < ||x|| < 1 \quad \text{and} \quad 0 < ||y|| < 1.$$

Then $(||x||, ||y||)$ belongs to one of the compact line segments joining two consecutive vertices of the first-quadrant polygonal chain of $S_{(\mathbb{R}^2,N)}$. Hence there exist $r \in \{1, \ldots, R-1\}$ and $\theta \in [0,1]$ such that

$$(||x||, ||y||) = \theta(a_r, b_r) + (1-\theta)(a_{r+1}, b_{r+1}).$$

Equivalently,

$$(x,y) = \theta\left(a_r \frac{x}{||x||}, b_r \frac{y}{||y||}\right) + (1-\theta)\left(a_{r+1} \frac{x}{||x||}, b_{r+1} \frac{y}{||y||}\right).$$

Choose $u \in \operatorname{aconv}(C)$ arbitrarily close to $x/||x||$ and $v \in \operatorname{aconv}(D)$ arbitrarily close to $y/||y||$.

If $a_r b_r > 0$, Claim 1 gives

$$(a_r u, b_r v) \in \operatorname{aconv}(A_r) \subset \operatorname{aconv}(A).$$

If $a_r b_r = 0$, then necessarily $(a_r, b_r) = (1,0)$ (since $r \leq R-1$), and Claim 2 applies instead. A similar argument applies to (a_{r+1}, b_{r+1}). Hence

$$\theta(a_r u, b_r v) + (1-\theta)(a_{r+1} u, b_{r+1} v) \in \operatorname{aconv}(A).$$

Moreover, one may choose this point to be arbitrarily close to (x,y). Thus

$$(x,y) \in \overline{\operatorname{aconv}}(A).$$

We have shown that

$$S_{X\oplus_N Y} \subset \overline{\mathrm{aconv}}(A).$$

Since $\overline{\mathrm{aconv}}(A)$ is closed and absolutely convex, it follows that

$$\overline{\mathrm{aconv}}(A) = B_{X\oplus_N Y}.$$

This proves condition (iii), and hence the family $\{(u_\xi, u_\xi^*) : \xi \in \Xi\}$ witnesses property α of $X \oplus_N Y$ with constant $\lambda < 1$. ☐

Corollary 3.3. *If X and Y have property α, then both $X \oplus_1 Y$ and $X \oplus_\infty Y$ have property α.*

Proof. Apply Theorem 3.2 to the absolute norms

$$N_1(a,b) = |a| + |b| \quad \text{and} \quad N_\infty(a,b) = \max\{|a|, |b|\}. \qquad \square$$

The forward stability result above depends on a polyhedral description of the unit sphere of $(\mathbb{R}^2, N)$. In the converse direction, the relevant geometry is different. Indeed, to recover a summand from an absolute sum, it is enough to assume that the corresponding coordinate point of the unit sphere is an extreme point. This yields a thin-slice phenomenon near that point, allowing us to isolate vectors whose mass is concentrated in a single coordinate.

Lemma 3.4. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$. Then for every $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that, whenever $(a,b) \in S_{(\mathbb{R}^2,N)}$ satisfies $|a| > 1-\delta$, we have $|b| < \varepsilon$.*

Proof. We argue by contradiction. Suppose that there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$, one can find $(a_n, b_n) \in S_{(\mathbb{R}^2,N)}$ with

$$|a_n| > 1 - \frac{1}{n} \quad \text{and} \quad |b_n| \geq \varepsilon_0.$$

Since N is absolute, we also have $(|a_n|, |b_n|) \in S_{(\mathbb{R}^2,N)}$, so replacing (a_n, b_n) by $(|a_n|, |b_n|)$ we may assume that $a_n, b_n \geq 0$ for all n.

The unit sphere $S_{(\mathbb{R}^2,N)}$ is compact, hence after passing to a subsequence, we may assume that

$$(a_n, b_n) \longrightarrow (a,b) \in S_{(\mathbb{R}^2,N)}.$$

Since $a_n \to 1$, we get $a = 1$, and since $b_n \geq \varepsilon_0$ for all n, we get $b \geq \varepsilon_0 > 0$. Thus $(1,b) \in S_{(\mathbb{R}^2,N)}$ with $b > 0$.

Now

$$(1,0) = \frac{1}{2}(1,b) + \frac{1}{2}(1,-b),$$

and, by absoluteness of N, both $(1,b)$ and $(1,-b)$ belong to $S_{(\mathbb{R}^2,N)}$. Since $(1,b) \neq (1,-b)$, this contradicts the fact that $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$.

Therefore, the stated implication holds for some $\delta > 0$. Replacing δ by $\min\{\delta, \varepsilon\}$ if necessary, we may also assume that $\delta < \varepsilon$. ☐

Proposition 3.5. *Let X and Y be Banach spaces and let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$. If $X \oplus_N Y$ has property α, then X has property α.*

Proof. Set

$$Z = X \oplus_N Y.$$

Assume that Z has property α, witnessed by a family

$$\{(z_i, z_i^*) : i \in I\} \subset S_Z \times S_{Z^*},$$

with constant $\lambda_0 \in [0,1)$, where $z_i = (x_i, y_i) \quad \text{and} \quad z_i^* = (x_i^*, y_i^*)$.

Let

$$P_X \colon Z \to X \quad \text{and} \quad P_X(x,y) = x.$$

Then

$$B_X = P_X(B_Z) = P_X\big(\overline{\operatorname{aconv}}(\{z_i : i \in I\})\big) \subset \overline{\operatorname{aconv}}(\{x_i : i \in I\}).$$

The reverse inclusion is obvious because each x_i belongs to B_X. Hence

$$\overline{\operatorname{aconv}}(\{x_i : i \in I\}) = B_X. \tag{1}$$

Choose $\varepsilon > 0$ so small that

$$\lambda = \frac{\lambda_0 + \varepsilon}{(1-\varepsilon)^2} < 1.$$

By Lemma 3.4, there exists $\delta \in (0, \varepsilon)$ such that for every $(a,b) \in S_{(\mathbb{R}^2,N)}$,

$$|a| > 1 - \delta \quad \Longrightarrow \quad |b| < \varepsilon.$$

Define

$$I_X = \{i \in I : \|x_i\| > 1 - \delta\}.$$

Since (1) holds, Lemma 2.4 yields

$$\overline{\operatorname{aconv}}(\{x_i : i \in I_X\}) = B_X. \tag{2}$$

For $i \in I_X$ we have $(\|x_i\|, \|y_i\|) \in S_{(\mathbb{R}^2,N)}$ and $\|x_i\| > 1 - \delta$, hence

$$\|y_i\| < \varepsilon. \tag{3}$$

For $i \in I_X$ define

$$u_i = \frac{x_i}{\|x_i\|} \in S_X \quad \text{and} \quad u_i^* = \frac{x_i^*}{\|x_i^*\|} \in S_{X^*}.$$

We claim that the family

$$\{(u_i, u_i^*) : i \in I_X\} \subset S_X \times S_{X^*}$$

witnesses property α of X with constant λ. Let

$$A_X = \{u_i : i \in I_X\} \quad \text{and} \quad A_X^* = \{u_i^* : i \in I_X\}.$$

We first check condition (iii). By (2), the set $\{x_i : i \in I_X\}$ already absolutely generates B_X. Since for every $i \in I_X$,

$$x_i = \|x_i\| \frac{x_i}{\|x_i\|} \quad \text{and} \quad 0 < 1 - \delta < \|x_i\| \leq 1,$$

we have

$$\{x_i : i \in I_X\} \subset \operatorname{aconv}(A_X).$$

Therefore

$$B_X = \overline{\operatorname{aconv}}(\{x_i : i \in I_X\}) \subset \overline{\operatorname{aconv}}(A_X) \subset B_X,$$

so

$$\overline{\operatorname{aconv}}(A_X) = B_X.$$

We now verify condition (i). Fix $i \in I_X$. Since $z_i = (x_i, y_i) \in S_Z$ and $z_i^* = (x_i^*, y_i^*) \in S_{Z^*}$, we have

$$1 = z_i^*(z_i) = x_i^*(x_i) + y_i^*(y_i).$$

Also,

$$x_i^*(x_i) + y_i^*(y_i) \leq \|x_i^*\| \, \|x_i\| + \|y_i^*\| \, \|y_i\| \leq 1,$$

because $(\|x_i\|, \|y_i\|) \in S_{(\mathbb{R}^2, N)}$ and $(\|x_i^*\|, \|y_i^*\|) \in B_{(\mathbb{R}^2, N^*)}$. Equality hence holds throughout and, in particular,

$$x_i^*(x_i) = \|x_i^*\| \, \|x_i\|.$$

Therefore

$$\frac{x_i^*}{\|x_i^*\|} \left(\frac{x_i}{\|x_i\|} \right) = 1.$$

We next record a useful lower bound. For $i \in I_X$, using (3) and the trivial estimate $\|y_i^*\| \leq 1$, we get

$$1 = x_i^*(x_i) + y_i^*(y_i) \leq \|x_i^*\| + \|y_i^*\| \, \|y_i\| \leq \|x_i^*\| + \varepsilon,$$

so

$$\|x_i^*\| \geq 1 - \varepsilon. \tag{4}$$

Since $\delta < \varepsilon$, we also have

$$\|x_i\| > 1 - \delta > 1 - \varepsilon \quad \text{for all } i \in I_X. \tag{5}$$

Finally, we verify condition (ii). Take distinct $i, j \in I_X$. Then

$$|z_i^*(z_j)| \leq \lambda_0$$

and therefore

$$|x_i^*(x_j)| \leq |z_i^*(z_j)| + |y_i^*(y_j)| \leq \lambda_0 + \|y_i^*\| \, \|y_j\| \leq \lambda_0 + \varepsilon,$$

where we used (3). Combining this with (4) and (5), we get

$$\left| \frac{x_i^*}{\|x_i^*\|} \left(\frac{x_j}{\|x_j\|} \right) \right| \leq \frac{\lambda_0 + \varepsilon}{\|x_i^*\| \, \|x_j\|} \leq \frac{\lambda_0 + \varepsilon}{(1 - \varepsilon)^2} = \lambda < 1.$$

Thus, all three conditions (i)–(iii) in the definition of property α hold for the family $\{(u_i, u_i^*) : i \in I_X\}$. Consequently, X has property α. ☐

Corollary 3.6. *Let X and Y be Banach spaces and let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$. If $X \oplus_N Y$ has property α, then Y has property α.*

Proof. Apply Proposition 3.5 to the norm

$$\widetilde{N}(a,b) = N(b,a)$$

and to the space $Y \oplus_{\widetilde{N}} X$, which is canonically isometric to $X \oplus_N Y$. ☐

Corollary 3.7. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal and both $(1,0)$ and $(0,1)$ are extreme points of $B_{(\mathbb{R}^2,N)}$. Then*

$$X \oplus_N Y \text{ has property } \alpha \quad \Longleftrightarrow \quad X \text{ and } Y \text{ have property } \alpha.$$

Proof. The reverse implication is Theorem 3.2. The forward implication follows from Proposition 3.5 and Corollary 3.6. ☐

Corollary 3.8. *The space $X \oplus_1 Y$ has property α if and only if both X and Y have property α.*

Remark 3.9. Proposition 3.5 does not apply to ℓ_∞-sums, since neither $(1,0)$ nor $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2,\|\cdot\|_\infty)}$.

4. Property β

We first record that summands inherit property β from their ℓ_∞-sum.

Proposition 4.1. *Let*

$$Z = \Big(\bigoplus_{\gamma\in\Gamma} X_\gamma\Big)_{\ell_\infty}.$$

If Z has property β, then X_γ has property β for every $\gamma \in \Gamma$.

Proof. Fix $\gamma \in \Gamma$ and set

$$Y_\gamma = \Big(\bigoplus_{\eta\in\Gamma\setminus\{\gamma\}} X_\eta\Big)_{\ell_\infty}.$$

Then Z is canonically isometric to

$$X_\gamma \oplus_\infty Y_\gamma.$$

Assume that Z has property β, witnessed by a family

$$\{(z_i, z_i^*) : i \in I\} \subset S_Z \times S_{Z^*},$$

with constant $\lambda \in [0,1)$. With respect to the decomposition $Z = X_\gamma \oplus_\infty Y_\gamma$, write

$$z_i = (x_i, y_i), \quad z_i^* = (x_i^*, y_i^*).$$

Fix

$$\theta \in \Big(\frac{1+\lambda}{2}, 1\Big)$$

and define

$$I_\gamma := \{i \in I : \|x_i^*\| > \theta\}.$$

For $i \in I_\gamma$, condition (i) of property β gives

$$1 = x_i^*(x_i) + y_i^*(y_i) \le \|x_i^*\|\,\|x_i\| + \|y_i^*\|\,\|y_i\| \le \|x_i^*\| + \|y_i^*\| = 1.$$

Therefore,

$$x_i^*(x_i) = \|x_i^*\|\,\|x_i\| = \|x_i^*\|,$$

so $\|x_i\| = 1$. For each $i \in I_\gamma$, we may hence define

$$u_i = x_i \in S_{X_\gamma}, \quad u_i^* = \frac{x_i^*}{\|x_i^*\|} \in S_{X_\gamma^*}.$$

We claim that the family

$$\{(u_i, u_i^*) : i \in I_\gamma\}$$

witnesses property β of X_γ.

Indeed, for every $i \in I_\gamma$,

$$u_i^*(u_i) = \frac{x_i^*(x_i)}{\|x_i^*\|} = 1,$$

so condition (i) holds.

Next, let $i, j \in I_\gamma$ with $i \ne j$. Since

$$|x_i^*(x_j) + y_i^*(y_j)| = |z_i^*(z_j)| \le \lambda,$$

we obtain

$$|x_i^*(x_j)| \le \lambda + |y_i^*(y_j)| \le \lambda + \|y_i^*\| = \lambda + 1 - \|x_i^*\|.$$

Hence

$$|u_i^*(u_j)| = \frac{|x_i^*(x_j)|}{\|x_i^*\|} \le \frac{\lambda + 1 - \|x_i^*\|}{\|x_i^*\|} < \frac{\lambda + 1 - \theta}{\theta} < 1.$$

Thus, condition (ii) holds.

Finally, let $x \in X_\gamma$. Since $Z = X_\gamma \oplus_\infty Y_\gamma$, we have

$$\|(x, 0)\| = \|x\|.$$

By condition (iii) for property β in Z,

$$\|x\| = \|(x, 0)\| = \sup_{i \in I} |z_i^*(x, 0)| = \sup_{i \in I} |x_i^*(x)|.$$

If $i \notin I_\gamma$, then $\|x_i^*\| \le \theta$, so for $x \ne 0$,

$$|x_i^*(x)| \le \|x_i^*\|\,\|x\| \le \theta\|x\| < \|x\|.$$

Therefore,

$$\|x\| = \sup_{i \in I_\gamma} |x_i^*(x)|.$$

Since $x_i^* = \|x_i^*\|\, u_i^*$ and $\|x_i^*\| \leq 1$,

$$\|x\| = \sup_{i \in I_\gamma} |x_i^*(x)| \leq \sup_{i \in I_\gamma} |u_i^*(x)| \leq \|x\|.$$

Hence

$$\|x\| = \sup_{i \in I_\gamma} |u_i^*(x)|.$$

Condition (iii) therefore holds as well.

X_γ thus has property β. Since $\gamma \in \Gamma$ was arbitrary, the proof is complete. ☐

We now turn to the forward stability result for finite absolute sums.

Theorem 4.2. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal. If X and Y have property β, then $X \oplus_N Y$ has property β.*

Proof. Fix witness families

$$\{(x_i, x_i^*) : i \in I\} \subset S_X \times S_{X^*} \quad \text{and} \quad \big\{(y_j, y_j^*) : j \in J\big\} \subset S_Y \times S_{Y^*}$$

for property β of X and Y with respective constants $\lambda_X, \lambda_Y \in [0,1)$.

We now use the polygonal structure of the first-quadrant part of $S_{(\mathbb{R}^2,N)}$ to construct a witness family for $X \oplus_N Y$. To do so, we consider the set E of the vertices of the first-quadrant polygonal chain of $S_{(\mathbb{R}^2,N)}$. More precisely, let

$$E' = \{(u_k, v_k) : k = 1, \ldots, K\}$$

be the set of first-quadrant vertices of the polygonal chain $S_{(\mathbb{R}^2,N)}$, numbered so that

$$u_1 > \cdots > u_K \qquad \text{and} \qquad v_1 < \cdots < v_K.$$

We now augment this list, if necessary, with points from the axes. In particular, define a sequence

$$(a_0, b_0), \ldots, (a_M, b_M)$$

of points of the first quadrant of $S_{(\mathbb{R}^2,N)}$ as follows:

- if $(1,0) \in E'$, let $(a_0, b_0) = (1,0)$; otherwise prepend the point $(1,0)$ to the list of vertices;
- if $(0,1) \in E'$, let $(a_M, b_M) = (0,1)$; otherwise append the point $(0,1)$ to the list of vertices;
- the remaining points are the vertices of E' in their natural order.

Thus

$$a_0 = 1, \quad b_M = 1,$$

and every point of the first-quadrant polygonal chain of $S_{(\mathbb{R}^2,N)}$ lies on one of the line segments joining two consecutive points

$$(a_r, b_r),\ (a_{r+1}, b_{r+1}), \quad r = 0, \ldots, M-1.$$

For each $r \in \{0,\dots,M-1\}$, choose a supporting functional $(c_r,d_r) \in S_{(\mathbb{R}^2,N)^*}$ with $c_r,d_r \ge 0$ such that

$$c_r a_r + d_r b_r = 1, \quad c_r a_{r+1} + d_r b_{r+1} = 1,$$

and

$$c_r a_s + d_r b_s < 1 \quad \text{for every } s \notin \{r, r+1\}\,.$$

Now define the midpoint of the r-th edge by

$$(m_r,n_r) = \left(\frac{a_r + a_{r+1}}{2}, \frac{b_r + b_{r+1}}{2}\right) \in S_{(\mathbb{R}^2,N)}.$$

Since consecutive vertices on the first-quadrant polygonal chain of $S_{(\mathbb{R}^2,N)}$ are distinct, we have

$$m_r > 0 \quad \text{and} \quad n_r > 0 \qquad \text{for every } r = 0,\dots,M-1.$$

Moreover, $c_r m_r + d_r n_r = 1$ for every such r.

Define a signed copy of the witness family of X by setting

$$I^{\pm} = I \cup (-I),$$

and for $i \in I^{\pm}$,

$$z_i = \begin{cases} x_i, & \text{if } i \in I, \\ -x_{-i}, & \text{if } i \in -I, \end{cases} \qquad \text{and} \qquad z_i^* = \begin{cases} x_i^*, & \text{if } i \in I, \\ -x_{-i}^*, & \text{if } i \in -I. \end{cases}$$

Fix $i_0 \in I$ and $j_0 \in J$. We now define the candidate family edge by edge, according to which coordinates the supporting functional (c_r,d_r) sees. For each $r = 0,\dots,M-1$, define an index set Ξ_r as follows:

$$\Xi_r = \begin{cases} I^{\pm} \times J, & \text{if } c_r d_r > 0, \\ I \times \{j_0\}\,, & \text{if } c_r > 0 \text{ and } d_r = 0, \\ \{i_0\} \times J, & \text{if } c_r = 0 \text{ and } d_r > 0. \end{cases}$$

Set

$$\Xi = \{(r,i,j) : 0 \le r \le M-1,\ (i,j) \in \Xi_r\}$$

and for $\xi = (r,i,j) \in \Xi$,

$$u_\xi = (m_r z_i, n_r y_j) \quad \text{and} \quad u_\xi^* = (c_r z_i^*, d_r y_j^*).$$

For each $r = 0,\dots,M-1$, set

$$A_r = \{(m_r z_i, n_r y_j) : (i,j) \in \Xi_r\} \subset S_{X \oplus_N Y}$$

and

$$A_r^* = \left\{(c_r z_i^*, d_r y_j^*) : (i,j) \in \Xi_r\right\} \subset S_{(X \oplus_N Y)^*}.$$

We show that the family

$$\left\{(u_\xi, u_\xi^*) : \xi \in \Xi\right\} \subset S_{X \oplus_N Y} \times S_{(X \oplus_N Y)^*}$$

witnesses property β of $X \oplus_N Y$. Let

$$A = \{u_\xi : \xi \in \Xi\} \quad \text{and} \quad A^* = \left\{u_\xi^* : \xi \in \Xi\right\}.$$

Condition (i) is immediate from the definitions. Indeed, for every $\xi \in \Xi$ one has

$$u_\xi^*(u_\xi) = 1.$$

We next verify condition (ii). Let $P = \{r : 0 \le r \le M-1,\ c_r d_r > 0\}$. In the following definitions, the maximum over the empty set is understood to be 0. Define

$$\begin{aligned}
\gamma_0 &= \max\{c_r m_s + d_r n_s : 0 \le r, s \le M-1,\ r \neq s\},\\
\gamma_1 &= \max\{c_r m_r \lambda_X + d_r n_r : r \in P\},\\
\gamma_2 &= \max\{c_r m_r + d_r n_r \lambda_Y : r \in P\},\\
\gamma_3 &= \max\{|-c_r m_r + d_r n_r| : r \in P\}.
\end{aligned}$$

All these constants are strictly smaller than 1:

- If $r \neq s$, then at least one endpoint of the s-th edge does not belong to the r-th edge. Hence, the supporting functional (c_r, d_r) takes strictly smaller values than 1 on at least one endpoint of the s-th edge, and therefore also at the midpoint (m_s, n_s). Thus
$$c_r m_s + d_r n_s < 1$$
and hence $\gamma_0 < 1$.
- For every $r \in P$, we have $c_r > 0$, $d_r > 0$, $m_r > 0$, and $n_r > 0$. Hence both $c_r m_r$ and $d_r n_r$ are strictly positive. Since
$$c_r m_r + d_r n_r = 1,$$
we get
$$c_r m_r \lambda_X + d_r n_r = 1 - c_r m_r (1 - \lambda_X) < 1$$
and
$$c_r m_r + d_r n_r \lambda_Y = 1 - d_r n_r (1 - \lambda_Y) < 1.$$
Thus $\gamma_1 < 1$ and $\gamma_2 < 1$. Moreover,
$$|-c_r m_r + d_r n_r| < c_r m_r + d_r n_r = 1,$$
so $\gamma_3 < 1$.

Set

$$\lambda = \max\{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \lambda_X, \lambda_Y\} < 1.$$

Take two distinct elements $(r, i, j), (r', i', j') \in \Xi$, then

$$(m_r z_i, n_r y_j) \in A_r \quad \text{and} \quad (c_{r'} z_{i'}^*, d_{r'} y_{j'}^*) \in A_{r'}^*.$$

If $r \neq r'$, then

$$\left|(c_r z_i^*, d_r y_j^*)(m_{r'} z_{i'}, n_{r'} y_{j'})\right| \le c_r m_{r'} + d_r n_{r'} \le \gamma_0 < 1.$$

Assume now that $r = r'$.

Consider first the case $c_rd_r > 0$. If $j \neq j'$, then

$$|(c_rz_i^*, d_ry_j^*)(m_rz_{i'}, n_ry_{j'})| \leq c_rm_r + d_rn_r\lambda_Y \leq \gamma_2 < 1.$$

If $j = j'$ and $i \neq i'$, then there are two subcases. If $i' \neq -i$, then

$$\left|(c_rz_i^*, d_ry_j^*)(m_rz_{i'}, n_ry_j)\right| \leq c_rm_r\lambda_X + d_rn_r \leq \gamma_1 < 1.$$

If $i' = -i$, then

$$\left|(c_rz_i^*, d_ry_j^*)(m_rz_{-i}, n_ry_j)\right| = |-c_rm_r + d_rn_r| \leq \gamma_3 < 1.$$

If $c_r > 0$ and $d_r = 0$, then $j = j_0$ is fixed and $i, i' \in I$. For $i \neq i'$,

$$|(c_rz_i^*, 0)(m_rz_{i'}, n_ry_{j_0})| = c_rm_r|x_i^*(x_{i'})| \leq \lambda_X < 1.$$

If $c_r = 0$ and $d_r > 0$, then $i = i_0$ is fixed and $j, j' \in J$. For $j \neq j'$,

$$|(0, d_ry_j^*)(m_rz_{i_0}, n_ry_{j'})| = d_rn_r|y_j^*(y_{j'})| \leq \lambda_Y < 1.$$

Condition (ii) thus holds with constant λ.

It remains to verify condition (iii). Fix $(x, y) \in X \oplus_N Y$. If $(x, y) = (0, 0)$, there is nothing to prove, so assume $\|(x, y)\|_N > 0$. Then

$$\left(\frac{\|x\|}{\|(x,y)\|_N}, \frac{\|y\|}{\|(x,y)\|_N}\right) \in S_{(\mathbb{R}^2,N)}.$$

Hence there exists $r \in \{0, \dots, M-1\}$ such that this point lies on the segment joining (a_r, b_r) and (a_{r+1}, b_{r+1}). Since (c_r, d_r) supports that segment, we get

$$c_r\frac{\|x\|}{\|(x,y)\|_N} + d_r\frac{\|y\|}{\|(x,y)\|_N} = 1,$$

that is,

$$c_r\|x\| + d_r\|y\| = \|(x,y)\|_N. \tag{6}$$

If $c_rd_r > 0$, then for every $j \in J$,

$$\sup_{i \in I^{\pm}} \left|c_rz_i^*(x) + d_ry_j^*(y)\right| = c_r \sup_{i \in I} |x_i^*(x)| + d_r|y_j^*(y)|.$$

Indeed, by passing from I to $I^{\pm} = I \cup (-I)$, we may choose the sign of the X-term so that it agrees with the sign of $y_j^*(y)$. Taking the supremum over $j \in J$ and using property β of X and Y, we obtain

$$\begin{aligned}
\sup_{\xi \in \Xi} |u_\xi^*(x,y)| &\geq \sup_{(i,j) \in \Xi_r} \left|(c_rz_i^*, d_ry_j^*)(x,y)\right| \\
&\geq \sup_{j \in J} \sup_{i \in I^{\pm}} |c_rz_i^*(x) + d_ry_j^*(y)| \\
&= c_r \sup_{i \in I} |x_i^*(x)| + d_r \sup_{j \in J} |y_j^*(y)| \\
&= c_r\|x\| + d_r\|y\| \\
&\overset{(6)}{=} \|(x,y)\|_N.
\end{aligned}$$

If $c_r > 0$ and $d_r = 0$, then

$$\begin{aligned}\sup_{\xi\in\Xi}|u_{\xi^*}(x,y)| &\geq \sup_{(i,j)\in\Xi_r}|(c_r x_i^*,0)(x,y)| = \sup_{i\in I}|c_r x_i^*(x)| \\ &= c_r\|x\| = c_r\|x\| + d_r\|y\| = \|(x,y)\|_N.\end{aligned}$$

If $c_r = 0$ and $d_r > 0$, then likewise

$$\sup_{\xi\in\Xi}|u_{\xi^*}(x,y)| \geq \sup_{j\in J}|d_r y_j^*(y)| = d_r\|y\| = c_r\|x\| + d_r\|y\| = \|(x,y)\|_N.$$

Since every u_ξ^* has norm 1, the reverse inequality is automatic. Thus

$$\|(x,y)\|_N = \sup_{u_\xi^*\in A^*}|u_\xi^*(x,y)|.$$

This proves condition (iii), and hence the family $\{(u_\xi, u_\xi^*) : \xi \in \Xi\}$ witnesses property β of $X \oplus_N Y$. ☐

Corollary 4.3. *If X and Y have property β, then both $X \oplus_1 Y$ and $X \oplus_\infty Y$ have property β.*

Proof. Apply Theorem 4.2 to the absolute norms

$$N_1(a,b) = |a| + |b| \quad \text{and} \quad N_\infty(a,b) = \max\{|a|,|b|\}. \qquad \square$$

Remark 4.4. The real assumption is essential in the ℓ_1-case. Indeed, complex ℓ_1^m does not have property β; see [3].

Remark 4.5. In the β-setting, the geometry is complementary to that of the α-case: instead of assuming that a coordinate point is extreme in the unit ball of $(\mathbb{R}^2, N)$, one naturally assumes that it is not extreme. This leads to the partial converse proved below.

Lemma 4.6. *Let N be an absolute normalised norm on $\mathbb{R}^2$. If $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$, then $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$.*

Proof. Since $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$, there exist (a,b), $(c,d) \in B_{(\mathbb{R}^2,N)}$, $(a,b) \neq (c,d)$, such that

$$(1,0) = \frac{1}{2}(a,b) + \frac{1}{2}(c,d).$$

Because N is an absolute normalised norm, we have

$$\max\{|a|,|b|\} \leq N(a,b) \leq 1 \quad \text{and} \quad \max\{|c|,|d|\} \leq N(c,d) \leq 1.$$

Hence $|a| \leq 1$ and $|c| \leq 1$. Since

$$1 = \frac{a+c}{2},$$

it follows that $a = c = 1$. Therefore $d = -b$, and since $(a,b) \neq (c,d)$, we get $b \neq 0$.

By absoluteness,

$$(1, |b|) \in B_{(\mathbb{R}^2,N)}.$$

On the other hand,

$$1 = \|(1,0)\|_\infty \leq N(1, |b|) \leq 1,$$

so in fact

$$N(1, |b|) = 1.$$

Set

$$t = |b| > 0.$$

Thus

$$(1, t) \in S_{(\mathbb{R}^2,N)}.$$

We now show that if $(1, s) \in B_{(\mathbb{R}^2,N^*)}$, then necessarily $s = 0$. Indeed, by the formula for the dual norm of an absolute normalised norm,

$$N^*(1, s) = \sup_{N(u,v)\leq 1} \big(|u| + |s||v|\big).$$

Since $(1, t) \in B_{(\mathbb{R}^2,N)}$, we obtain

$$1 \geq N^*(1, s) \geq 1 + |s|t.$$

Because $t > 0$, this implies $s = 0$.

It remains to prove that $(1, 0)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$. Suppose

$$(1, 0) = \frac{1}{2}(u_1, v_1) + \frac{1}{2}(u_2, v_2)$$

with $(u_1, v_1), (u_2, v_2) \in B_{(\mathbb{R}^2,N^*)}$. As above, every point of $B_{(\mathbb{R}^2,N^*)}$ satisfies

$$|u_k| \leq N^*(u_k, v_k) \leq 1 \quad (k = 1, 2).$$

Since

$$1 = \frac{u_1 + u_2}{2},$$

we must have $u_1 = u_2 = 1$. But then, by the claim proved above, $(1, v_k) \in B_{(\mathbb{R}^2,N^*)}$ implies $v_k = 0$ for $k = 1, 2$. Hence

$$(u_1, v_1) = (u_2, v_2) = (1, 0),$$

and therefore $(1, 0)$ is an extreme point of $B_{(\mathbb{R}^2,N^*)}$. □

Proposition 4.7. *Let X and Y be Banach spaces and let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(1, 0)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$. If $X \oplus_N Y$ has property β, then X has property β.*

Proof. Set

$$Z = X \oplus_N Y.$$

Assume that Z has property β, witnessed by a family

$$\{(z_i, z_i^*) : i \in I\} \subset S_Z \times S_{Z^*}$$

with constant $\lambda_0 \in [0,1)$, where

$$z_i = (x_i, y_i) \quad \text{and} \quad z_i^* = (x_i^*, y_i^*).$$

By Lemma 4.6, the point $(1,0)$ is an extreme point of $B_{(\mathbb{R}^2, N^*)}$. Hence Lemma 3.4, applied to the absolute normalised norm N^*, yields the following: for every $\varepsilon > 0$ there exists $\delta \in (0,\varepsilon)$ such that for every $(a,b) \in S_{(\mathbb{R}^2,N^*)}$,

$$|a| > 1 - \delta \quad \Longrightarrow \quad |b| < \varepsilon.$$

Choose $\varepsilon > 0$ so small that

$$\mu := \frac{\lambda_0 + \varepsilon}{(1-\varepsilon)^2} < 1,$$

and let $\delta \in (0,\varepsilon)$ be given by the previous paragraph. Define

$$I_X = \{ i \in I : \|x_i^*\| > 1 - \delta \}.$$

Choose $x \in S_X$. Property β of Z gives

$$1 = \|x\| = \|(x,0)\|_Z = \sup_{i \in I} |z_i^*(x,0)| = \sup_{i\in I} |x_i^*(x)|.$$

For every $i \in I \setminus I_X$ we have

$$|x_i^*(x)| \leq \|x_i^*\| \, \|x\| \leq 1 - \delta < 1,$$

so the above supremum cannot be taken only over $I \setminus I_X$. Hence $I_X \neq \varnothing$.

Now fix $i \in I_X$. Since

$$Z^* = X^* \oplus_{N^*} Y^*,$$

we have

$$N^*(\|x_i^*\|, \|y_i^*\|) = \|z_i^*\| = 1.$$

Because $\|x_i^*\| > 1 - \delta$, the choice of δ gives

$$\|y_i^*\| < \varepsilon. \tag{7}$$

Also,

$$1 = z_i^*(z_i) = x_i^*(x_i) + y_i^*(y_i) \leq \|x_i^*\| \, \|x_i\| + \|y_i^*\| \, \|y_i\|.$$

Since $(x_i, y_i) \in S_Z$, we have $\|y_i\| \leq 1$, and thus by (7),

$$1 \leq \|x_i^*\| \, \|x_i\| + \varepsilon.$$

Therefore

$$\|x_i^*\| \, \|x_i\| \geq 1 - \varepsilon.$$

As both $\|x_i^*\|$ and $\|x_i\|$ are at most 1, this implies

$$\|x_i^*\| \geq 1 - \varepsilon \quad \text{and} \quad \|x_i\| \geq 1 - \varepsilon. \tag{8}$$

In particular, $x_i \neq 0$ for every $i \in I_X$.

For each $i \in I_X$, define

$$u_i = \frac{x_i}{\|x_i\|} \in S_X \quad \text{and} \quad u_i^* = \frac{x_i^*}{\|x_i^*\|} \in S_{X^*}.$$

We claim that

$$\{(u_i, u_i^*) : i \in I_X\} \subset S_X \times S_{X^*}$$

witnesses property β of X with constant $\mu < 1$.

We first verify condition (i). Fix $i \in I_X$. Since

$$1 = z_i^*(z_i) = x_i^*(x_i) + y_i^*(y_i) \leq ||x_i^*||\,||x_i|| + ||y_i^*||\,||y_i|| \leq 1,$$

equality holds throughout, and in particular

$$x_i^*(x_i) = ||x_i^*||\,||x_i||.$$

Hence

$$u_i^*(u_i) = \frac{x_i^*(x_i)}{||x_i^*||\,||x_i||} = 1.$$

Next we verify condition (ii). Let $i, j \in I_X$ with $i \neq j$. Since

$$|z_i^*(z_j)| \leq \lambda_0,$$

we get

$$|x_i^*(x_j)| \leq |z_i^*(z_j)| + |y_i^*(y_j)| \leq \lambda_0 + ||y_i^*||\,||y_j|| \leq \lambda_0 + \varepsilon,$$

where we used (7) and the fact that $||y_j|| \leq 1$. Combining this with (8), we obtain

$$|u_i^*(u_j)| = \frac{|x_i^*(x_j)|}{||x_i^*||\,||x_j||} \leq \frac{\lambda_0 + \varepsilon}{(1-\varepsilon)^2} = \mu < 1.$$

Condition (ii) thus holds with constant μ.

Finally, we verify condition (iii). Fix $x \in X$. Since

$$||x|| = ||(x, 0)||_Z,$$

property β of Z gives

$$||x|| = \sup_{i \in I} |z_i^*(x, 0)| = \sup_{i \in I} |x_i^*(x)|.$$

For every $i \in I \setminus I_X$ we have

$$|x_i^*(x)| \leq ||x_i^*||\,||x|| \leq (1-\delta)||x|| < ||x||$$

whenever $x \neq 0$. Hence $||x|| = \sup_{i \in I_X} |x_i^*(x)|$.
Therefore

$$||x|| = \sup_{i \in I_X} |x_i^*(x)| \leq \sup_{i \in I_X} |u_i^*(x)| \leq ||x||,$$

because every u_i^* has norm 1. Thus

$$||x|| = \sup_{i \in I_X} |u_i^*(x)|,$$

so condition (iii) holds.

We have proved that the family

$$\{(u_i, u_i^*) : i \in I_X\}$$

satisfies conditions (i)–(iii) of property β for X. □

Corollary 4.8. *Let X and Y be Banach spaces and let N be an absolute normalised norm on $\mathbb{R}^2$ such that $(0,1)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$. If $X \oplus_N Y$ has property β, then Y has property β.*

Proof. Apply Proposition 4.7 to the norm

$$\widetilde{N}(a,b) = N(b,a)$$

and to the space $Y \oplus_{\widetilde{N}} X$, which is canonically isometric to $X \oplus_N Y$. Since $(1,0)$ is not an extreme point of $B_{(\mathbb{R}^2,\widetilde{N})}$ exactly when $(0,1)$ is not an extreme point of $B_{(\mathbb{R}^2,N)}$, the conclusion follows. ☐

Corollary 4.9. *Let N be an absolute normalised norm on $\mathbb{R}^2$ such that $S_{(\mathbb{R}^2,N)}$ is polygonal and neither $(1,0)$ nor $(0,1)$ is an extreme point of $B_{(\mathbb{R}^2,N)}$. Then*

$$X \oplus_N Y \text{ has property } \beta \iff X \text{ and } Y \text{ have property } \beta.$$

Proof. If X and Y have property β, then $X \oplus_N Y$ has property β by Theorem 4.2. Conversely, if $X \oplus_N Y$ has property β, then X has property β by Proposition 4.7, and Y has property β by Corollary 4.8. ☐

Corollary 4.10. *The space $X \oplus_\infty Y$ has property β if and only if both X and Y have property β.*

Acknowledgements

This work was supported by the Estonian Research Council grant PRG1901. The author thanks Sheldon Dantas and Mingu Jung for sharing their expertise on the topic, Rainis Haller for assistance with structuring the paper, and the anonymous referee for insightfully pointing out some gaps in the proofs of the original version of this paper.

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