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Correction to: Convergence analysis of an inertial method for a system of general quasi-variational inequalities under mild conditions

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Abstract. This note concerns the paper by S. Jabeen, S. Macías, J. E. Macías-Díaz and S. Ullah, *Convergence analysis of an inertial method for a system of general quasi-variational inequalities under mild conditions*, Acta Comment. Univ. Tartu. Math. **29** (2025), no. 2, 243–258. We note that the fixed-point reformulation used in that paper does not follow from the stated assumptions, and give a simple two-dimensional example. We also record a separate difficulty in the proof of the convergence theorem: the assumptions on the operators T_1 and T_2 concern only the first variable, whereas the proof uses estimates in which both variables vary. A one-dimensional example shows that the convergence statement, as stated, is not valid even when the constraint set is the whole real line and the auxiliary mapping is the identity.

1. The fixed-point reformulation

The system of general quasi-variational inequalities considered in [1] asks for $x^*, y^* \in C(x^*, y^*)$ satisfying

$$\begin{cases} \langle \rho_1 T_1(y^*, x^*) + x^* - \eth(y^*),\, \eth(x) - x^* \rangle \geq 0, & x \in H : \ \eth(x) \in C(y^*, x^*), \\ \langle \rho_2 T_2(x^*, y^*) + y^* - \eth(x^*),\, \eth(x) - y^* \rangle \geq 0, & x \in H : \ \eth(x) \in C(x^*, y^*). \end{cases}$$

Notice, however, that the first inequality involves the constraint set $C(y^*, x^*)$, while the second involves $C(x^*, y^*)$. Thus the natural membership conditions associated with the displayed inequalities would be $x^* \in C(y^*, x^*)$ and $y^* \in C(x^*, y^*)$. It is then claimed that this problem is equivalent to the fixed-point

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system

$$x^* = \Pi_{C(y^*,x^*)}\big[ð(y^*) - \rho_1 T_1(y^*,x^*)\big],$$
$$y^* = \Pi_{C(x^*,y^*)}\big[ð(x^*) - \rho_2 T_2(x^*,y^*)\big].$$

Recall that the metric projection onto a closed convex set C in a Hilbert space is characterised by

$$z = \Pi_C(w)$$

if and only if

$$\langle z - w, v - z\rangle \geq 0 \quad \text{for all } v \in C.$$

In the system above, however, the inequalities are tested only at points of the form $ð(x)$. Thus the projection argument would require an additional assumption, for example

$$C(y^*,x^*) \subset \operatorname{Ran}(ð) \quad \text{and} \quad C(x^*,y^*) \subset \operatorname{Ran}(ð),$$

or some equivalent condition. No such assumption is made in the paper.

The following example shows the difficulty. Let

$$H = \mathbb{R}^2, \quad B = \{z \in \mathbb{R}^2 : \|z\| \leq 1\},$$

and put

$$C(x,y) = B \quad (x, y \in H).$$

Define

$$ð(u_1,u_2) = (u_1, 0), \quad T_1(x,y) = T_2(x,y) = (0,1),$$

and take

$$\rho_1 = \rho_2 = 1, \quad x^* = y^* = (0,0).$$

Then $x^*, y^* \in B$. Moreover, every admissible test point $ð(x)$ is of the form $(s,0)$, with $|s| \leq 1$. Hence

$$\big\langle \rho_1 T_1(y^*,x^*) + x^* - ð(y^*),\, ð(x) - x^*\big\rangle = \langle (0,1),(s,0)\rangle = 0,$$

and similarly

$$\big\langle \rho_2 T_2(x^*,y^*) + y^* - ð(x^*),\, ð(x) - y^*\big\rangle = \langle (0,1),(s,0)\rangle = 0.$$

Thus (x^*,y^*) satisfies the system of general quasi-variational inequalities as formulated in [1].

On the other hand,

$$ð(y^*) - \rho_1 T_1(y^*,x^*) = (0,-1),$$

and since $(0,-1) \in B$, we have

$$\Pi_B(0,-1) = (0,-1) \neq (0,0) = x^*.$$

Thus the first asserted fixed-point identity fails. The second one fails in the same way.

The example also satisfies Assumption 1 of [1], since $C(x,y)$ is constant and the corresponding projection does not depend on the parameters.

It follows that the stated system of general quasi-variational inequalities and the displayed fixed-point system are not equivalent under the assumptions of the paper. Consequently, the arguments depending on this equivalence require additional hypotheses or a different proof.

2. A point concerning the convergence theorem

A second issue concerns the estimates used in the proof of the convergence theorem. Theorem 1 assumes that

$$T_1, T_2 : H \times H \to H$$

are relaxed cocoercive and Lipschitz continuous in the first variable. However, the proof estimates differences such as

$$T_1(\omega_n, x_n) - T_1(y^*, x^*) \quad \text{and} \quad T_2(x_{n+1}, \omega_n) - T_2(x^*, y^*),$$

where both variables change. Conditions in the first variable alone do not give such estimates.

Let

$$H = \mathbb{R}, \quad C(x, y) = \mathbb{R}, \quad \eth = I,$$

and define

$$T_1(s, t) = s + 10t \quad \text{and} \quad T_2(s, t) = s.$$

Take

$$\rho_1 = \rho_2 = 1, \quad \alpha_n = 1, \quad \Theta_n = 0 \quad (n \geq 1).$$

The corresponding system has a unique solution

$$x^* = y^* = 0.$$

Indeed, since $C = \mathbb{R}$, the first inequality becomes

$$\big(T_1(y^*, x^*) + x^* - y^*\big)(x - x^*) \geq 0 \quad (x \in \mathbb{R}),$$

that is,

$$11x^*(x - x^*) \geq 0 \quad (x \in \mathbb{R}),$$

which forces $x^* = 0$. The second inequality similarly gives $y^* = 0$.

The assumptions of Theorem 1 are satisfied. For each fixed second variable, both $s \mapsto T_1(s, t)$ and $s \mapsto T_2(s, t)$ are the identity map plus a constant. Hence they are 1-Lipschitz in the first variable and relaxed (ξ_i, r_i)-cocoercive in the first variable, for instance with

$$\xi_i = \varepsilon, \quad r_i = 1 + \varepsilon, \quad \eta_i = 1 \quad (i = 1, 2),$$

where $0 < \varepsilon < 1$. The same constants may be used for $\eth = I$. Since $C(x, y) = \mathbb{R}$, Assumption 1 holds with any $\upsilon > 0$. Taking $\upsilon = 1/4$, we get

$$\kappa = \sqrt{1 + 2\varepsilon - 2(1 + \varepsilon) + 1} + \frac{1}{4} = \frac{1}{4} < 1.$$

Furthermore, for $i = 1, 2$,

$$r_i - \xi_i \eta_i^2 = 1,$$

and with $\rho_i = 1$ the numerical condition in Theorem 1 gives

$$\left|\rho_i - \frac{r_i - \xi_i \eta_i^2}{\eta_i^2}\right| = 0 < \sqrt{1-\kappa(2-\kappa)} = \frac{3}{4},$$

while

$$r_i = 1 + \varepsilon > \varepsilon + \sqrt{\kappa(2-\kappa)} = \varepsilon + \frac{\sqrt{7}}{4}.$$

Thus the stated numerical assumptions are fulfilled.

However, the iterative scheme treated in Theorem 1 gives, for $y_1 = 0$,

$$\omega_n = y_n,$$

$$x_{n+1} = \omega_n - T_1(\omega_n, x_n) = y_n - (y_n + 10x_n) = -10x_n,$$

and

$$y_{n+1} = x_{n+1} - T_2(x_{n+1}, \omega_n) = x_{n+1} - x_{n+1} = 0.$$

Starting with $x_1 = 1$ and $y_1 = 0$, one obtains

$$x_n = (-10)^{n-1} \quad \text{and} \quad y_n = 0 \quad (n \geq 1),$$

and the sequence therefore does not converge to $(0,0)$.

This shows that the hypotheses of Theorem 1 do not suffice for the asserted convergence. To obtain a correct statement one would need substantially stronger assumptions, for example that T_1 and T_2 do not depend on their second variable, or suitable Lipschitz conditions in the second variable together with new smallness assumptions.

3. Related occurrences

The same projection difficulty occurs in several related papers by some of the same authors: [2], [3], [4], [5]. These papers are not discussed further here.

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