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On the absolute weighted mean summability factors

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1. Introduction.

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) . Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty \quad as \quad n \to \infty, (P_{-i} = p_{-i} = 0, i \geq 1).$$

The sequence to sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu s_\nu$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n) , generated by the sequence of coefficients (p_n) . The series $\sum a_n$ is said to be summable $| \overline{N}, p_n |_k, k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty,$$

and it is said to be summable $| \overline{N}, p_n; \delta |_k, k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} \mid t_n - t_{n-1} \mid^k < \infty.$$

In the special case when $\delta = 0$ (resp. $\delta = 0$ and $p_n = 1$ for all values of n), $| \overline{N}, p_n; \delta |_k$ summability is the same as $| \overline{N}, p_n |_k$ (resp. $| C, 1 |_k$) summability.

Mishra and Srivastava [4] proved the following theorem for $| C, 1 |_k$ summability.

Theorem A. *Let* (x_n) *be a positive non-decreasing sequence and let there be sequences* (β_n) *and* (λ_n) *such that*

$$| \Delta \lambda_n | \leq \beta_n, \qquad (1.1)$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{1.2}$$

$$| \lambda_n | x_n = 0(1) \quad as \quad n \to \infty, \tag{1.3}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid x_n < \infty. \tag{1.4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} \mid s_n \mid^k = O(x_m) \quad as \quad m \to \infty, \tag{1.5}$$

then the series $\sum a_n \lambda_n$ *is summable* $| C, 1 |_k$, $k \geq 1$.

The author has generalized Theorem A for $| \overline{N}, p_n |_k$ summability in the form of the following theorem (see [3]).

Theorem B. *Let* (x_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *such that conditions (1.1)-(1.4) of Theorem A are satisfied. Furthermore, if* (p_n) *is a sequence of positive numbers such that*

$$P_n = O(np_n), \tag{1.6}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \mid s_n \mid^k = O(x_m) \quad as \quad m \to \infty, \tag{1.7}$$

then the series $\sum a_n \lambda_n$ *is summable* $| \overline{N}, p_n |_k$, $k \geq 1$.

It should be noted that if we take $p_n = 1$ for all values of n , then condition (1.7) will be reduced to condition (1.5). Also, it can be noticed that in this case condition (1.6) is obvious.

2. The main result

The aim of this paper is to generalize Theorem B for $| \overline{N}, p_n; \delta |_k$ summability methods. Now, we shall prove the following theorem.

Theorem. *Let* (x_n) *be a positive non-decreasing sequence and the sequences* (β_n) *and* (λ_n) *such that conditions* (1.1)-(1.4) *of Theorem A are satisfied. If* (p_n) *is a sequence of positive numbers such that condition* (1.6) *of Theorem B is satisfied and*

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k - 1} \mid s_n \mid^k = O(x_m) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=\nu}^{\infty} (P_n/p_n)^{\delta k - 1} \frac{1}{P_{n-1}} = O\{(P_\nu/p_\nu)^{\delta k} \frac{1}{P_\nu}\}, \tag{2.2}$$

then the series $\sum a_n \lambda_n$ *is summable* $| \overline{N}, p_n; \delta |_k$, $k \geq 1$ *and* $0 \leq \delta k < 1$.

Remark. It may be noted that if we take $\delta = 0$ in this Theorem, then we get Theorem B. In this case condition (2.1) reduces to condition (1.7) and condition (2.2) reduces to

$$\sum_{n=\nu}^{\infty} \frac{p_n}{P_n P_{n-1}} = \sum_{n=\nu}^{\infty} \left(\frac{1}{P_{n-1}} - \frac{1}{P_n}\right) = O(1/P_\nu),$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma ([4]). *If the conditions (1.1)-(1.4) on $(x_n), (\beta_n)$ and (λ_n) are satisfied, then*

$$n\beta_n x_n = O(1) \quad as \quad n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \beta_n x_n < \infty.$$

3. Proof of the Theorem

Let (T_n) be the $(\overline{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r \lambda_r = \frac{1}{P_n} \sum_{\nu=0}^{n} (P_n - P_{\nu-1}) a_\nu \lambda_\nu.$$

Then, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu \lambda_\nu.$$

Using Abel's transformation, we get

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} \Delta(P_{\nu-1}\lambda_\nu) s_\nu + \frac{p_n}{P_n} s_n \lambda_n = -\frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu s_\nu \lambda_\nu$$

$$+ \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n-1} P_\nu s_\nu \Delta\lambda_\nu + \frac{p_n}{P_n} s_n \lambda_n = T_{n,1} + T_{n,2} + T_{n,3},$$

say. To complete the proof of the Theorem by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} (P_n/p_n)^{\delta k + k - 1} \mid T_{n,r} \mid^k < \infty, \quad for \quad r = 1, 2, 3. \tag{3.1}$$

Now, when $k > 1$, applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} \mid T_{n,1} \mid^k$$

$$\leq \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} (P_{n-1})^{-k} \{\sum_{\nu=1}^{n-1} p_\nu \mid s_\nu \mid\mid \lambda_\nu \mid\}^k$$

$$\leq \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu \mid s_\nu \mid^k \mid \lambda_\nu \mid^k \{\frac{1}{P_{n-1}} \sum_{\nu-1}^{n-1} p_\nu\}^{k-1}$$

$$= O(1) \sum_{\nu=1}^{m} p_\nu \mid s_\nu \mid^k \mid \lambda_\nu \mid^k \sum_{n=\nu+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{\nu=1}^{m} (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k \mid \lambda_\nu \mid\mid \lambda_\nu \mid^{k-1}$$

$$= O(1) \sum_{\nu=1}^{m} (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k \mid \lambda_\nu \mid$$

$$= O(1) \sum_{\nu=1}^{m-1} \triangle \mid \lambda_\nu \mid \sum_{r=1}^{\nu} (P_r/p_r)^{\delta k-1} \mid s_r \mid^k$$

$$+ O(1) \mid \lambda_m \mid \sum_{\nu=1}^{m} (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k$$

$$= O(1) \sum_{\nu=1}^{m-1} \mid \triangle\lambda_\nu \mid x_\nu + O(1) \mid \lambda_m \mid x_m$$

$$= O(1) \sum_{\nu=1}^{m-1} \beta_\nu x_\nu + O(1) \mid \lambda_m \mid x_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the Theorem and Lemma.

Using the fact that $\mid \triangle\lambda_n \mid \leq \beta_n$ and $P_n = O(np_n)$, and after applying Hölder's inequality, we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} \mid T_{n,2} \mid^k$$

$$\leq \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} (P_{n-1})^{-k} \{\sum_{\nu=1}^{n-1} P_\nu \mid \triangle\lambda_\nu \mid\mid s_\nu \mid\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} (P_{n-1})^{-k} \{\sum_{\nu=1}^{n-1} \nu p_\nu \beta_\nu \mid s_\nu \mid\}^k$$

$$= O(1) \sum_{n=2}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} (\nu\beta_\nu)^k p_\nu \mid s_\nu \mid^k \{\frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} p_\nu\}^{k-1}$$

$$= O(1) \sum_{\nu=1}^{m} (\nu\beta_\nu)^k p_\nu \mid s_\nu \mid^k \sum_{n=\nu+1}^{m+1} (P_n/p_n)^{\delta k-1} \frac{1}{P_{n-1}}$$

$$= O(1) \sum_{\nu=1}^{m} (\nu\beta_\nu)^k (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k .$$

Since $\nu\beta_\nu = O(1/x_\nu) = O(1)$, by (2.3), we have that

$$\sum_{n=2}^{m+1} (P_n/p_n)^{\delta k+k-1} \mid T_{n,2} \mid^k = O(1) \sum_{\nu=1}^{m} (\nu\beta_\nu)^{k-1} \nu\beta_\nu (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k$$

$$= O(1) \sum_{\nu=1}^{m} \nu\beta_\nu (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k$$

$$= O(1) \sum_{\nu=1}^{m-1} \Delta(\nu\beta_\nu) \sum_{r=1}^{\nu} (P_r/p_r)^{\delta k-1} \mid s_r \mid^k$$

$$+ O(1) m\beta_m \sum_{\nu=1}^{m} (P_\nu/p_\nu)^{\delta k-1} \mid s_\nu \mid^k$$

$$= O(1) \sum_{\nu=1}^{m-1} \mid \Delta(\nu\beta_\nu) \mid x_\nu + O(1) m\beta_m x_m$$

$$= O(1) \sum_{\nu=1}^{m-1} \nu \mid \Delta\beta_\nu \mid x_\nu + O(1) \sum_{\nu=1}^{m-1} \beta_{\nu+1} x_\nu + O(1) m\beta_m x_m = O(1)$$

as $m \to \infty$, by virtue of the hypotheses of the theorem and lemma.

Finally, as in $T_{n,1}$, we get that

$$\sum_{n=1}^{m} (P_n/p_n)^{\delta k+k-1} \mid T_{n,3} \mid^k = O(1) \sum_{n=1}^{m} (P_n/p_n)^{\delta k-1} \mid s_n \mid^k \mid \lambda_n \mid = O(1)$$

as $m \to \infty$. Therefore, we get (3.1). This completes the proof of the theorem.

References

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