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On bounded spectral systems

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1. Introduction

A *spectral system* $\tilde{\sigma}$ on a complex unital Banach algebra A [11] is a map which assigns a non-empty compact subset $\tilde{\sigma}(\mathbf{a})$ of $\mathbb{C}^n$ to every commuting n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ ($n \in \mathbb{N}$); and a spectral system $\tilde{\sigma}$ on A is a *spectroid* on A (cf. [28]) if for each commuting n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ the conditions

(I) $\tilde{\sigma}(\mathbf{a}) \subset \prod\limits_{k=1}^{n} \sigma_A(a_k)$ ($\sigma_A(a_k)$ stands for the spectrum of the element a_k in A);

(II) $\tilde{\sigma}(a_1 + \lambda_1 e, a_2 + \lambda_2 e, \ldots, a_n + \lambda_n e) = \tilde{\sigma}(\mathbf{a}) + (\lambda_1, \lambda_2, \ldots, \lambda_n)$

are satisfied.

Moreover, a spectral system $\tilde{\sigma}$ on A is called a ***subspectrum*** (cf. [28]) if for any commuting n-tuple $\mathbf{a} = (a_1, a_2, ..., a_n) \in A^n$ and m-tuple of polynomials $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ in n indeterminates the condition **(I)** and the spectral mapping property

(III) $\tilde{\sigma}(\mathbf{p}(\mathbf{a})) = \mathbf{p}(\tilde{\sigma}(\mathbf{a}))$ (here $\mathbf{p}(\mathbf{a}) = (p_1(\mathbf{a}), p_2(\mathbf{a}), \ldots, p_m(\mathbf{a}))$)

are satisfied.

A well-known functional representation due to W. Żelazko [28] says that if $\tilde{\sigma}$ is a subspectrum defined on a complex unital Banach algebra A and B is any maximal commutative subalgebra of A, then there exists a compact subset $\Delta(\tilde{\sigma}, B)$ in the maximal ideal space of B such that

$$\tilde{\sigma}(a_1, a_2, \ldots, a_n) = \{(\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n)) : \Lambda \in \Delta(\tilde{\sigma}, B)\}$$

for all $(a_1, a_2, \ldots, a_n) \in B^n$.

Now, in this note we shall deal with the class of spectral systems that can be characterized by means of certain subsets of maximal ideal spaces of finitely generated subalgebras. More precisely, we shall consider the class of *bounded* spectral systems $\tilde{\sigma}$ on a complex unital Banach algebra A, i.e. we shall consider the class of spectral systems on A such that[1]

$$\tilde{\sigma}(\mathbf{a}) \subset \{(\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n)) : \Lambda \in \mathrm{Hom}[\mathbf{a}]\}$$

[1] Here $[\mathbf{a}]$ stands for the closed subalgebra of A generated by the elements $a_1, a_2, \ldots, a_n$ and the unit e; and $\mathrm{Hom}[\mathbf{a}]$ is the maximal ideal space of $[\mathbf{a}]$.

for any commuting n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$.

We first obtain the fact that a spectral system $\tilde{\sigma}$ on a complex unital Banach algebra A is bounded if and only if

(IV) $p(\tilde{\sigma}(\mathbf{a})) \subset \sigma_{[\mathbf{a}]}(p(\mathbf{a}))$ for any commuting $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ and any polynomial p in n indeterminates.

Next we prove that many of the important spectral systems coincide on some type of commuting n-tuples of a given Banach algebra elements . In particular, we show that if $\tilde{\sigma}$ is a bounded spectral system on a complex unital Banach algebra A and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ is a commuting n-tuple with $\sigma_A(a_k) \subset \mathbb{R}$ $(k = 1, 2, \ldots, n)$, then $\tilde{\sigma}(\mathbf{a})$ is always contained in the joint approximate point spectrum $\tau(\mathbf{a})$ of $\mathbf{a}$.

Then we apply these results giving some characterizations of joint spectra of doubly commuting systems in Banach algebras and, finally, of joint normal approximate point spectra in C^*-algebras.

2. Preliminaries

Throughout this note, all algebras are assumed to be associative, unital and over the complex field $\mathbb{C}$.

Let A be an algebra. The set of all commuting n-tuples $\mathbf{a} = (a_1, \ldots, a_n)$ of elements of A with length n $(n = 1, 2, \ldots)$ will be denoted by A_{com}, and for $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$, $< \mathbf{a} > = < a_1, a_2, \ldots, a_n >$ will be the subalgebra of A generated by the elements $a_1, a_2, \ldots, a_n$ and the unit e. If, in addition, A is a Banach algebra, then the closure of $< \mathbf{a} >$ in A will be written as $[\mathbf{a}]$. The set of all non-zero multiplicative linear functionals on A, endowed with the weak $*$-topology, will be denoted by $\operatorname{Hom} A$. Furthermore, when $\operatorname{Hom} A$ is non-empty and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$, we shall let $\hat{\mathbf{a}}$ denote the Gelfand transform of $\mathbf{a}$, i.e.

$$\hat{\mathbf{a}}(\Lambda) = (\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n))$$

for every Λ in $\operatorname{Hom} A$. Finally, if A is a commutative Banach algebra, $\Gamma(A)$ is the Shilov boundary of A.

The following proposition is given in a somewhat different form in [16].

Proposition 1. *Let A be a Banach algebra, let $\tilde{\sigma}$ be a spectral system on A satisfying the property* **(IV)** *and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$. For any n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \tilde{\sigma}(\mathbf{a})$ there is $\Lambda(\alpha) \in \operatorname{Hom}[\mathbf{a}]$ with $\Lambda(\alpha)(a_k) = \alpha_k$ $(k = 1, 2, \ldots, n)$ and the mapping $\alpha \longrightarrow \Lambda(\alpha)$ is a homeomorphism of $\tilde{\sigma}(\mathbf{a})$ into* $\operatorname{Hom}[\mathbf{a}]$.

Proof. Analogously as in the proof of Theorem 1 in [16] one can show that there is a homeomorphism $\alpha \longrightarrow \lambda(\alpha)$ of $\tilde{\sigma}(\mathbf{a})$ into $\operatorname{Hom} < a_1, a_2, \ldots, a_n >$ with $\lambda(\alpha)(a_k) = \alpha_k$ $(k = 1, 2, \ldots, n)$. Now, in view of the property **(IV)** we note that $\lambda(\alpha)$ is continuous on $< a_1, a_2, \ldots, a_n >$ and so it has an extension to a multiplicative linear functional $\Lambda(\alpha) \in$ Hom[**a**]. Now, taking into account the fact that $\tilde{\sigma}(\mathbf{a})$ is compact, the desired result follows.

Remark 2. An easy calculation shows that most of the classical spectral systems in Banach algebras, such as the left and the right joint spectra, the Harte joint spectrum, the joint Browder spectra, the commutant and bicommutant spectra, the joint approximate point spectrum, the rationally convex joint spectrum, etc. are bounded spectral systems.

Remark 3. If $\tilde{\sigma}$ is a spectroid on a Banach algebra A and $\mathbf{a} \in A_{\mathrm{com}}$ is a commuting n-tuple, then the *geometric spectral radius* of $\mathbf{a}$ relative to $\tilde{\sigma}$ is defined by the formula (see [9, 10])

$$r_{\tilde{\sigma}}(\mathbf{a}) = \max\{|z| : z \in \tilde{\sigma}(\mathbf{a})\}$$

where

$$|z| = |(z_1, z_2, \ldots, z_n)| = (\sum_{i=1}^{n} |z_i|^2)^{1/2}.$$

The class of all spectroids $\tilde{\sigma}$ such that

$$r_{\tilde{\sigma}}(\mathbf{a}) = \max\{|\hat{\mathbf{a}}(\Lambda)| : \Lambda \in \mathrm{Hom}[\mathbf{a}]\}$$

for any $\mathbf{a} \in A_{\mathrm{com}}$ is denoted in [10] by Σ_0 , and it is proved in [24], among others, that for an arbitrary subspectrum $\tilde{\sigma}$ defined on a Banach algebra A, $\tilde{\sigma}$ is of class Σ_0 if and only if $\hat{\mathbf{a}}(\Gamma([\mathbf{a}])) \subset \tilde{\sigma}(\mathbf{a})$ for all $\mathbf{a} \in A_{\mathrm{com}}$. Note that an analogous argument gives us that if $\tilde{\sigma}$ is a bounded spectroid such that $\hat{\mathbf{a}}(\Gamma([\mathbf{a}])) \subset \tilde{\sigma}(\mathbf{a})$ for all $\mathbf{a} \in A_{\mathrm{com}}$ then $\tilde{\sigma}$ is of class Σ_0.

3. Coincidence of spectral systems on some classes of commuting n-tuples

Now we turn to specifying some n-tuples of Banach algebra elements for which many of the important spectral systems coincide.

Let B be a commutative Banach algebra. Following W. Żelazko [29], let us denote by $\mathcal{L}(B)$ the set of all those $\Lambda \in$HomB for which $\ker \Lambda$ is an ideal consisting of joint topological divisors of zero. In other words, $\Lambda \in \mathcal{L}(B)$ if and only if there is a net $(z_j) \subset B$, $\|z_j\| = 1$ such that $\lim_j z_j x = \theta_A$ for all $x \in \ker \Lambda$.

Now suppose that we are given $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$ in a Banach algebra A such that $\sigma_A(a_k) \subset \mathbf{R}$ $(k = 1, 2, \ldots, n)$ and take any $\Lambda \in$ Hom[**a**]. If B is any commutative Banach algebra extension of [**a**], then (see, for example, [17], p. 62)

$$0 \in \sigma_B(\sum_{k=1}^{n}(a_k - \Lambda(a_k)e)^2)$$

and, consequently, there is a functional $\bar{\Lambda}$ $\in$Hom$B$ with

$$\bar{\Lambda}(\sum_{k=1}^{n}(a_k - \Lambda(a_k)e)^2) = 0.$$

Because $\bar{\Lambda}(a_k) \in \mathbf{R}$, it immediately follows that $\bar{\Lambda}(a_k) = \Lambda(a_k)$ $(k = 1, 2, \ldots, n)$ or, equivalently, Λ has an extension to a multiplicative linear functional $\bar{\Lambda}$ $\in$Hom$B$. Hence by [29], p. 66, we have Hom[**a**] $= \mathcal{L}([\mathbf{a}])$.

Now recall that the *joint approximate point spectrum* $\tau(\mathbf{a})$ for an arbitrary n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ (not necessarily commuting) is defined to be the set of all those $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{C}^n$ for which there is a sequence $(z_j) \subset A$ with $\|z_j\| = 1$, such that either $\lim_j \|(a_k - \alpha_k e)z_j\| = 0$ $(k = 1, 2, ..., n)$ or $\lim_j \|z_j(a_k - \alpha_k e)\| = 0$ $(k = 1, 2, \ldots, n)$. So, if $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a commuting n-tuple in a Banach algebra A, then

$$\tau(\mathbf{a}) \subset \hat{\sigma}(\mathbf{a}) = \hat{\mathbf{a}}(\mathrm{Hom}[\mathbf{a}]);$$

and if, in addition, $\sigma_A(a_k) \subset \mathbf{R}$ for each $k = 1, 2, ..., n$, then, in view of the discussion above, $\tau(\mathbf{a}) = \hat{\sigma}(\mathbf{a})$.

Following A. McIntosh, A. Pryde and W. J. Ricker [19,20], let us denote by $\gamma(\mathbf{a})$ the *spectral set*

$$\gamma(\mathbf{a}) = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{R}^n : 0 \in \sigma_A(\sum_{k=1}^{n}(a_k - \alpha_k e)^2)\}$$

of an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$. As it is established in [20] (see also [21]), for commuting n-tuples of linear operators with real spectra some of the important joint spectra and the spectral set coincide. As a matter of fact, a similar result can be obtained for all bounded spectral systems.

Theorem 4. *Let* $\mathbf{a} = (a_1, a_2, ..., a_n)$ *be a commuting* n*-tuple in a complex unital Banach algebra* A *and suppose that* $\tilde{\sigma}$ *is a bounded spectral system on* A*. Then the following conditions are equivalent:*

(a) $\sigma_A(a_k) \subset \mathbf{R}$ *for each* $k = 1, 2, ..., n$*;*

(b) $\tilde{\sigma}(\mathbf{a}) \subset \tau(\mathbf{a}) = \hat{\sigma}(\mathbf{a}) = \gamma(\mathbf{a})$.

Proof. (a)$\Rightarrow$(b). As we showed above, $\hat{\sigma}(\mathbf{a}) = \tau(\mathbf{a})$ for any n-tuple $\mathbf{a}$ $\in A_{\mathrm{com}}$ which satisfies the condition (a). Furthermore, by the assertions, $\tilde{\sigma}(\mathbf{a}) \subset \hat{\sigma}(\mathbf{a})$. Moreover, if $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \gamma(\mathbf{a})$ then,

$$\sum_{k=1}^{n}(a_k - \alpha_k e)^2$$

is not invertible in [**a**] and thus $(\alpha_1, \alpha_2, \ldots, \alpha_n) = \hat{\mathbf{a}}(\Lambda)$ for a suitable Λ in Hom[**a**]. Hence $\gamma(\mathbf{a}) \subset \hat{\sigma}(\mathbf{a})$. On the other hand, if $\Lambda \in$ Hom[**a**], then a similar argument shows that

$$0 \in \sigma_A\Big(\sum_{k=1}^{n}(a_k - \Lambda(a_k)e)^2\Big),$$

which, in turn, yields that $\hat{\mathbf{a}}(\Lambda) \in \gamma(\mathbf{a})$.

(b)$\Rightarrow$(a). Note that $\sigma_A(a_k) \subset \hat{a_k}(\mathrm{Hom}[\mathbf{a}]) \subset \mathbf{R}$ for each $k = 1, 2, \ldots, n$ because $(\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n)) \in \gamma(\mathbf{a}) \subset \mathbf{R}^n$ for any Λ in Hom[**a**].

Remark 5. A straightforward calculation shows that for every two or three complex numbers $z_1, \ldots, z_l$ $(l = 2, 3)$ there exist real numbers $\alpha_1, \ldots, \alpha_l$ $(l = 2, 3)$ such that

$$\sum_{k=1}^{l}(z_k - \alpha_k)^2 = 0.$$

It immediately follows that for any $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$ $(n \geq 2)$ and $\Lambda \in \mathrm{Hom}\, M$ where M stands for a maximal commutative subalgebra of a Banach algebra A containing the elements a_k $(k = 1, 2, \ldots, n)$, there are $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{R}^n$ with

$$\Lambda\Big(\sum_{k=1}^{n}(a_k - \alpha_k e)^2\Big) = 0.$$

So we easily get the fact that $\gamma(\mathbf{a})$ is non-empty for every n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$ with $n \geq 2$ (in this respect see [22,23]).

An analogous situation to Theorem 4 occurs when one considers commuting elements with totally disconnected spectra. In fact, when $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$ is such that $\sigma_A(a_k)$ is totally disconnected for every $k = 1, 2, \ldots, n$, then, because of the equality $\sigma_{[\mathbf{a}]}(a_k) = \sigma_A(a_k)$ (see [26], p. 496), Hom[**a**] can be considered as a closed subset of the totally disconnected space $\prod_{k=1}^{n} \sigma_A(a_k)$. But this implies the fact that Hom[**a**] is totally disconnected as well (see [13], p. 251), which together with Theorem 6.5 from [12] (see also [27]) yields that $\tau(\mathbf{a}) = \hat{\sigma}(\mathbf{a})$. So we have the following result.

Theorem 6. *Let A be a Banach algebra and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{\mathrm{com}}$ be such that $\sigma_A(a_k)$ is totally disconnected for each $k = 1, 2, \ldots, n$.*

If $\tilde{\sigma}$ is a bounded spectral system on A, then

$$\tilde{\sigma}(\mathbf{a}) \subset \tau(\mathbf{a}) = \hat{\sigma}(\mathbf{a}).$$

4. Joint approximate point spectra of some classes of commuting systems

Let A be a Banach algebra, let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in A_{\text{com}}$ and put $c_k = a_k + ib_k$, $\bar{c}_k = a_k - ib_k$ $(k = 1, 2, ..., n)$. Moreover, let us suppose that for the n-tuple $(c_1, c_2, ..., c_n)$ the following condition

(V) if $(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \tau(c_1, c_2, \ldots, c_n)$ then $(\gamma_1, \ldots, \gamma_n, \bar{\gamma}_1, \ldots, \bar{\gamma}_n) \in \tau(c_1, c_2, \ldots, c_n, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$

is satisfied[2] . So, if $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in \mathbf{R}$ are such that

$$(\alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \ldots, \alpha_n + i\beta_n) \in \tau(c_1, c_2, \ldots, c_n),$$

then

$$(\alpha_1 + i\beta_1, \alpha_2 + i\beta_2, \ldots, \alpha_n + i\beta_n, \alpha_1 - i\beta_1, \alpha_2 - i\beta_2, \ldots, \alpha_n - i\beta_n) \in \tau(c_1, c_2, \ldots, c_n, \bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n)$$

and, analogously as it was done in the proof of Proposition 1, one can show that there is a functional $\Lambda \in \mathrm{Hom}[a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n]$ with $\Lambda(a_k + ib_k) = \alpha_k + i\beta_k$. Now it readily follows that $\Lambda(a_k) = \alpha_k$ and $\Lambda(b_k) = \beta_k$ $(k = 1, 2, \ldots, n)$. Hence, in accordance with Theorem 4, we have the following result.

Theorem 7. *Let A be a Banach algebra and let $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n) \in A_{\text{com}}$ be such that $\sigma_A(a_k), \sigma_A(b_k) \subset \mathbf{R}$ $(k = 1, 2, ..., n)$. Moreover, let $\mathbf{c} = (c_1, c_2, ..., c_n)$, $c_k = a_k + ib_k$, $\bar{c}_k = a_k - ib_k$ $(k = 1, 2, ..., n)$ and suppose that the condition* **(V)** *is satisfied.*

1) If $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_n \in \mathbf{R}$ are such that $(\alpha_1 + i\beta_1, ..., \alpha_n + i\beta_n) \in \tau(c_1, ..., c_n)$, then $(\alpha_1, \alpha_2, ..., \alpha_n) \in \gamma(\mathbf{a})$ and $(\beta_1, \beta_2, ..., \beta_n) \in \gamma(\mathbf{b})$.

2) The following conditions are equivalent:

(a) every $\Lambda \in \mathrm{Hom}[\mathbf{a}]$ has an extension $\bar{\Lambda} \in \mathrm{Hom}[\mathbf{a}, \mathbf{b}]$ such that $\hat{\mathbf{c}}(\bar{\Lambda}) \in \tau(\mathbf{c})$;

(b) for each n-tuple $(\mu_1, \mu_2, ..., \mu_n) \in \gamma(\mathbf{a})$ there is $(\nu_1, \nu_2, ..., \nu_n) \in \gamma(\mathbf{b})$ such that

$$(\mu_1 + i\nu_1, \mu_2 + i\nu_2, ..., \mu_n + i\nu_n) \in \tau(\mathbf{c}).$$

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[2] Here $\bar{\gamma}_k$ is the conjugate of γ_k.

Remark 8. Let X be a complex Banach space, X^* be the dual space of X and let $B(X)$ be the algebra of all bounded linear operators on X. Recall that the *spatial numerical range* $V(T)$ of an operator $T \in B(X)$ is defined as the set

$$V(T) = \{f(Tx) : (x, f) \in X \times X^*, \|f\| = f(x) = \|x\| = 1\}$$

and if $V(T) \subset \mathbf{R}$, then T is called *hermitian.* Furthermore, an operator $T \in B(X)$ is called *hyponormal* if there are hermitian operators $A, B \in B(X)$ such that $T = A + iB$ and $V(i(AB - BA)) \subset \mathbf{R}^+ = \{\alpha \in \mathbf{R} : \alpha \geq 0\}$.

A commuting n-tuple $\mathbf{T} = (T_1, T_2, \ldots, T_n) \in B(X)^n$ is said to be *doubly commuting* if there are hermitian operators A_j, B_j in $B(X)$ such that $T_j = A_j + iB_j$ and $\bar{T}_j T_k = T_k \bar{T}_j$ $(j \neq k)$ where $\bar{T}_j = A_j - iB_j$.

Now suppose that $\mathbf{c} = (c_1, c_2, ..., c_n)$ is a doubly commuting n-tuple of hyponormal operators in a uniformly convex Banach space X and let a_k, b_k $(k = 1, 2, ..., n)$ be hermitian operators on X such that $c_k = a_k + ib_k$, $\bar{c}_k = a_k - ib_k$ $(k = 1, 2, ..., n)$. As it follows from [18, Theorem 2.7], if (x_j) is a bounded sequence in X with $c_k x_j \longrightarrow 0$ $(k = 1, 2, ..., n)$, then $a_k x_j \longrightarrow 0$ and $b_k x_j \longrightarrow 0$ $(k = 1, 2, ..., n)$. It readily follows that for the n-tuple $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ the condition **(V)** is satisfied, so that Theorem 7 is closely related to many of the results concerning the joint spectra of doubly commuting n-tuples of hyponormal operators on uniformly convex spaces (see, for example, [2,3,4,5,6,7,8,25]).

In conclusion we would briefly like to consider the case of C^*-algebras.

Let A be a C^*-algebra, comm$A$ be the commutator ideal of A, that is comm$A$ is the closed ideal in A generated by the set of all commutators $\{ab - ba : a, b \in A\}$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an arbitrary n-tuple in A.

The *joint normal approximate point spectrum* $\tau_n(\mathbf{a})$ of an n-tuple $\mathbf{a} \in A^n$ is defined to be the set of all those $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \tau(\mathbf{a})$ for which

$$(\alpha_1, \alpha_2, \ldots, \alpha_n, \bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_n) \in \tau(a_1, a_2, \ldots, a_n, a_1^*, a_2^*, \ldots, a_n^*).$$

Theorem 9. *Let A be a C^*-algebra and $\mathbf{a} = (a_1, a_2, ..., a_n)$, $\mathbf{b} = (b_1, b_2, ..., b_n) \in A_{\text{com}}$. If a_k, b_k are self-adjoint for all $k = 1, 2, ..., n$, then the following assertions are equivalent:*

(a) every $\Lambda \in \mathrm{Hom}[\mathbf{a}]$ has an extension $\tilde{\Lambda} \in \mathrm{Hom}[\mathbf{a}, \mathbf{b}]$;

(b) for each n-tuple $(\mu_1, \mu_2, ..., \mu_n) \in \gamma(\mathbf{a})$ there is $(\nu_1, \nu_2, ..., \nu_n) \in \gamma(\mathbf{b})$ such that

$$(\mu_1 + i\nu_1, \mu_2 + i\nu_2, ..., \mu_n + i\nu_n) \in \tau_n(a_1 + ib_1, a_2 + ib_2, ..., a_n + ib_n);$$

(c) $[\mathbf{a}]\bigcap \text{comm}[\mathbf{a},\mathbf{b}] = \{\theta_A\}$.

Proof. First note that (a)$\Leftrightarrow$(b) is true due to the fact that for an arbitrary n-tuple $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ in A we have the equality[3]

$$\tau_n(\mathbf{c}) = \{\hat{\mathbf{c}}(\Lambda) : \Lambda \in \text{Hom}[\mathbf{c}, \mathbf{c}^*] = \text{Hom}[c_1, c_2, \ldots, c_n, c_1^*, c_2^*, \ldots, c_n^*]\}.$$

The equivalence (a)$\Leftrightarrow$(c) comes immediately from the following lemma.

Lemma 10. *Let A be a C^*-algebra and B be a C^*-subalgebra of A. Then* $\text{comm}B = B\bigcap \text{comm}A$ *if and only if every* $\Lambda \in \text{Hom}B$ *has an extension* $\bar{\Lambda} \in \text{Hom}A$.

Proof of Lemma 10. Suppose that $B\bigcap\text{comm}A = \text{comm}B$. Clearly we can assume that comm$A \neq A$ since otherwise Hom$B$ is empty. So, we can consider the quotient algebra $A/\text{comm}A$ which is a commutative C^*-algebra. Now, denoting by π the quotient homomorphism of A onto $A/\text{comm}A$, we see that $\pi(B)$ is a commutative C^*-algebra as well. Moreover, if $\Lambda \in$Hom$B$, then obviously there is $\lambda \in \text{Hom}(\pi(B))$ with $\Lambda = \lambda \circ \pi$ and because λ is extendable to a $\bar{\lambda} \in \text{Hom}(A/\text{comm}A)$, we conclude that there is $\bar{\Lambda} \in$Hom$A$ with $\Lambda(b) = \bar{\Lambda}(b)$ $(b \in B)$.

Now suppose that every $\Lambda \in$Hom$B$ has an extension $\bar{\Lambda} \in$Hom$A$ and that there is an element $a_0 \in (B\bigcap\text{comm}A)\backslash\text{comm}B$. As

$$\text{comm}B = \bigcap\{\ker\Lambda : \Lambda \in \text{Hom}B\}$$

(see [1], p. 193), we can find a functional $\Lambda_0 \in \text{Hom}B$ such that $\Lambda_0(a_0) \neq 0$. So, by the hypothesis, $\bar{\Lambda}_0(a_0) \neq 0$ for a certain $\bar{\Lambda}_0 \in \text{Hom}A$ which clearly is impossible. Hence $B\bigcap\text{comm}A = \text{comm}B$. The proof of the lemma is completed.

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[3] This equality can be easily derived from the results in [14,15].

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Tõkestatud spektraalsetest süsteemidest

Resümee

Käesolevas artiklis on uuritud tõkestatud spektraalsete süsteemide funktsionaalseid esitusi. On tõestatud, et Banachi algebrates langevad paljud tähtsad tõkestatud spektraalsed süsteemid kokku lõplike süsteemide korral, mille komponendid on reaalse spektriga. Seejärel rakendatakse saadud tulemusi normaalse aproksimatiivse ühisspektri uurimiseks C^*-algebrates.

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