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On summability in measure with speed

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1. Definitions and former results.

In this paper* we consider the F-space $M[a,b]$ of all measurable functions defined on $[a,b]$. It is well-known that the Frechet-norm of f in $M[a,b]$ is

$$\| f \| = \inf_{\alpha>0} (\text{mes}\{t : | f(t) | \geq \alpha\} + \alpha)$$

and the convergence of sequence (f_n) to f in $M[a,b]$ is the convergence in measure, i.e. for every $\alpha > 0$

$$\lim_{n\to\infty} \text{mes}\{t : | f_n(t) - f(t) | \geq \alpha\} = 0.$$

Let $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$, $(0 < \lambda_k \nearrow \infty,\ 0 < \mu_k \nearrow \infty)$ be speeds.

We have defined the convergence in measure with speed in the following form in [1].

Definition 1. *A sequence* (f_n) *is called* mesλ *-convergent on* $[a,b]$ *to* f *if for every* $\alpha > 0$

$$\lim_{n\to\infty} \text{mes}\{t : \lambda_k \mid f_n(t) - f(t) \mid \geq \alpha\} = 0.$$

Definition 2. *A sequence* (f_n) *is called* λmes *-convergent on* $[a,b]$ *to* f *if for every* $\alpha > 0$ *there exists*

$$\lim_{n\to\infty} \lambda_n \text{mes}\{t : | f_n(t) - f(t) | \geq \alpha\}.$$

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Definition 3. *A sequence* (f_n) *is called* $\lambda\text{mes}\mu$ *-convergent on* $[a,b]$ *to* f *if for every* $\alpha > 0$ *there exists*

$$\lim_{n\to\infty} \lambda_n \text{mes}\{t : \mu_n \mid f_n(t) - f(t) \mid \geq \alpha\}.$$

By $c_{\text{mes}\lambda}(c_{\lambda\text{mes}}, c_{\lambda\text{mes}\mu})$ we denote the set of all $\text{mes}\lambda$ -convergent (λmes -convergent, $\lambda\text{mes}\mu$ -convergent) sequences in $M[a,b]$ and by $c^0_{\text{mes}\lambda}(c^0_{\lambda\text{mes}}, c^0_{\lambda\text{mes}\mu})$ we denote the set of all to $f = 0$ $\text{mes}\lambda$ -convergent (λmes -convergent, $\lambda\text{mes}\mu$ -convergent) sequences in $M[a,b]$.

By $c^\lambda(c^\lambda_0)$ we denote the set of all λ-convergent (λ-convergent to 0) sequences (see [3]).

If (f_n) is $\text{mes}\lambda$ -convergent (λmes -convergent, $\lambda\text{mes}\mu$ -convergent $\mu\text{mes}\lambda$ -convergent, $\lambda\text{mes}\lambda$ -convergent) for every λ , then we say that (f_n) is $\text{mes}\infty$ -convergent (∞mes -convergent, $\infty\text{mes}\mu$ -convergent, $\mu\text{mes}\infty$ -convergent, $\infty\text{mes}\infty$ -convergent).

By $c_{\text{mes}\infty}(c_{\infty\text{mes}}, c_{\lambda\text{mes}\infty}, c_{\infty\text{mes}\lambda}, c_{\infty\text{mes}\infty})$ we denote the set of all $\text{mes}\infty$ -convergent (∞mes -convergent, $\lambda\text{mes}\infty$ -convergent, $\infty\text{mes}\lambda$ - -convergent, $\infty\text{mes}\infty$ -convergent) sequences in $M[a,b]$.

In [1] we have shown that the λmes -convergence, $\text{mes}\lambda$ - convergence and $\lambda\text{mes}\mu$ -convergence of sequences (f_n) are different and we have proved the following theorems

Theorem A. *If* $(f_n) \in c_{\lambda\text{mes}\mu}$ *and*

$$\sum \frac{1}{\lambda_k} < \infty, \tag{1}$$

then

$$\lim_n \mu_n \mid f_n(t) - f(t) \mid = 0$$

a.e. in $[a,b]$ *(i.e. the sequence* (f_n) *is* c^μ *-convergent a.e. on* $[a,b]$ *).*

Corollary A. *If* $(f_n) \in c^0_{\lambda\text{mes}\mu}$, *where*

$$\sum \frac{1}{\mu_k} < \infty$$

and (1) *holds, then*

$$\sum \mid f_n(t) \mid < \infty$$

a.e. on $[a,b]$.

Theorem B. *For every μ and λ with*

$$\sum \frac{1}{\lambda_k} = \infty \tag{2}$$

there exists a sequence $(f_n) \in c^0_{\lambda\text{mes}\mu}$ such that for every $t_0 \in [a,b]$ the sequence $(f_n(t_0))$ is divergent.

In [1] we have formulated the following problem.

Let $A = (a_{nk})$ be a summability method which transforms a sequence $F = (f_n)$ into the sequence $G = (g_n)$, where

$$g_n = \sum a_{nk} f_k,$$

i.e. $G = (A_n F) = AF$.

A sequence F is said to be $A_{\text{mes}\lambda}$ -summable ($A_{\lambda\text{mes}}$ - summable, $A_{\lambda\text{mes}\mu}$ -summable) if $AF \in c_{\text{mes}\lambda}$ $(c_{\lambda\text{mes}}, c_{\lambda\text{mes}\mu})$.

By $c_{A\text{mes}\lambda}(c_{A\lambda\text{mes}}, c_{A\lambda\text{mes}\mu})$ we denote the set of all $A\text{mes}\lambda$ -summable ($A\lambda\text{mes}$ -summable, $A\lambda\text{mes}\mu$ -summable) sequences.

Let α be λmes (mesλ, or λmesμ) and β be Amesλ ($A\lambda$mes, or $A\lambda$mesμ) . The problem is:

Under which necessary and sufficient conditions for A is the inclusion

$$c_\alpha \subset c_\beta$$

true?

In this paper we have solved this problem partly – we consider only the following inclusions:

$$c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu}, c_{\lambda\text{mes}} \subset c_{A\text{mes}}, c_{\infty\text{mes}\infty} \subset c_{A\mu\text{mes}}, c_{\infty\text{mes}} \subset c_{A\mu\text{mes}}.$$

If $c_{\text{mes}\lambda} \subset c_{A\text{mes}\lambda}(c_{\lambda\text{mes}} \subset c_{A\lambda\text{mes}})$, then we say that the summability method is mesλ -convergence (λmes -convergence) preserving. Particularly, if $c \subset c_A$, then we say that it is convergence preserving.

2. Counter examples.

Denote

$$m^0_{\lambda\text{mes}\mu} := \{F : \lambda_n \text{mes}\{t : \mu_n \mid f_n \mid \geq \alpha\} = O(1), \quad \alpha > 0\}.$$

We demonstrate a simple proof of Theorem B in the following formulation

Theorem C. *For every* λ *with* (2) *there exists* $F \in m^0_{\lambda \mathrm{mes} \infty}$ *such that for every* $t_0 \in [0,1]$ *the sequence* $(f_n(t_0))$ *is divergent.*

Proof. Let the natural numbers k_i be defined by the inequalities

$$0 < \Lambda_{k_1-1} < 1 \leq \Lambda_{k_1},$$

$$1 < \Lambda_{k_2-1} < 2 \leq \Lambda_{k_2},$$

$$\dots\dots\dots\dots\dots\dots$$

$$i-1 < \Lambda_{k_i-1} < i \leq \Lambda_{k_i},$$

where

$$\Lambda_k = \sum_{l=1}^{k} \frac{1}{\lambda_l}.$$

Denote

$$e_1 := [0, \frac{1}{\lambda_1}) = [0, \Lambda_1);$$

$$e_2 := [\frac{1}{\lambda_1}, \frac{1}{\lambda_1} + \frac{1}{\lambda_k}) = [\Lambda_1, \Lambda_2);$$

$$e_k := [\Lambda_{k-1}, \Lambda_k);$$

$$\dots\dots\dots\dots\dots\dots$$

$$e_{k_1-1} := [\Lambda_{k_1-2}, \Lambda_{k_1-1});$$

$$e_{k_1} := [\Lambda_{k_1-1}, 1];$$

$$e_{k_1+1} := [0, \Lambda_{k_1} - 1);$$

$$e_{k_1+2} := [\Lambda_{k_1} - 1, \Lambda_{k_1+1} - 1);$$

$$e_{k_2} := [\Lambda_{k_2-1} - 1, 1];$$

$$\dots\dots\dots\dots\dots\dots$$

$$e_{k_i-1} := [\Lambda_{k_i-2} - i + 1, \Lambda_{k_i-1} - i + 1);$$

$$e_{k_i} := [\Lambda_{k_i-1} - i + 1, 1];$$

$$e_{k_i+1} = [0, \Lambda_{k_i} - i);$$

$$\dots\dots\dots\dots\dots\dots$$

Also

$$\bigcup_{s=1}^{k_1} e_s = [0,1];$$

$$\bigcup_{s=k_1+1}^{k_2} e_s = [0,1];$$

$$\dots\dots\dots\dots\dots$$

$$\bigcup_{s=k_i+1}^{k_{i+1}} e_s = [0,1];$$

$$\dots\dots\dots\dots\dots$$

Let $t_0 \in [0,1]$ be a fixed number. Then for every natural number i there exist the natural numbers s_i^0 and s_i^1 such that $t_0 \in e_{s_i^0}$ and $t_0 \notin e_{s_i^1}$. Let now

$$f_i(t) := \chi_{e_i}(t). \tag{3}$$

Then for every $0 < \alpha \leq 1$ and for every $\mu_i > 0$ we have

$$\text{mes}\{t : \mu_i :\mid f_i(t) \mid \geq \alpha\} = \text{mes } e_i \leq \frac{1}{\lambda_i},$$

i.e. $(f_i) \in m^0_{\lambda\text{mes}\infty}$.

Now $f_{s_i^0}(t_0) \equiv 1$ and $f_{s_i^1}(t_0) \equiv 0$ for every i , i.e. the limit

$$\lim_i f_i(t_0)$$

doesn't exist. The proof of Theorem C is completed.

Corollary C. *For every* λ *with* (2) *there exists* $F \in m^0_{\lambda\text{mes}\infty}$ *such that the series*

$$\sum_{s=1}^{\infty} f_s(t) \tag{4}$$

is divergent in measure on $[0,1]$.

Proof. Let $F = (f_v)$ be defined by (3). Then $F \in m^0_{\lambda\text{mes}\infty}$ and for every $0 < \alpha \leq 1$

$$\text{mes}\{t :\mid \sum_{s=k_i+1}^{k_{i+1}} f_s(t) \mid \geq \alpha\} = 1,$$

i.e. the series (4) is divergent in measure.

From the corollaries C and A now follows that the next theorem holds.

Theorem 1. *If* $F \in c^0_{\lambda \mathrm{mes}\mu}$ *, then the series* (4) *is convergent in measure on* $[a,b]$ *iff* (1) *holds and*

$$\sum \frac{1}{\mu_k} < \infty.$$

3. The representations of the sequences F from the sets $c^0_{\lambda \mathrm{mes}\infty}, c^0_{\infty \mathrm{mes}\lambda}$ and $c^0_{\infty \mathrm{mes}\infty}$.

Theorem 2. *The sequence* $F = (f_n)$ belongs to $c^0_{\lambda \mathrm{mes}\infty}$ iff

$$\lim_n \lambda_n \mathrm{mes\ supp} f_n = 0. \tag{5}$$

Proof. Sufficiency. According to the assumption for every $\alpha > 0$ and for every speed λ

$$\lim_n \mathrm{mes}\{t : \lambda_n \mid f_n(t) \mid \geq \alpha\} = 0. \tag{6}$$

Against the statement suppose that

$$\lim_n \mathrm{mes\ supp} f_n \neq 0. \tag{7}$$

Then we can find $\epsilon_0 > 0$ such that for every natural number k there exists n_k such that

$$\mathrm{mes\ supp} f_{n_k} > 5\epsilon_0.$$

Let us fix the natural number k_0 . Since $f_{n_{k_0}}$ is the measurable function, then according to Luzin's theorem there exists the measurable set $T_{\epsilon_0} \subset e$ such that on the set CT_{ϵ_0} the function $f_{n_{k_0}}$ is continuous and

$$\mathrm{mes}(CT_{\epsilon_0} \bigcap \mathrm{supp} f_{n_{k_0}}) > 4\epsilon_0.$$

Since $CT_{\epsilon_0} \cap \mathrm{supp} f_{n_{k_0}}$ is measurable, then there exists an open set

$$B_0 \subset (CT_{\epsilon_0} \bigcap \mathrm{supp} f_{n_{k_0}})$$

such that

$$\mathrm{mes} B_0 > 3\epsilon_0.$$

On the other hand there exist the structural intervals (α_i, β_i) such that

$$B_0 = \bigcup_i (\alpha_i, \beta_i).$$

Let us find s_0 such that

$$\operatorname{mes} \bigcup_{i=1}^{s_0} (\alpha_i, \beta_i) > 2\epsilon_0.$$

Further on we can find the segments $[\gamma_i, \delta_i] \subset (\alpha_i, \beta_i)$ such that

$$\operatorname{mes} \bigcup_{i=1}^{s_0} (\gamma_i, \delta_i) > \epsilon_0.$$

Let us fix i_1 freely from the set $1 \leq i_1 \leq s_0$ and observe the segment $[\gamma_{i_1}, \delta_{i_1}]$.

Since $[\gamma_{i_1}, \delta_{i_1}] \subset CT_{\epsilon_0}$, then on that segment the function $f_{n_{k_0}}(t)$ is continuous. According to Weierstrass theorem this function is bounded on that segment and has the extremal values.

Let

$$| f_{n_{k_0}}(t^1) | = \min_{t \in [\gamma_{i_1}, \delta_{i_1}]} | f_{n_{k_0}}(t) | .$$

Since

$$[\gamma_{i_1}, \delta_{i_1}] \subset \operatorname{supp} f_{n_{k_0}},$$

then

$$| f_{n_{k_0}}(t^1) | > 0.$$

Let us now find the number $\lambda^1_{n_{k_0}}$ such that

$$\lambda^1_{n_{k_0}}(t^1) \mid f_{n_{k_0}}(t') \mid > 1.$$

Since the natural number i_1 was chosen arbitrarily with the condition $i_1 \leq s_0$, therefore our discussion about the segment $[\gamma_{i_1}, \delta_{i_1}]$ is correct for every segment $[\gamma_i, \delta_i]$ that is

$$\lambda^m_{n_{k_0}} \mid f_{n_{k_0}}(t^m) \mid > 1,$$

where

$$t^m \in [\gamma_m, \delta_m], \quad | f_{n_{k_0}}(t^m) | = \min_{t \in [\gamma_{i_1}, \delta_{i_1}]} | f_{n_{k_0}}(t) | \quad \textit{and} \quad 1 \leq m \leq s_0.$$

Let

$$\lambda_{n_{k_0}} := \max_{1 \le m \le s_0} \lambda^m_{n_{k_0}}.$$

Then for every $m = 1, \ldots, s_0$

$$\lambda_{n_{k_0}} \mid f_{n_{k_0}}(t^m) \mid > 1.$$

Since for every $t \in \cup_{i=1}^{s_o} [\gamma_i, \delta_i]$

$$\mid f_{n_{k_0}}(t) \mid > \mid f_{n_{k_0}}(t^m) \mid,$$

hence

$$\lambda_{n_{k_0}} \mid f_{n_{k_0}}(t) \mid > 1,$$

and consequently on that

$$\text{mes}\{t : \lambda_{n_{k_0}} \mid f_{n_{k_0}}(t) \mid \ge \frac{1}{2}\} \ge \text{mes} \bigcup_{i=1}^{s_0} [\gamma_i, \delta_i] > \epsilon_0.$$

Since k_0 was an arbitrary number, then for every natural number k

$$\text{mes}\{t : \lambda_{n_k} \mid f_{n_k}(t) \mid \ge \frac{1}{2}\} > \epsilon_0$$

and hence the equality (6) doesn't hold for every speed λ.

Consequently the equality (7) is not true and holds the statement (5).

Necessity is clear since for every speed λ and for every $\alpha > 0$

$$\{t : \lambda_n \mid f_n(t) \mid \ge \alpha\} \subset \text{supp}\lambda_n f_n = \text{supp} f_n.$$

Theorem 3. *The sequence* $F \in c^0_{\infty \text{mes}\lambda}$ *iff there exists a measurable subset* $E \subset [a, b]$ *such that* $\text{mes}E = 0$ *and* $\lim \lambda_n f_n(t) = 0$ *uniformly on* $[a, b] \setminus E$.

Proof. Necessity. If $F \in c^0_{\infty\text{mes}\lambda}$, then $(\lambda f_n) \in c^0_{\infty\text{mes}}$, i.e. for every μ

$$\mu_n \text{mes}\{f : \lambda_n \mid f_n(t) \mid \ge \alpha\} = O(1).$$

Hence for every $\alpha > 0$ there exists n_α such that for $n > n_\alpha$

$$\text{mes}\{t : \lambda_n \mid f_n(t) \mid \ge \alpha\} = 0.$$

Denote by

$$E^n_\alpha := \{t : \lambda_n \mid f_n(t) \mid \ge \alpha\}.$$

Then for $n > n_\alpha$ $\text{mes}E_\alpha^n = 0$ and for fixed $\alpha > 0$ we have $\text{mes}E_\alpha = 0$, where

$$E_\alpha = \bigcup_{n>n_\alpha} E_\alpha^n.$$

If $t \in [a,b] \setminus E_\alpha$, then $\lambda_n \mid f_n(t) \mid < \alpha$ if $n > n_\alpha$, i.e.

$$\lim_{n\to\infty} \lambda_n f_n(t) = 0$$

uniformly on $[a,b] \setminus E_\alpha$.

Sufficiency. If $\lim \lambda_n f_n = 0$ uniformly on $[a,b] \setminus E$ where $\text{mes}E = 0$, then for $n > n_\alpha$ we have $\lambda_n \mid f_n \mid < \alpha$ for all $t \in [a,b] \setminus E$, i.e.

$$\{t : \lambda_n \mid f_n \mid \geq \alpha\} = \varnothing$$

and therefore $F \in c^0_{\infty\text{mes}\lambda}$.

Theorem 4. *The sequence* $F \in c^0_{\infty\text{mes}\infty}$ *iff there exists a natural number* n_0 *such that for* $n > n_0$ $f_n(t) = 0$ *a.e. on* $[a,b]$.

Proof. Necessity. If $F \in c^0_{\infty\text{mes}\infty}$, then for every $\lambda = (\lambda_n)$ $F \in c^0_{\lambda\text{mes}\infty}$ and therefore from theorem 2 it follows that

$$\lambda_n \text{mes supp} f_n = O(1)$$

for every $\lambda = (\lambda_n)$. Hence there exists a natural number n_0 such that for $n > n_0$

$$\text{mes supp} f_n = 0,$$

i.e. $f_n(t) = 0$ a.e. on $[a,b]$.

Sufficiency is clear.

4. The inclusion $c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu}$.

By c_{mes} ($c_{A\text{mes}}$) we denote the set of all sequences $F = (f_n)$ which are convergent in measure to f (A -summable in measure to f , i.e. the sequence $A_n F = \sum_k a_{nk} f_k$ is convergent in measure). In [2,4] the inclusion $c_{\text{mes}} \subset c_{A\text{mes}}$ has been considered. For example, in [2,4] a class of summability methods of finite type is defined.

We say that the summability method A is of finite type if there exists a natural number L such that the number of nonzero elements in each row is less than L.

In [2,4] the following theorems are proved.

Theorem D. *For every sequence of real numbers (c_k) with $c_k \neq 0$ there exists a sequence $(g_k) \in c^0_{\text{mes}}$ such that the series*

$$\sum c_k g_k(t) \tag{8}$$

is divergent in measure.

On the ground of the theorem D it follows (see [4])

Theorem E. *Inclusion $c_{\text{mes}\infty} \subset c_{A\text{mes}}$ holds iff*

1^0 $\quad c \subset c_A$.

2^0 $\quad A$ *is of finite type.*

From corollary C we obtain

Corollary E. *For every sequence of real numbers (c_k) with $c_k \neq 0$ and for every λ with (2) there exists $(g_k) \in c^0_{\lambda\text{mes}\infty}$ such that the series (8) is divergent.*

Proof. Let

$$g_k(t) = \frac{1}{c_k} f_k(t)$$

where $(f_k) \in c^0_{\text{mes}\lambda}$ is defined by (3). Then for every $\alpha > 0$

$$\text{mes}\{t : \mid \sum_{s=k_i+1}^{k_{i+1}} c_s g_s(t) \mid \geq \alpha\} = \text{mes}\{t : \mid \sum_{s=k_i+1}^{k_{i+1}} f_s(t) \mid \geq \alpha\} = 1.$$

Proof of the corollary E is completed.

Now, if $c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu}$ then $c_{\text{mes}\infty} \subset c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu} \subset c_{A\text{mes}}$, i.e., on the ground of the theorem E we obtain that A must be a finite type. Therefore we can prove the following theorem.

Theorem 5. *The inclusion $c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu}$ holds iff*

$1^0 \quad c^\lambda \subset c_A^\mu$,

$2^0 \quad$ *A is of finite type.*

Proof. Necessity of 2^0 has already been proved.

Since $c^\lambda \subset c_{\text{mes}\lambda}$, then from $c_{\text{mes}\lambda} \subset c_{A\text{mes}\mu}$ it follows that $c^\lambda \subset c_{A\text{mes}\mu}$. Now for every $(c_k) \in c^\lambda$ we have that $(d_n) \in c^\mu$, where

$$d_n = \sum_{k=0}^{\infty} a_{nk} c_k,$$

i.e. $c^\lambda \subset c_A^\mu$.

Sufficiency. Assume that A is of finite type, for example L -type. Observe first the transformation

$$y_n = \sum_{i=1}^{L} a_{nk_i(n)} f_{k_i(n)},$$

where $(f_k) \in c_{\text{mes}\lambda}^0$, i.e. $f_k(t) = c_k \tau_k(t)$, where $(\tau_k) \in c_{\text{mes}}^0$ and $(c_k) \in c_0^\lambda$.

Now

$$\text{mes}\{t : \mu_n \mid y_n(t) \mid\geq \sigma\} \leq \text{mes}\{t : \sum_{i=1}^{L} \mid \frac{a_{nk_i(n)}\mu_n}{\lambda_{k_i(n)}} \tau_{k_i(n)}(t) \mid\geq \sigma\} \leq$$

$$\leq \sum_{i=1}^{L} \text{mes}\{t : \frac{\mid a_{nk_i(n)} \mid \mu_n}{\lambda_{k_i(n)}} \mid \tau_{k_i(n)}(t) \mid\geq \frac{\sigma}{2^L}\}.$$

Because $c^\lambda \subset c_A^\mu$, then from [3] we have

$$\mid \frac{a_{nk}\mu_n}{\lambda_k} \mid= O(1).$$

Thus, if $1 \leq i \leq L$, we obtain

$$\text{mes}\{t : \frac{\mid a_{nk_i(n)} \mid \mu_n}{\lambda_{k_i(n)}} \mid \tau_{k_i(n)}(t) \mid\geq \frac{\sigma}{2^L}\} \leq \frac{\epsilon}{L}$$

and hence

$$\text{mes}\{t : \mu_n \mid y_n(t) \mid\geq \sigma\} \leq \sum_{i=1}^{L} \frac{\epsilon}{L} = \epsilon,$$

i.e. $(y_n) \in c_{\text{mes}\mu}^0$.

If in second case $(f_n) \in c_{\mathrm{mes}\lambda}$, then $f_n \to f$ in measure on $[a,b]$ and

$$y_n = \sum_{i=1}^{L} a_{nk_i(n)}(f_{k_i(n)} - f) + f \sum_{i=1}^{L} a_{nk_i(n)}.$$

Since $(f_{k_i(n)} - f) \in c^0_{\mathrm{mes}\lambda}$, then

$$\Big(\sum_{i=1}^{L} a_{nk_i(n)}(f_{k_i(n)} - f)\Big) \in c^0_{\mathrm{mes}\mu}.$$

Since $c^\lambda \subset c^\mu_A$ and $e = (1,1,\ldots) \in c^\lambda$, then there exists the limit

$$\lim_n \sum_{i=1}^{L} a_{nk_i(n)} = a.$$

Therefore

$$\lim_n f \sum_{i=1}^{L} a_{nk_i(n)} = fa$$

and

$$(y_n - fa) \in c^0_{A\mathrm{mes}\mu},$$

i.e.

$$(y_n) \in c_{A\mathrm{mes}\mu}.$$

5. The inclusion $c^0_{\lambda\mathrm{mes}} \subset c^0_{A\mathrm{mes}}$.

We will show that the inclusion

$$c^0_{\lambda\mathrm{mes}} \subset c^0_{A\mathrm{mes}}$$

doesn't imply that A must be a finite type.

Let for λ the (1) be fulfilled. On the ground of the theorem A we have for every $F \in c^0_{\lambda\mathrm{mes}}$ that a.e. on $[a,b]$

$$\lim f_k(t) = 0.$$

If $c^0 \subset c^0_A$, where A is not a finite type, then a.e. on $[a,b]$

$$\lim (A_nF)(t) = 0$$

and therefore $AF \in c^0_{\text{mes}}$, i.e. $c^0_{\lambda\text{mes}} \subset c^0_{A\text{mes}}$.

Let for λ the condition (2) be fulfilled. Then we consider the summability method A which is c^0-convergence preserving and for every natural number n

$$a_{nk_i} \neq 0,$$

where

$$\sum \frac{1}{\lambda_{k_i}} < \infty.$$

Then the summability method A is not a finite type, but

$$A_n F = \sum_{i=1}^{\infty} a_{nk_i} f_{k_i},$$

where $f_{k_i}(t) \to 0$ a.e. on $[a,b]$. Therefore a.e. on $[a,b]$

$$\lim A_n F(t) = 0,$$

i.e. $c^0_{\lambda\text{mes}} \subset c^0_{A\text{mes}}$.

6. The inclusion $c^0_{\text{mes}} \subset c^0_{A\mu\text{mes}}$.

We use the following

Theorem 6. *For every sequence* (α_n) , *where* $\alpha_n \neq 0$ *and for every speed* $\mu = (\mu_n)$ *there exists a measurable function* f *on* $[0,1]$ *such that*

$$\lim_n \mu_n \text{mes}\{t :| \alpha_n f(t) |\geq 1\} = \infty. \tag{9}$$

Proof. If $\alpha_n \neq 0$, then there exists a sequence $\theta = (\theta_n)$, where $0 < \theta_n \nearrow \infty$ such that

$$\frac{1}{\theta_n} \leq | \alpha_n | .$$

Let $\sigma = 1$ and let $f(t) = \lambda(\mu^{-1}(\frac{1}{t^2}))$, where $\lambda = \lambda(t)$ and $\mu = \mu(t)$ are monotonically increasing functions with $\lambda(n) = \lambda_n$, $\mu(n) = \mu_n$ and μ^{-1} denotes the inverse function of μ . Then

$$\text{mes}\{t :| \alpha_n f(t) |\geq 1\} \geq \text{mes}\{t : \frac{\lambda(\mu^{-1}(\frac{1}{t^2}))}{\lambda_n} \geq 1\} =$$

$$= \text{mes}\{t : \lambda(\mu^{-1}(\frac{1}{t^2})) \geq \lambda(n)\} = \text{mes}\{t : \frac{1}{t^2} \geq \mu_n\} = \text{mes}\{t : t \leq \frac{1}{\sqrt{\mu_n}}\} =$$

$$= \frac{1}{\sqrt{\mu_n}},$$

i.e.

$$\lim_n \mu_n \text{mes}\{t : | \alpha_n f(t) | \geq 1\} \geq \lim \sqrt{\mu_n} = \infty.$$

Corollary 6. *If*

$$c^0_{\infty\text{mes}\infty} \subset c^0_{A\mu\text{mes}},$$

then for every natural number k *there exists a natural number* n_k *such that for* $n \geq n_k$

$$a_{nk} = 0.$$

Proof. Let for every natural numbers i *and* k *there exists a natural number* n_i *such that*

$$a_{n_i k} \neq 0.$$

Let k *be fixed,* $F^0 = (f^0_v) = (f^0 \delta_{kv})$ *, where* δ_{kv} *is Kronecker's symbol and* f^0 *is a function from Theorem 6 with* $\alpha_i = a_{n_i k}$ *. Then*

$$AF = (A_{n_i} F^0) = (a_{n_i k} f^0)$$

and

$$\lim \mu_n \text{mes}\{t : | a_{nk} f^0(t) | \geq 1\} = \infty,$$

i.e.

$$AF^0 \notin c^0_{\mu\text{mes}}.$$

Since $F^0 \in c^0_{\infty\text{mes}\infty}$ *, then we have obtained that* $c^0_{\infty\text{mes}\infty} \not\subset c^0_{A\mu\text{mes}}$ *. The proof is completed.*

Theorem 7. *The inclusion* $c^0_{\text{mes}\infty} \subset c^0_{A\mu\text{mes}}$ *holds iff there exists a natural number* M *such that for* $k, n > M$

$$a_{nk} \equiv 0.$$

Proof. Necessity. Since $c^0_{A\mu\text{mes}} \subset c^0_{A\text{mes}}$ *, then from Theorem C we get that* A *is a finite type. From Corollary 6 it follows that for every natural number* k *there exists a natural number* n_k *such that* $a_{nk} = 0$ *if* $n \geq n_k$ *.*

Assume that for natural numbers N *and* K *there exist* $k > K$ *and* $n > N$ *such that* $a_{nk} \neq 0$.

Since A *is of finite type, then we can find* k_1 *and* n_1 *such that* $a_{n_1 k_1} \neq 0$ *, while* $a_{n_1 k} = 0$ *for every* $k > k_1$ *. From Corollary 6 there exists a natural number* m_2 *such that* $a_{nk_1} = 0$ *if* $n \geq m_2$ *. According to our assumption we can find* $n_2 \geq m_2$ *and* $k_2 \geq k_1 + 1$ *such that* $a_{n_2 k_2} \neq 0$ *, but* $a_{n_2 k} = 0$ *, if* $k > k_2$ *. Continuing analogously we construct the sequences of natural numbers* (n_i) *and* (k_i) *such that* $a_{n_i k_i} \neq 0$ *,* $a_{n_i k} = 0$ *if* $k > k_i$ *and* $a_{nk_i - 1} = 0$ *if* $n > n_i$ *.*

Let

$$f_k^0 = f^0 \delta_{kk_i},$$

where f^0 *is from Theorem 6, and for which*

$$\lim_i \mu_{n_i} \mathrm{mes}\{t : | a_{n_i k_i} f^0(t) | \geq 1\} = \infty.$$

Then

$$A_{n_i} F = a_{n_i k_i} f^0$$

and

$$\lim \mu_{n_i} \mathrm{mes}\{t : | A_{n_i} F | \geq 1\} = \infty,$$

i.e.

$$AF \notin c^0_{\mu \mathrm{mes}}.$$

Hence we have obtained that at least one of the following possibilities must be true:

I: there exists a natural number K *such that* $a_{nk} = 0$ *if* $k > K$

or

II: there exists a natural number N *such that* $a_{nk} = 0$ *, if* $n > N$ *.*

In the first case we obtain from Lemma 6 that there exists a natural number N_K *such that* $a_{nk} = 0$ *, if* $n > N_K$ *and* $k > K$ *. Therefore we can find a natural number* M *such that* $a_{nk} = 0$ *for* $k, n > M$ *.*

Since A *is of finite type we obtain in second case that for every natural number* n *there exists a natural number* k_n *such that* $a_{nk_n} = 0$ *, if* $k > k_n$ *. Hence, if*

$$M = \max_{n \leq N} k_n,$$

then we have that $a_{nk} = 0$ *if* $n, k > M$ *.*

The proof of the necessity of Theorem 7 is completed.

Sufficiency. Let the condition of Theorem 7 be fulfilled. Then for every $F \in c^0_{\text{mes}\infty}$ *we have*

$$A_n F = \begin{cases} \sum_{k=0}^{M} a_{nk} f_k(t) & \text{if } n \le M, \\ 0 & \text{if } n > M. \end{cases}$$

Therefore we have that $(A_n F) \in c^0_{\infty \text{mes}\infty}$, *i.e.* $F \in c^0_{A\infty\text{mes}\infty} \subset c^0_{A\mu\text{mes}}$. *The proof of Theorem 7 is completed.*

Remark. *On the ground of Theorem 7 we have that* $c^0_{\text{mes}\infty} \subset c^0_{A\mu\text{mes}}$ *holds iff* $c^0_{\text{mes}\infty} \subset c^0_{A\infty\text{mes}\infty}$.

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