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Lebesgue's functions and summability of functional series with speed almost everywhere

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Dedicated to Professor Karoly Tandory on the Occasion of his 70th Birthday

1. We consider* the series in form

$$\sum \xi_k \varphi_k(t), \tag{1}$$

where $\varphi = \{\varphi_k\}$ is a system of integrable functions in $e = [a, b]$, $\lambda = (\lambda_k)$ is a sequence with $0 < \lambda_k \nearrow$ and $x = (\xi_k) \in l^2_\lambda$, i.e.

$$\sum \xi_k^2 \lambda_k^2 < \infty.$$

The following definitions are based on [6].

Let $A = (a_{nk})$ be a triangular summability method and $z = (\zeta_k) \in c$ with $\lim \zeta_k = \zeta$.

It is said that a sequence $z \in c$ is convergent (bounded) with speed λ or λ-convergent (λ-bounded), if the limit

$$\lim_n \lambda_n(\zeta_n - \zeta)$$

exists $(\lambda_n(\zeta_n - \zeta) = O(1))$.

The set of all λ-convergent (λ-bounded) sequences is denoted by $c^\lambda (m^\lambda)$.

It is said that a sequence z is A-summable (A-bounded) with speed λ or A^λ-summable (A^λ-bounded), if $y \in c^\lambda (y \in m^\lambda)$ where $y = (\eta_n)$ and

$$\eta_n = \sum_{k=0}^{n} a_{nk}\zeta_k.$$

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It is said that a summability method A is λ-convergence (λ-boundedness) preserving if every element of the set c^λ is A^λ-summable (A^λ-bounded).

It is said that a series (1) is A^λ-summable (A^λ-bounded) almost everywhere (a.e.) in e, if the limits

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) = f_x(t)$$

and

$$\lim_n \beta_n(x,t)$$

exist a.e. in e where

$$\beta_n(x,t) = \lambda_n\Big(\sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t) - f_x(t)\Big)$$

and

$$\alpha_{nk} - \alpha_{n,k+1} = a_{nk}.$$

The aim of the present article is to determine necessary and sufficient conditions for the A^λ-summability a.e. in e of the series (1) where $x \in l^2_\lambda$.

Let the series (1) be A^λ-summable a.e. in e for every $x \in l^2_\lambda$, then the series (1) is A-summable a.e. in e for every $x \in l^2_\lambda$. It follows that the limit

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\xi_k\varphi_k(t)$$

exists in measure in e for every $x \in l^2_\lambda$ or, the limit

$$\lim_n \sum_{k=0}^{n} \alpha_{nk}\theta_k \frac{\varphi_k(t)}{\lambda_k}$$

exists in measure in e for every $\theta = (\theta_k) \in l^2$, where $\theta_k = \xi_k\lambda_k$.

It follows from [10] (see Lemma 2 on page 70) that the system $\varphi \,/\, \lambda = (\varphi_k \,/\, \lambda_k)$ is the convergence system in measure in l^2. By Theorem 7 of [8] we get that for every $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ with $\mathrm{mes} T_\epsilon > \mathrm{mes} e - \epsilon$ and the constant M_ϵ such that for every $\theta \in l^2$

$$\int_{T_\epsilon} \Big|\sum_{k=0}^{n} \xi_k\lambda_k \frac{\varphi_k(t)}{\lambda_k}\Big|\, dt \le M_\epsilon\Big(\sum_{k=0}^{n} \xi_k^2\lambda_k^2\Big)^{1/2}$$

is valid, i.e.

$$\int_{T_\epsilon} \Big|\sum_{k=0}^{n} \xi_k\varphi_k(t)\Big|\, dt \le M_\epsilon\Big(\sum_{k=0}^{n} \xi_k^2\lambda_k^2\Big)^{1/2},$$

for every $x \in l^2_\lambda$.

In the case $\lambda_n \equiv 1$, $\alpha_{nk} \equiv 1$ or $\alpha_{nk} = (1 - \frac{k}{n+1})$ we get the well-known results of Kaczmarz [2,3].

Theorem A. *The orthogonal series* (1) *is a.e. in e convergent for every* $x \in l^2$, *if in e*

$$L_n(\varphi, t) = O(1),$$

where

$$L_n(\varphi, t) = \int_e \mid \sum_{k=0}^{n} \varphi_k(t)\varphi_k(\tau) \mid d\tau.$$

Theorem B. *The orthogonal series* (1) *is a.e. in e* C^1*- summable for every* $x \in l^2$, *if in e*

$$L_n(C^1, \varphi, t) = O(1),$$

where

$$L_n(C^1, \varphi, t) = \int_e \mid \sum_{k=0}^{n} \Big(1 - \frac{k}{n+1}\Big) \varphi_k(t)\varphi_k(\tau) \mid d\tau.$$

On the other hand Móricz and Tandori have proved that there exist a regular triangular summability method A_0, the orthogonal system φ_0 in e and the $x_0 \in l^2$ such that for every $t \in e$

$$L_n(A_0, \varphi_0, t) = O(1),$$

while the series (1) is not A_0-summable by $\varphi = \varphi_0$ and $x = x_0$ a.e. in e.

Here (see [4])

$$L_n(A_0, \varphi_0, t) = \int_e \mid \sum_{k=0}^{n} \alpha^0_{nk}\varphi^0_k(t)\varphi^0_k(\tau) \mid d\tau.$$

This motivates the introduction of the class of summability methods, for which from the condition

$$L_n(A, \varphi, t) = O_t(1)$$

a.e. in e follows that the series (1) is A-summable a.e. in e for every $x \in l^2$ (see [11]).

On the other hand the following result is proved in [11].

Theorem C. *Let for every* $k \in \mathbb{N}$

$$\lim_n \alpha_{nk} = 1. \tag{2}$$

The series (1) is A-summable a.e. in e for every $x \in l^2$ *iff for every* $\epsilon > 0$ *there exist a measurable subset* $T_\epsilon \subset e$ *where* $\mathrm{mes} T_\epsilon > \mathrm{mes} e - \epsilon$ *and a constant* $M_\epsilon > 0$ *such that the inequality*

$$\mid \int_{T_\epsilon}\int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(u) \sum_{p=0}^{m} \chi_{mp}(v) \sum_{k=0}^{v(n,p)} \alpha_{nk}\alpha_{pk}\varphi_k(u)\varphi_k(v)dudv \mid \leq M_\epsilon$$

holds uniformly for all decompositions

$$\mathfrak{M} =$$

$$= \{\mathfrak{M}_{mn} : n = 0, 1, \ldots, m; \mathfrak{M}_{mk} \bigcap \mathfrak{M}_{mn} = \varnothing \; if \; k \neq n; \bigcup_{n=0}^{m} \mathfrak{M}_{mn} = e\} \quad (3)$$

of e where

$$\chi_{mn} = \chi_{\mathfrak{M}_{mn}}$$

and

$$v(n,p) = \min(n,p).$$

From Theorem C it follows

Theorem D. *Let the condition* (2) *be fulfilled for summability method* A. *If a.e. in* e

$$\Lambda_n(A, \varphi, t) = O_t(1)$$

where

$$\Lambda_n(A, \varphi, t) = \int_e \sup_{p \geq n} \mid \sum_{k=0}^{n} \alpha_{nk}\alpha_{pk}\varphi_k(t)\varphi_k(\tau) \mid d\tau,$$

then the series (1) *is A-summable a.e. in e for every* $x \in l^2$.

The following theorem is proved in [5].

Theorem E. *The series* (1) *is* λ*-convergent a.e. in e for all* $x \in l^2_\lambda$ *iff for each* $\epsilon > 0$ *there exist a measurable subset* $T_\epsilon \subset e$ *where* $\text{mes}T_\epsilon \geq b - a - \epsilon$ *and a constant* $M_\epsilon > 0$ *such that the inequality*

$$\mid \int_{T_\epsilon} \int_{T_\epsilon} \sum_{n=0}^{m-1} \chi_{mn}(t)\lambda_n \sum_{p=0}^{m-1} \chi_{mp}(\tau)\lambda_p \sum_{k=\mu(n,p)}^{m} \lambda_k^{-2}\varphi_k(t)\varphi_k(\tau)dtd\tau \mid \leq M_\epsilon \quad (4)$$

holds uniformly for all decompositions (3) where $\mu(n,p) = \max(n,p) + 1$.

Corollary E. *If the inequality*

$$\sup_{m,n} \lambda_n^2 \int_e \mid \sum_{k=n+1}^{m} \frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2} \mid d\tau < \infty$$

holds a.e. in e, then the series (1) *is* λ*-convergent a.e. in e for all* $x \in l^2_\lambda$.

What follows is the principal result of this paper.

Theorem. *Let summability method A be λ^2-convergence preserving and let (2) be valid. The series (1) is A^λ-summable a.e. in e for all $x \in l^2_\lambda$ iff*

1^0 the series (1) is A-summable a.e. in e for every $\lambda \in l^2_\lambda$,

2^0 for each $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ where $\mathrm{mes} T_\epsilon >$ *$b - a - \epsilon$ and a constant $M_\epsilon > 0$ such that the inequality*

$$\left| \int_{T_\epsilon} \int_{T_\epsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \sum_{v=0}^{m} \varphi_v(t)\varphi_v(\tau) D^m_{npk} dt d\tau \right| \leq M_\epsilon \qquad \text{(A)}$$

holds uniformly for all decomposition (3) where

$$D^m_{npv} = \begin{cases} (\alpha_{mv} - \alpha_{nv})(\alpha_{mv} - \alpha_{pv}) \dfrac{\lambda_n \lambda_p}{\lambda_v^2} & \text{if } 0 \leq v \leq n < p \leq m, \\ 3\alpha_{mv}(\alpha_{mv} - \alpha_{pv}) \dfrac{\lambda_n \lambda_p}{\lambda_v^2} & \text{if } n < v \leq p \leq m, \\ \alpha^2_{mv} \dfrac{\lambda_n \lambda_p}{\lambda_v^2} & \text{if } n \leq p < v \leq m. \end{cases}$$

Corollary. *Let A be λ^2-convergence preserving. If the inequality (2) holds, the series (1) is A-summable a.e. in e for every $x \in l^2_\lambda$ and a.e. in e holds*

$$\Lambda^m_n(A, \varphi, t) = O_t(1)$$

where

$$\Lambda^m_n(A, \varphi, t) = \int_e \sup_{p \geq n} \left| \sum_{v=0}^{m-1} \varphi_v(t)\varphi_v(\tau) D^m_{npv} \right| d\tau,$$

then the series (1) is a.e. in e A^λ-summable for all $x \in l^2_\lambda$.

2. The following definitions and results are due to [8]. Let M_e denote the space of all measurable and a.e. in e finite functions with

$$\| f \| = \inf_{\alpha > 0} (\alpha + \mathrm{mes}\{t \in e : | f(t) | > \alpha\}).$$

It is said that the operator A from l^2 into M_e is openlinear if for every $x \in l^2$ there exists a linear operator T_x from l^2 into M_e such that

$$1^0 \quad T_x(x) = A(x),$$

$$2^0 \quad | T_x(y, t) | \leq | A(y, t) |$$

a.e. in e for every $y \in l^2$.

It is said that the set $Q \subset M_e$ is bounded in measure if for each $\epsilon > 0$ there exists a constant $R_\epsilon > 0$ such that

$$\text{mes}\{t \in e : | y(t) | \geq R_\epsilon\} \leq \epsilon$$

for all $y \in Q$.

It is said that the operator A from l^2 into M_e is bounded in measure if the set

$$\{A(x) : \| x \| \leq 1\}$$

is bounded in measure.

Lemma 1. (see [8], p. 137). *Let A be a bounded openlinear operator from l^2 into M_e. Then for each $\epsilon > 0$ there exists a measurable subset $T_\epsilon \subset e$ where $\text{mes}T_\epsilon > b - a - \epsilon$ and a constant $M_\epsilon > 0$ such that*

$$\int_{T_\epsilon} | A(x,t) | \, dt \leq M_\epsilon \| x \|$$

for all $x \in l^2$.

Lemma 2. (see [12], p. 142). *Let f be a measurable function in e. Then the inequality*

$$| f(t) | < \infty$$

holds a.e. in e iff for each $\epsilon > 0$ there exists a measurable subset $T_\epsilon \subset e$ where $\text{mes}T_\epsilon > b - a - \epsilon$ such that

$$\int_{T_\epsilon} | f(t) | \, dt < \infty.$$

Lemma 3. (see [12], p. 142). *Let (f_n) be a sequence of functions integrable in e. Then the inequality*

$$\sup_n | f_n(t) | < \infty$$

holds a.e. in e iff for each $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ where $\text{mes}T_\epsilon > b - a - \epsilon$ and a constant $M_{\epsilon > 0}$ such that the inequality

$$\left| \int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(t) f_n(t) dt \right| \leq M_\epsilon$$

holds uniformly for every decomposition (3) of e.

Lemma 4. (see [1], p. 361). *Let $D_m (m \in \mathbb{N})$ be the continuous homogenous operators from l^2 into M_e and suppose that the inequality*

$$| D_m(x_1 + x_2, t) | \leq | D_m(x_1, t) | + | D_m(x_2, t) |$$

holds. Then if

$$1^0 \quad \sup_n \mid D_n(x,t) \mid < \infty$$

a.e. in e for any $x \in l^2$ and there exists

$$2^0 \quad \lim_n D_n(\hat{x},t)$$

a.e. in e for any $\hat{x}$ from a total set in l^2, then there exists

$$\lim_n D_n(x,t)$$

a.e. in e for each $x \in l^2$.

Lemma 5. *Let the series (1) be A^λ-summable a.e. in e for each $x \in l^2_\lambda$. Then the operator D from l^2 into M_e, where*

$$D(z,t) = \sup_n \mid D_n(z,t) \mid,$$

$$D_n(z,t) = \lambda_n \Big(\sum_{k=0}^{n} \alpha_{nk} \xi_k \lambda_k \frac{\varphi_k(t)}{\lambda_k} - f_z(t) \Big),$$

$$f_z(t) = \lim_n \sum_{k=0}^{n} \alpha_{nk} \xi_k \lambda_k \frac{\varphi_k(t)}{\lambda_k}$$

and

$$z = (\xi_k \lambda_k),$$

is openlinear and bounded in measure.

Proof. It was shown by Nikishin that D is openlinear (see [8], p. 135). We shall prove the boundedness of D.

Since the inequality

$$D_n(z,t) \le D(z,t)$$

holds a.e. in e for every $z \in l^2$, then from the A^λ- summability of the series (1) a.e. in e for every $x \in l^2_\lambda$ we get that the equality

$$\lim_{\beta \to 0} \beta D_m(z,t) = 0$$

holds a.e. in e for all $z \in l^2$ uniformly in m.

Consequently, in space M_e for all $z \in l^2$ we have

$$\lim_{\beta \to 0} \beta D_m(z) = 0$$

uniformly in m.

By the principle of equicontinuity we have that in M_e

$$\lim_{z\to 0} D_m(z) = 0$$

uniformly in m.

It follows that there exists a constant $M > 0$ such that for all z with $\| z \| \leq 1$

$$\| D_m(z) \| \leq M$$

holds in M_e uniformly in m. Since in M_e $\lim_m D_m(z) = D(z)$ we have $\lim_{\beta\to 0} \beta D(z) = \theta$ uniformly in the unit sphere of l^2.

It follows that for each $\epsilon > 0$ there exists a constant β_ϵ such that for all z from unit sphere of l^2

$$\inf_\alpha (\alpha + \text{mes}\{t \in e : D(z,t) > \frac{\alpha}{\beta_\epsilon}\}) < \epsilon \,/\, 2$$

holds. Furthermore, there exists a constant α_ϵ such that

$$\inf_\alpha (\alpha + \text{mes}\{t \in e : D(z,t) > \frac{\alpha}{\beta_\epsilon}\}) >$$

$$> \alpha_\epsilon + \text{mes}\{t \in e : D(z,t) > \frac{\alpha}{\beta_\epsilon}\} - \epsilon \,/\, 2.$$

If we denote $R_\epsilon = \alpha \,/\, \beta_\epsilon$, then we have

$$\text{mes}\{t \in e : D(z,t) > R_\epsilon\} < \epsilon$$

for all z from the unit sphere of l^2. D is bounded in measure in l^2 to M_e.

Lemma 6. (see [10], p. 70). *Let the condition* (2) *be fulfilled. If the series* (1) *for all* $x \in l^2$ *is A-summable in measure in* e, *then the series* (1) *is convergent in measure in* e *for all* $x \in l^2$.

Lemma 7. (see [8], p. 158). *The series* (1) *is convergent in measure in* e *for all* $x \in l^2$ *iff for every* $\epsilon > 0$ *there exist a measurable subset* $T_\epsilon \subset e$ *with* $\text{mes}T_\epsilon > b - a - \epsilon$ *and a constant* $M_\epsilon > 0$ *such that the inequality*

$$\int_{T_\epsilon} \mid \sum_{k=0}^{N} a_k \varphi_k(t) \mid dt \leq M_\epsilon \{\sum_{k=0}^{N} a_k^2\}^{1/2}$$

holds for every natural number N *and for every real number* $\{a_0, a_1, \ldots, a_N\}$.

Lemma 8. (see [9], p. 338). *If the series* (1) *is convergent in measure for all* $x \in l^2$, *then there exist a measurable subset* T_ϵ *with* $\text{mes}T_\epsilon > b-a-\epsilon$, *a constant* N_ϵ *and a orthonormal system* $g = (g_k)$ *on* e, *such that* $\varphi_n(t) = M_\epsilon g_n(t)$ *a.e. in* T_ϵ.

3. *Proof of Theorem. Neccessity.* Let the series (1) be A^λ-summable a.e. in e for every $x \in l^2_\lambda$. Since

$$\sum_{m=0}^{\infty} \sum_{k=0}^{m} \bar{\alpha}_{mk} \xi_k \varphi_k(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \alpha_{nk} \xi_k \varphi_k(t) = f_x(t)$$

where

$$\bar{\alpha}_{mk} = \alpha_{mk} - \alpha_{m-1,k},$$

then

$$D_n(z,t) = \lambda_n \sum_{m=n+1}^{\infty} \sum_{k=0}^{m} \bar{\alpha}_{mk} \xi_k \lambda_k \frac{\varphi_k(t)}{\lambda_k}.$$

By using Lemmas 5 and 1 we have that for each $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ with $\mathrm{mes}T_\epsilon > b - a - \epsilon$ and a constant $M_\epsilon > 0$ such that

$$\int_{T_\epsilon} \sup_n \mid D_n(z,t) \mid dt \leq M_\epsilon \parallel z \parallel \tag{5}$$

holds for every $z \in l^2$.

From Levi's Theorem and Lemma 2 it follows that the condition (5) is equivalent to the following condition:

for every $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ with $\mathrm{mes}T_\epsilon > b - a - \epsilon$ and a constant $M_\epsilon > 0$ such that the inequality

$$\mid A_m z \mid := \mid \int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(t) D_n(z,t) dt \mid \leq M_\epsilon \parallel z \parallel \tag{6}$$

holds uniformly for every decomposition (3) of e and for all $z \in l^2$.

Now

$$\mid A_m z \mid - \mid A_m^2 z \mid \leq \mid A_m^1 z \mid \leq \mid A_m z \mid + \mid A_m^2 z \mid \tag{7}$$

where

$$A_m^1 z = \int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(t) \lambda_n \sum_{p=n+1}^{m} \sum_{k=0}^{p} \bar{\alpha}_{pk} \xi_k \varphi_k(t) dt$$

and

$$A_m^2 z = \int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(t) \lambda_n \sum_{p=m+1}^{\infty} \sum_{k=0}^{p} \bar{\alpha}_{pk} \xi_k \varphi_k(t).$$

By using Lemmas 6 and 7 we have

$$\mid A_m^2 z \mid = \mid \int_{T_\epsilon} \sum_{n=0}^{m} \chi_{mn}(t) \lambda_n \Big(\sum_{k=0}^{m} \alpha_{mk} \xi_k \varphi_k(t) - f_x(t) \Big) dt \mid \leq$$

$$\leq \int_e \sum_{n=0}^{m} \chi_{mn}(t) \lambda_m \mid \sum_{k=0}^{m} \alpha_{mk} \xi_k \varphi_k(t) - f_x(t) \mid dt \leq$$

$$\le \sqrt{b-a}\Big(\lambda_m^2 \int_e \Big| \sum_{k=0}^{m} (\alpha_{mk}-1)\xi_k\varphi_k(t) + \sum_{k=0}^{m} \xi_k\varphi_k(t) - f_x(t)\Big|^2 dt\Big)^{1/2} \le$$

$$\le \sqrt{b-a}\Big[\Big(\lambda_m \sum_{k=0}^{m} \frac{(\alpha_{mk}-1)^2}{\lambda_k^2}\xi_k^2\lambda_k^2\Big)^{1/2} + \Big(\lambda_m^2 \sum_{k=m+1}^{\infty} \xi_k^2\lambda_k^2\lambda_k^{-2}\Big)^{1/2}\Big] \le$$

$$\le \sqrt{b-a}\Big[\sup_{k\le m} \frac{\lambda_m \mid \alpha_{mk}-1 \mid}{\lambda_k}\Big(\sum_{k=0}^{m} \xi_k^2\lambda_k^2\Big)^{1/2} + \Big(\sum_{k=m+1}^{\infty} \xi_k^2\lambda_k^2\Big)^{1/2}\Big].$$

From [7] (see Lemma 3) it follows that

$$\frac{\lambda_m \mid \alpha_{mk}-1 \mid}{\lambda_k} = O(1).$$

Therefore, we have

$$A_m^2 z = O(\| z \|). \tag{8}$$

It follows from (6), (7) and (8) that the condition (5) is equivalent to the following condition:

for every $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ with mes$T_\epsilon > b-a-\epsilon$ and a constant $N_\epsilon > 0$ such that the inequality

$$\mid A_m^1 z \mid \le N_\epsilon \| z \| \tag{9}$$

holds uniformly for every decomposition (3) of e and for all $z \in l^2$.

If we denote

$$A_k^m(t) = \sum_{n=0}^{k-1} \chi_{mn}(t)\lambda_n,$$

then we have

$$A_m^1 z = \int_{T_\epsilon} \sum_{k=1}^{m}\sum_{v=0}^{k} \bar\alpha_{kv}\xi_v\varphi_v(t)A_k^m(t)dt = B_m + C_m$$

where

$$B_m = \sum_{v=1}^{m} \xi_v \int_{T_\epsilon} \sum_{k=v}^{m} \bar\alpha_{kv} A_k^m(t)dt \tag{10}$$

and

$$C_m = \int_{T_\epsilon} \sum_{k=1}^{m} \bar\alpha_{k0}\xi_0\varphi_0(t)A_k^m(t),$$

i.e.

$$\mid A_m^1 z \mid - \mid C_m \mid \le \mid B_m z \mid \le \mid A_m^1 z \mid + \mid C_m \mid .$$

Since

$$\mid C_m \mid = \Big| \int_{T_\epsilon} \sum_{n=1}^{m} \chi_n(t)\lambda_{n-1}(\alpha_{m0}-\alpha_{n-1,0})\xi_0\varphi_0(t)dt \Big|$$

and, therefore

$$| C_m | \leq (\lambda_{m-1} \mid \alpha_{m0} - 1 \mid + \max_{k \leq m} \lambda_{k-1} \mid \alpha_{k-1,0} - 1 \mid) \int_e \mid \xi_0 \varphi_0(t) \mid dt,$$

then from [7] (see Lemma 3) it follows that

$$\lambda_m \mid \alpha_{mk} - 1 \mid = O(1).$$

So, we have that

$$C_m = O(1)$$

and the inequality (9) holds iff

$$| B_m z | \leq N_\epsilon^1 \| z \| . \quad (11)$$

By (10) we may consider B_m as a bounded linear functional in l^2 for fixed $\epsilon > 0$ and $m \in \mathbb{N}$.

Using the principle of uniform boundedness, we get that the inequality (10) holds iff

$$\| B_m \| = O_\epsilon(1). \quad (12)$$

Since

$$\| B_m \|^2 = \sum_{v=1}^{m} B_{mv}^2$$

where

$$B_{mv} = \int_{T_\epsilon} \sum_{k=v}^{m} \bar{\alpha}_{kv} A_k^m \frac{\varphi_v(t)}{\lambda_v},$$

then the (12) holds iff

$$\sum_{v=0}^{m} B_{mv}^2 =$$

$$= \int_{T_\epsilon} \int_{T_\epsilon} \sum_{v=0}^{m} \frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2} \sum_{k=v}^{m} \bar{\alpha}_{kv} A_k^m(t) \sum_{p=v}^{m} \bar{\alpha}_{pv} A_p^m(\tau) dt d\tau = O_\epsilon(1). \quad (13)$$

We have

$$\sum_{k=v}^{m} \bar{\alpha}_{kv} A_k^m(t) = \alpha_{mv} A_v^m(t) + \sum_{n=v}^{m-1} \chi_n(t) \lambda_n A_{nv}^m$$

where

$$A_{nv}^m = \alpha_{mv} - \alpha_{nv}$$

and, therefore

$$\sum_{v=0}^{m} B_{mv}^2 = B_m^1 + 2B_m^2 + B_n^3$$

where

$$B_m^1 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{v=0}^{m}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\alpha_{mv}^2 A_v^m(t)A_v^m(\tau)dtd\tau,$$

$$B_m^2 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{v=0}^{m-1}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\alpha_{mv}A_v^m(t)\sum_{p=v+1}^{m}\chi_{mp}(\tau)\lambda_p A_{pv}^m dtd\tau$$

and

$$B_m^3 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{v=0}^{m-1}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\sum_{n=v+1}^{m}\chi_{mn}(t)\lambda_n A_{nv}^m\sum_{p=v+1}^{m}\chi_{mp}(\tau)\lambda_p A_{pv}^m dtd\tau.$$

On the other hand,

$$B_m^1 = 2\int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\lambda_n\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\lambda_p\sum_{v=n+1}^{m}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\alpha_{mv}^2 dtd\tau+$$

$$+\int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-1}\chi_{mn}(t)\lambda_n^2\chi_{mn}(\tau)\sum_{v=n+1}^{m}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\alpha_{mv}^2 dtd\tau,$$

$$B_m^2 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\lambda_n\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\lambda_p\sum_{v=n+1}^{p}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}A_{nv}^m A_{pv}^m dtd\tau$$

and

$$B_m^3 = 2\int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-2}\chi_n(t)\lambda_n\sum_{p=n+1}^{m-1}\chi_n(\tau)\lambda_p\sum_{v=0}^{p}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}A_{nv}^m A_{pv}^m dtd\tau+$$

$$+\int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-1}\chi_n(t)\chi_n(\tau)\lambda_n^2\sum_{v=0}^{n}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}(A_{nv}^m)^2 dtd\tau.$$

Now we have that

$$\sum_{v=0}^{m} B_{mv}^2 = B_m^4 + B_m^5 + B_m^6 \qquad (14)$$

where

$$B_m^4 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\sum_{v=n+1}^{m}\varphi_v(t)\varphi_v(\tau)D_{npv}^m dtd\tau$$

where D_{npv}^m is as defined in Theorem,

$$B_m^5 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-1}\chi_{mn}(t)\chi_{mn}(\tau)\sum_{v=n+1}^{m}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}\alpha_{mn}^2\lambda_n^2 dtd\tau$$

and

$$B_m^6 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-1}\chi_{mn}(t)\chi_{mn}(\tau)\sum_{v=0}^{n}\frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2}(A_{nv}^m)^2\lambda_n^2 dtd\tau.$$

Since for $v > n$, $A_{nv}^m = \alpha_{mv}$, then

$$B_m^5 + B_m^6 = B_n^7 \tag{15}$$

where

$$B_m^7 = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-1}\chi_{mn}(t)\chi_{mn}(\tau)\sum_{v=0}^{m}\varphi_v(t)\varphi_v(\tau)(\frac{\lambda_n A_{nv}^m}{\lambda_v})^2 dtd\tau =$$

$$= \sum_{v=0}^{m}\sum_{n=0}^{m-1}(\frac{\lambda_n A_{nv}^m}{\lambda_v})^2(\int_{T_\epsilon}\varphi_v(t)\chi_{mn}(t)dt)^2 =$$

$$= \sum_{n=0}^{m-1}\sum_{v=0}^{m}(\frac{\lambda_n A_{nv}^m}{\lambda_v})^2(\int_{T_\epsilon}\varphi_v(t)\chi_{mn}(t)dt)^2.$$

We have

$$\mid \frac{\lambda_n A_{nv}^m}{\lambda_v} \mid \le \frac{\lambda_n \mid \alpha_{mv} - 1 \mid}{\lambda_v} + \frac{\lambda_n \mid \alpha_{nv} - 1 \mid}{\lambda_v} = O(1).$$

Therefore

$$B_m^7 = O(1)\sum_{n=0}^{m}\sum_{v=0}^{\infty}(\int_{T_\epsilon}\varphi_v(t)\chi_{mn}(t)dt)^2.$$

Using Lemma 8 and Bessel's inequality, we get that

$$\sum_{v=0}^{\infty}(\int_{T_\epsilon}\varphi_v(t)\chi_{mn}(t)dt)^2 = N_\epsilon^2\sum_{v=0}^{m}(\int_e \chi_{T_\epsilon}(t)\chi_{mn}(t)g_v(t)dt)^2 \le$$

$$\le N_\epsilon^2\int_e \chi_{T_\epsilon}^2(t)\chi_{mn}^2(t)dt \le N_\epsilon^2\int_e \chi_{mn}(t)dt = N_\epsilon^2 \mathrm{mes}\mathfrak{M}_{mn}.$$

Also,

$$B_m^7 = O(1)N_\epsilon^2\sum_{n=0}^{m}\mathrm{mes}\mathfrak{M}_{mn} = O(1)N_\epsilon^2(b-a). \tag{16}$$

It follows from (14), (15) and (16) that the condition (13) is equivalent to the following condition:

for every $\epsilon > 0$ there exist a measurable subset $T_\epsilon \subset e$ with $\mathrm{mes}T_\epsilon > b - a - \epsilon$ and a constant $M_\epsilon > 0$ such that the inequality

$$\mid \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\sum_{v=0}^{m}\varphi_v(t)\varphi_v(\tau)D_{npv}^m dtd\tau \mid \le N_\epsilon$$

holds uniformly for all decompositions (3) of e.

The proof of the necessity of Theorem is complete.

Sufficiency. Let the conditions 1^0 and 2^0 of Theorem be fulfilled. From 2^0 it follows that the condition (13) holds. As the condition (13) is equivalent to the condition (9), then by 1^0 we have that the condition (8) holds. It follows that the condition (9) is equivalent to the condition (5). Now, from Lemma 2 it follows, that the condition 1^0 of Lemma 4 holds.

Since

$$D_n(e_i, t) = \lambda_n(1 - \alpha_{ni})\varphi_i(t),$$

and from [7] (see Lemma 3) it follows that

$$\lim_n \lambda_n(\alpha_{ni} - 1) = \lim \lambda_n(\sum_{k=0}^{n} \alpha_{nk}\delta_{ki} - 1) = 0,$$

(because A is λ^2- convergence preserving) and the set $\{e_i = (\delta_{ik})\ i \in \mathbb{N}\}$ is total in l^2_λ, then the condition 2^0 of Lemma 4 is fulfilled. In addition, the series (1) is A^λ-summable a.e. in e for every $x \in l^2_\lambda$. The proof is complete.

Remark. If we take in the inequality (A) in our Theorem $\alpha_{nv} \equiv 1$ if $v \leq n$ and $\alpha_{nv} \equiv 0$ if $v > n$, then we get the following inequalities

$$U^\epsilon_m := \left| \int_{T_\epsilon}\int_{T_\epsilon} \sum_{n=0}^{m-2} \chi_{mn}(t)\lambda_n \sum_{p=n+1}^{m-1} \chi_{mp}(\tau)\lambda_p \sum_{v=p+1}^{m} \frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2} dt d\tau \right| \leq M_\epsilon$$

and

$$V^\epsilon_m := \left| \int_{T_\epsilon}\int_{T_\epsilon} \sum_{p=1}^{m-1} \chi_{mp}(t)\lambda_p \sum_{n=0}^{p-1} \chi_{mn}(\tau)\lambda_n \sum_{v=p+1}^{m} \frac{\varphi_v(t)\varphi_v(\tau)}{\lambda_v^2} dt d\tau \right| \leq M_\epsilon.$$

On the other hand we can write the condition (4) in Theorem E in the form

$$\bar{U}^\epsilon_m + \bar{V}^\epsilon_m = O_\epsilon(1)$$

where

$$\bar{U}^\epsilon_m = \int_{T_\epsilon}\int_{T_\epsilon} \sum_{n=0}^{m-1} \chi_{mn}(t)\lambda_n \sum_{p=n+1}^{m-1} \chi_{mp}(\tau)\lambda_p \sum_{k=p+1}^{m} \frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2} dt d\tau$$

and

$$\bar{V}_m := \int_{T_\epsilon}\int_{T_\epsilon} \sum_{n=1}^{m-1} \chi_{mn}(t)\lambda_n \sum_{p=0}^{n} \chi_{mp}(\tau)\lambda_p \sum_{k=n+1}^{m} \frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2} dt d\tau.$$

We have

$$\bar{U}^\epsilon_m = U^\epsilon_m + \tilde{U}^\epsilon_m$$

and

$$\bar{V}^\epsilon_m = V^\epsilon_m + \bar{W}^\epsilon_m + W^\epsilon_m$$

where

$$\tilde{U}^\epsilon_m = \int_{T_\epsilon}\int_{T_\epsilon} \chi_{m,m-1}(t)\chi_{m,m-1}(\tau)\lambda^2_{m-1}\frac{\varphi_m(t)\varphi_m(\tau)}{\lambda^2_m}dtd\tau,$$

$$\bar{W}^\epsilon_m = \int_{T_\epsilon}\int_{T_\epsilon}\sum_{n=1}^{m-1}\chi_{mn}(t)\lambda_n\chi_{mn}(\tau)\lambda_n\sum_{k=n+1}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda^2_k}dtd\tau,$$

$$W^\epsilon_m = \int_{T_\epsilon}\int_{T_\epsilon}\chi_{mo}(t)\lambda_o\chi_{mo}(\tau)\lambda_o\sum_{k=1}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda^2_k}dtd\tau.$$

Since

$$\tilde{U}^\epsilon_m = O(1)(\int_e \mid \varphi_m(t) \mid dt)^2 = O(1),$$

$$W^\epsilon_m = \sum_{k=1}^{m}(\frac{\lambda_o}{\lambda_k})^2(\int_e \chi_{T_\epsilon}(t)\chi_{m,o}(t)\varphi_k(t)dt)^2 \leq$$

$$\leq \sum_{k=1}^{\infty}(\int_e \chi_{T_\epsilon}(t)\chi_{m,o}(t)\varphi_k(t))^2$$

and

$$\bar{W}^\epsilon_m = \sum_{n=1}^{m-1}\sum_{k=n+1}^{m}(\frac{\lambda_n}{\lambda_k})^2(\int_e \chi_{T_\epsilon}(t)\chi_{mn}(t)\varphi_k(t)dt)^2 \leq$$

$$\leq \sum_{n=0}^{m}\sum_{k=0}^{\infty}(\int_e \chi_{T_\epsilon}(t)\chi_{mn}(t)\varphi_k(t)dt)^2,$$

then by using Lemma 8 and Bessel's inequality, we get that

$$W^\epsilon_m \leq \int_e \chi^2_{T_\epsilon}(t)\chi^2_{m,o}(t)dt \leq b-a$$

and

$$\bar{W}^\epsilon_m \leq \sum_{n=1}^{m}\int_e \chi^2_{T_\epsilon}(t)\chi^2_{mn}(t)dt = \sum_{n=1}^{m}\operatorname{mes}\mathfrak{M}_{mn} = b-a.$$

Therefore we get $\bar{U}^\epsilon_m = O_\epsilon(1)$ and $\bar{V}^\epsilon_m = O_\epsilon(1)$, i.e. the condition (4) is fulfilled and we have that Theorem E follows from our Theorem.

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