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On the ideal structure and functional representation of the topological algebra $C(X,A)$

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Introduction. Let X be a completely regular space and A be a topological algebra over $\mathbb{C}$. By $C(X, A)$ we denote the algebra of all continuous functions from X to A and by $\Delta(A)$ the set of all continuous nontrivial complex homomorphisms on A. (We shall assume that $\Delta(A)$ is nonempty). The set $\Delta(A)$ endowed with the Gelfand topology will be called the carrier space of A. If above $A = \mathbb{C}$, then we shall write $C(X)$ instead of $C(X, A)$. For each point t_0 in X and $Y = X \setminus \{t_0\}$ let $C(Y)_\infty = \{g_{|Y} |\ g \in C(X), g(t_0) = 0\}$. Moreover, for each $x \in A$ let $\widehat{x}$ be the Gelfand function defined by $\widehat{x}(\tau) = \tau(x)$ for each $\tau \in \Delta(A)$ and let $\widehat{A} = \{\widehat{x}|\ x \in A\}$. The mapping $x \mapsto \widehat{x}$, for each $x \in A$, will be called the Gelfand transform. It is clear that the Gelfand transform is an algebra homomorphism from A into $C(\Delta(A))$.

The space $\Delta(C(X, A))$ has been studied in many papers under various kinds of topological assumptions on X , A and $C(X, A)$. In this paper we shall study the case in which A is a commutative locally m-pseudoconvex algebra over the field of complex numbers. Such kind of algebras are the generalizations of locally m-convex algebras (see [19] or [20]) and of p-normed algebras studied in [26]. Let $r_\lambda \in (0, 1]$ for each $\lambda \in \Lambda$ and let $\mathcal{Q} = \{q_\lambda |\ \lambda \in \Lambda\}$ be a family of r_λ-homogeneous submultiplicative seminorms defining a topology $T(\mathcal{Q})$ on A. Let $\mathcal{K}$ be a compact cover of X which is closed under finite unions. For each $K \in \mathcal{K}$ let $r_K \in (0, 1]$ and let $\mathcal{Q}(\mathcal{K}, \Lambda) = \{q_{(K,\lambda)}|\ K \in \mathcal{K}, \lambda \in \Lambda\}$ be a family of $r_K r_\lambda$-homogeneous seminorms on $C(X, A)$ where

$$q_{(K,\lambda)}(f) = [\sup_{t\in K} q_\lambda(f(t))]^{r_K},$$

$f \in C(X, A)$, $K \in \mathcal{K}$ and $\lambda \in \Lambda$.

The topology on $C(X, A)$ defined by the family $\mathcal{Q}(\mathcal{K}, \Lambda)$ we denote by $T(\mathcal{K}, \Lambda)$. If A does not have unit, let $A_e = A \times \mathbb{C}$ be an algebra with an adjoint unit. We can define a topology on A_e by using the family $\mathcal{Q}_e = \{Q_\lambda|\ \lambda \in \Lambda\}$ of seminorms where $Q_\lambda((x, \alpha)) = q_\lambda(x) + |\alpha|^{r_\lambda}$ for each $(x, \alpha) \in A_e$ and $\lambda \in \Lambda$. It can be shown that $\Delta(A_e) = \Delta(A) \cup \{\tau_\infty\}$ where $\tau_\infty(x, \alpha) = \alpha$ for all $(x, \alpha) \in A_e$ (see [10], p. 3). Let $N_\lambda = \ker q_\lambda$, $M_\lambda = \ker Q_\lambda$ for each $\lambda \in \Lambda$ and I be an ideal of A. The hull of I is the set $h(I) = \{\tau \in$

$\Delta(A) |\ \widehat{x}(\tau) = 0, x \in I\}$, the kernel of a nonempty subset E of $\Delta(A)$ is the set $k(E) = \{x \in A |\ \widehat{x}(\tau) = 0, \tau \in E\}$ and $k(E) = A$ if the subset E is empty. Moreover, for a given $K \in \mathcal{K}$ and $\lambda \in \Lambda$ let $Q_{(K,\lambda)}$ be a seminorm on $C(X, A_e)$ defined by

$$Q_{(K,\lambda)}(f) = [\sup_{t \in K} Q_\lambda(f(t))]^{r_K}$$

for each $f \in C(X, A_e)$. The family $\{Q_{(K,\lambda)} |\ K \in \mathcal{K}, \lambda \in \Lambda\}$ defines a locally m-pseudoconvex topology on $C(X, A_e)$ denoted by $T_e(\mathcal{K}, \Lambda)$. Now $(C(X, A), T(\mathcal{K}, \Lambda))$ is a closed ideal of $(C(X, A_e), T_e(\mathcal{K}, \Lambda))$. It is clear that $(C(X, A), T(\mathcal{K}, \Lambda))$ is not necessarily a maximal ideal of $(C(X, A_e), T_e(\mathcal{K}, \Lambda))$. For a given $q_\lambda \in \mathcal{Q}$ let p_λ be a mapping from A into $\mathbb{C}$ defined by $p_\lambda(x) = [q_\lambda(x)]^{\frac{1}{r_\lambda}}$ for each $x \in A$. We shall say that $(A, T(\mathcal{Q}))$ has the property (LC), if p_λ is a homogeneous seminorm on A for each $\lambda \in \Lambda$.

Given an element x of A, we denote the constant function, $t \mapsto x$, $t \in X$, by f_x. Thus f_e is the unit element of $C(X, A)$.

We shall say that $(A, T(\mathcal{Q}))$ is a square algebra, if $q_\lambda(x^2) = q_\lambda(x)^2$ for all $x \in A$ and $\lambda \in \Lambda$ (see [6] or [17]). It is known that each locally m-pseudoconvex square algebra has the property (LC) (see [10], Lemma 9). Furthermore, it can be shown that each square preserving seminorm is automatically submultiplicative. This has been shown for homogeneous seminorms in [8], Theorem 1. See also [11], Theorem 1. So if we deal with locally convex square algebras we do not need any extra assumption of multiplicativity of these seminorms.

Let now A be an algebra with an involution $x \mapsto x^*$, $x \in A$. Then we can define an involution $f \mapsto f^*$ on $C(X, A)$ by $f^*(t) = f(t)^*$, $t \in X$. We shall say that $(A, T(\mathcal{Q}))$ with an involution is a star algebra if $q_\lambda(xx^*) = q_\lambda(x)^2$ for all $x \in A$ and $\lambda \in \Lambda$. Obviously each commutative stara lgebra is a square algebra and thus is automatically locally m-pseudoconvex. It is known that for a complete star algebra A we have $\widehat{A} = C_\infty(\Delta(A))$ (see [10], Theorem 4). Such algebras were called full by [20]. There are also noncomplete full star algebras (see [10], Example 3).

On the ideal structure of $(\mathbf{C}(\mathbf{X}, \mathbf{A}), \mathbf{T}(\mathcal{K}, \Lambda))$. Let X be a completely regular space and let $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra. It is known that for a locally m-convex case (that is, the case where $r_K = r_\lambda = 1$ for all $K \in \mathcal{K}$ and $\lambda \in \Lambda$) the carrier space $\Delta(C(X, A), T(\mathcal{K}, \Lambda))$ is homeomorphic to $X \times \Delta(A)$ (see [6], Theorem 5, and [9], Theorem 2.2). The description of all closed ideals or all closed regular ideals of $(C(X, A), T(\mathcal{K}, \Lambda))$ has been studied in [6] and [9]. The ideal structure of $C(X, A)$ has been studied in many papers under various topological assumptions on X, A and $C(X, A)$ (see, for example, [1-6], [9], [12], [14-18], [21-23], [25]). In particular, in discribing the carrier space $\Delta(C(X, A))$, the denseness of linear hull $L(C(X), A) = \{\sum_{i=1}^n \alpha_i f_{x_i} |\ \alpha_i \in C(X)\ , x_i \in A\ , i = 1, ..., n\ , n \in \mathbb{N}\}$ of $C(X)$ and A in $(C(X, A), T(\mathcal{K}, \Lambda))$ has been used in most of the papers mentioned above (see, for example, [12], [13] or [8]). We shall now sketch the proof where this denseness property has not been used. First we prove the following useful result:

Lemma 1. *Let $(A, T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra, I be a closed ideal of it and I_0 be a closed regular (proper) ideal of $(I, T(\mathcal{Q})_I)$ where $T(\mathcal{Q})_I$ denotes the subspace topology on I. Then there is a closed regular ideal I_1 of $(A, T(\mathcal{Q}))$ such that $I \not\subset I_1$ and $I_0 = I_1 \cap I$. If I_0 is a closed maximal regular ideal in $(I, T(\mathcal{Q})_I)$, then I_1 is a closed maximal regular ideal in $(A, T(\mathcal{Q}))$.*

Proof. Let u be an identity element in I modulo I_0 (i.e. $ux - x \in I_0$ for all $x \in I$) and define $I_1 = \{x \in A | \; ux \in I_0\}$. Then I_1 satisfies the demanded conditions. For a proof see [5], Lemma 3.2. Clearly I_1 is closed ideal in $(A, T(\mathcal{Q}))$ since the multiplication is continuous in $(I, T(\mathcal{Q})_I)$ and in $(A, T(\mathcal{Q}))$. If I_2 is another closed regular ideal in $(A, T(\mathcal{Q}))$ such that $I_0 = I_2 \cap I$, then for an arbitrary $x \in I_2$ we have $ux \in I_2 \cap I = I_1 \cap I \subset I_1$. On the other hand we have $ux - x \in I_1$. (Note that u is also identity in A modulo I_1, since $u(ux - x) = u(ux) - ux \in I_0$ for all $x \in A$). So we have $x = ux - (ux - x) \in I_1$, and thus $I_2 \subset I_1$. Suppose that I_0 is a closed maximal regular ideal of $(I, T(\mathcal{Q}))$. Let now M be an arbitrary regular ideal of $(A, T(\mathcal{Q}))$ such that $I_1 \subset M$. Then $I_0 = I_1 \cap I \subset M \cap I$ and from the maximality of I_0 we get that $M \cap I = I_0$ or $M \cap I = I$. This latter condition is not possible, whence $M \cap I = I_1 \cap I = I_0$. Therefore $M \subset I_1$. Consequently, $M = I_1$ and thus I_1 is a closed maximal regular ideal of $(A, T(\mathcal{P}))$.

Corollary 1. *Let $(A, T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra the carrier space $\Delta(A)$ of which is nonempty and I be a closed ideal of it. If $\Delta(I) \neq \emptyset$ and $\omega \in \Delta(I)$ then there is a unique $\tau \in \Delta(A)$ such that $\tau_I = \omega$.*

Proof. Let $\omega \in \Delta(I)$ be given, then $\ker\omega$ is a closed maximal regular ideal of $(I, T(\mathcal{Q})_I)$. By Lemma 1 there is an unique closed maximal regular ideal I_1 of $(A, T(\mathcal{Q}))$ such that $\ker\omega = I_1 \cap I$. If we now choose an element τ from $\Delta(A, T(\mathcal{Q}))$ such that $\ker\tau = I_1$, we can see that τ satisfies the demanded condition.

Let now X and A be as above, J be an ideal of $C(X, A)$, t be a given point of X and $I(t) = cl(\{f(t) | \; f \in J\})$. Here cl means the closure in $(A, T(\mathcal{Q}))$ with respect to the topology $T(\mathcal{Q})$. Now it is easy to see that either $I(t)$ is a closed proper ideal of $(A, T(\mathcal{Q}))$ or otherwise $I(t) = A$. Note that if J is a regular ideal and $I(t)$ is a proper ideal of A, then $I(t)$ is also a regular ideal. Following Theorem is useful for our purposes to describe the ideal structure of $(C(X, A), T(\mathcal{K}, \Lambda))$.

Theorem 1. *Let X be a completely regular space, $(A, T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra with unit which has the property*

(LC) and J be a closed ideal of $(C(X,A),T(\mathcal{K},\Lambda))$. *Then there is at least one point in X such that* $I(t)$ *is a closed proper ideal of* $(A,T(\mathcal{Q}))$.

Proof. Suppose that $I(t) = A$ for all $t \in X$. Let $K \in \mathcal{K}$, $\lambda \in \Lambda$, $x \in A$, $\epsilon > 0$ be given and e be the unit element of A. Now for each $t_0 \in K$ there is $f_{t_0} \in J$ such that

$$p_\lambda(f_{t_0}(t_0) - e) = p_\lambda(f_{t_0}(t_0) - f_e(t_0)) < \epsilon^{\frac{1}{r_K r_\lambda}}.$$

Since f_{t_0} and f_e are continuous at t_0, there is a neighbourhood $U(t_0)$ of t_0 in X such that

$$p_\lambda(f_{t_0}(t) - f_e(t)) < \epsilon^{\frac{1}{r_K r_\lambda}} \tag{2}$$

for each $t \in U(t_0)$. The sets $\{U(t_0) | \, t_0 \in K\}$ form an open cover of the compact set K. So there is a finite subcover $U_1,...,U_n$. Let f_i be a such element of J for which inequality (2) holds for each $i = 1,...,n$. By Lemma 2.1.1 of [13] we can pick up elements $\alpha_i \in C(X)$ where $i = 1,...,n$ such that $0 \le \alpha_i(t) \le 1$ for all $t \in K$, supp$\alpha_i \subset U_i$ and $\sum_{i=1}^n \alpha_i(t) = 1$ for all $t \in K$. If $F = \sum_{i=1}^n \alpha_i f_i$, then $F \in J$ and it is easy to see that

$$p_\lambda(F(t) - f_e(t)) < \epsilon^{\frac{1}{r_K r_\lambda}}$$

for all $t \in K$. Thus we get $q_{(K,\lambda)}(F - f_e) < \epsilon$ which shows that $f_e \in cl(J) = J$. Hence, $J = C(X,A)$ and we get a contradiction.

Remark. Above we need the assumption that $(A,T(\mathcal{Q}))$ has the property (LC), otherwise the use of partition of unity in the proof would not be possible (compare [4], Theorem 1, and [24], Theorem 1).

For a given $t \in X$ and $\tau \in \Delta(A)$ let $\phi_{(t,\tau)}$ be a mapping from $C(X,A)$ into $\mathbb{C}$ defined by $\phi_{(t,\tau)}(f) = \tau(f(t))$ for each $f \in C(X,A)$. Clearly $\phi_{(t,\tau)} \in \Delta(C(X,A))$.

Lemma 2. *Let X be a completely regular space and* $(A,T(\mathcal{Q}))$ *be a locally m-pseudoconvex algebra with the property (LC). If N is a closed maximal regular ideal of* $(C(X,A),T(\mathcal{K},\Lambda))$, *then there are unique points* $t \in X$ *and* $\tau \in \Delta(A)$ *such that* $N = ker\phi_{(t,\tau)}$.

Proof. If A has unit, then this result can be shown just like as it has been done in the locally m-convex case (see [6], Lemma 4). If A does not have unit, then we study $C(X,A_e)$ instead of $C(X,A)$. It is known that $C(X,A)$ is isomorphic with a closed ideal of $C(X,A_e)$ by Corollary 1. Therefore there is a maximal closed ideal N_e of $(C(X,A_e),T_e(\mathcal{K},\Lambda))$ for which $N = N_e \cap C(X,A)$. Since $C(X,A_e)$ has unit, then we have $N_e = \ker\phi_{(t,\tau)}$ for some $t \in X$ and $\tau \in \Delta(A_e)$. As $\Delta(A_e) = \Delta(A) \cup \{\tau_\infty\}$, then $\tau \ne \tau_\infty$, but otherwise we would have $\phi_{(t,\tau)}(f) = 0$ for all $f \in C(X,A)$. It implies that $N = C(X,A)$, which is not possible.

We can now define a mapping $\varphi : X \times \Delta(A) \mapsto \Delta(C(X,A))$ by

$$\varphi(t,\tau) = \phi_{(t,\tau)} \tag{3}$$

for each $(t, \tau) \in X \times \Delta(A)$.

Theorem 2. *Let X be a completely regular space, $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra with the property (LC). Then the mapping φ defined in (3) is a bijection from $X \times \Delta(A)$ onto $\Delta(C(X, A))$. The inverse mapping φ^{-1} of it is continuous, but φ itself is continuous, if the carrier space $\Delta(A)$ of A is locally equicontinuous.*

Proof. For a proof see [6], Theorem 5 (see also [1] and [2]).

Corollary 2. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra with the property (LC). If the carrier space $\Delta(A)$ of A is locally equicontinuous, then the carrier space $\Delta(C(X, A))$ of the algebra $(C(X, A), T(\mathcal{K}, \Lambda))$ is homeomorphic to $X \times \Delta(A)$.*

Let I be an ideal of A and let t be a given point of X. We define an ideal $J_{(t,I)}$ of $C(X, A)$ by $J_{(t,I)} = \{f \in C(X, A) | \ f(t) \in I\}$. It is clear that $J_{(t,I)}$ is a closed ideal of $(C(X, A), T(\mathcal{K}, \Lambda))$, if I is a closed ideal of $(A, T(\mathcal{Q}))$.

Theorem 3. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra with the property (LC). If J is a proper closed regular ideal of $(C(X, A), T(\mathcal{K}, \Lambda))$, then there is a subset E of X and a family $\{I(t) | \ t \in E\}$ of proper closed regular ideals of $(A, T(\mathcal{Q}))$ such that $J = \bigcap_{t \in E} J_{(t, I(t))}$.*

Proof. For a proof see [6], Theorem 8.

The set E above is not necessarily closed (see [6], p. 315). Several conditions that quarantee E to be closed have been given in [6] (Theorem 9, p. 316).

On functional representation of $(\mathbf{C(X, A)}, \mathbf{T}(\mathcal{K}, \Lambda))$. Next we shall consider the Gelfand functions of $(C(X, A), T(\mathcal{Q}, \Lambda))$. If $f \in C(X, A)$, then its Gelfand function is defined by $\widehat{f}(\phi) = \phi(f)$ for each $\phi \in \Delta(C(X, A))$. Moreover, if $(A, T(\mathcal{Q}))$ has the property (LC), then by Theorem 3 each $\phi \in \Delta(C(X, A))$ is of the form $\phi = \phi_{(t,\tau)}$ for some $t \in X$ and $\tau \in \Delta(A)$. So

$$\widehat{f}(\phi) = \widehat{f}(\phi_{(t,\tau)}) = \phi_{(t,\tau)}(f) = \tau(f(t)).$$

This means that we can consider the Gelfand function $\widehat{f}$ also as a complex valued function defined on $X \times \Delta(A)$. From now let $\widehat{f}$ be the function defined by $\widehat{f}(t, \tau) = \widehat{f}(\phi_{(t,\tau)}) = \tau(f(t))$ for each $(t, \tau) \in X \times \Delta(A)$. If $\Delta(A)$ is locally equicontinuous, then $\widehat{f} \in C(X \times \Delta(A))$ (see [12], Theorem 4, or [16], Lemma 3). Thus, if $\Delta(A)$ is locally equicontinuous, the Gelfand transform $f \mapsto \widehat{f}$, $f \in C(X, A)$, is an algebra homomorphism from $C(X, A)$ into $C(X \times \Delta(A))$.

We shall now provide $C(X \times \Delta(A))$ with a locally m-pseudoconvex topology that makes the Gelfand transform of it continuous.

Lemma 3. *Let $(A, T(\mathcal{Q}))$ be a commutative locally m-pseudoconvex algebra with the property (LC). Then $\Delta(A) = \cup\{h(N_\lambda) | \ \lambda \in \Lambda\}$, where $h(N_\lambda)$*

is a locally compact subset (a compact subset, if A has unit) for each $\lambda \in \Lambda$. *Furthermore, if* $\tau \in h(N_\lambda)$, *then* $|\tau(x)|^{r_\lambda} \leq q_\lambda(x)$ *for all* $x \in A$.

Proof. For a proof see [10], Theorem 1.

Let now A be an algebra without unit and A_e be the algebra obtained from A by adjoining the unit element. Let $T_e(\mathcal{Q})$ be the topology on A_e defined by the seminorms $\{Q_\lambda | \lambda \in \Lambda\}$ where $Q_\lambda(x,\alpha) = q_\lambda(x) + |\alpha|^{r_\lambda}$ for each $(x,\alpha) \in A_e$. Denote by h_e the hull operation on A_e. Since $M_\lambda = kerQ_\lambda = \{(x,0)| x \in N_\lambda\}$ we see that $h_e(M_\lambda) = h(N_\lambda) \cup \{\tau_\infty\}$. Each $h_e(M_\lambda)$ is compact and thus $h_e(M_\lambda)$ is a one point compactification of $h(N_\lambda)$. Note that $h(N_\lambda)$ is compact if τ_∞ is an isolated point of $h(N_\lambda)$. Since we assumed that the family $\mathcal{P} = \{q_\lambda^{\frac{1}{r_\lambda}} | \lambda \in \Lambda\}$ is directed, then $\{h(N_\lambda)| \lambda \in \Lambda\}$ is a locally compact cover of $\Delta(A)$ by Lemma 3, which is closed under finite unions (see also [10], Theorem 1). We can now equip $\widehat{A}$ with a topology $T(\widehat{\mathcal{Q}})$ generated by the family of seminorms $\widehat{\mathcal{Q}} = \{\widehat{q}_\lambda | \lambda \in \Lambda\}$ where

$$\widehat{q}_\lambda(\widehat{x}) = [\sup_{\tau \in h(N_\lambda)} |\widehat{x}(\tau)|]^{r_\lambda}$$

for each $\widehat{x} \in \widehat{A}$ and $\lambda \in \Lambda$. It is easy to see that each $\widehat{q}_\lambda \in \widehat{Q}$ is r_λ-homogeneous. By Lemma 3 we have $\widehat{q}_\lambda(\widehat{x}) \leq q_\lambda(x)$ for all $x \in A$ and $\lambda \in \Lambda$. This shows that the Gelfand mapping $x \mapsto \widehat{x}$ is continuous from $(A, T(\mathcal{Q}))$ onto $(\widehat{A}, T(\widehat{\mathcal{Q}}))$. Note that $\widehat{A} \subset C_\infty(\Delta(A)) = \{g_{|\Delta(A)} | g \in C(\Delta(A_e)),\ g(\tau_\infty) = 0\}$ (it is easy to see that $C_\infty(\Delta(A)) = C(\Delta(A))$, if A has unit).

We can consider $C(X,A)$ as an ideal of $C(X,A_e)$. Since $\widehat{f}(t,\tau_\infty) = \tau_\infty(f(t)) = 0$ for all $t \in X$, we can see that the functions of $C(X,A)\hat{}$ vanish on the slice $X \times \{\tau_\infty\}$. Therefore

$$C(X,A)\hat{} \subset k(X \times \{\tau_\infty\}) = \{g \in C(X \times \Delta(A_e))| \ g(t,\tau_\infty) = 0 \text{ for all } t \in X\}.$$

We shall put

$$C_\infty(X \times \Delta(A)) = \{g_{|X \times \Delta(A)} | \ g \in C(X \times (\Delta(A_e)), g \in k(X \times \{\tau_\infty\})\}.$$

So we have $C(X,A)\hat{} \subset C_\infty(X \times \Delta(A))$. If A does not have unit, then we could also consider the algebra $C(X,A)_e$ instead of the algebra $C(X,A_e)$. But for our purposes it is better to study the latter algebra. Note that in general these two algebras are not identical. To show that the mapping $f \mapsto \widehat{f}$ from $(C(X,A), T(\mathcal{K}))$ into $(C(X \times \Delta(A)), T(\mathcal{K} \times \Lambda))$ is continuous we need the following results:

Lemma 4. *Let* X *be a completely regular space and* $(A, T(\mathcal{Q}))$ *be a locally m-pseudoconvex algebra with the property (LC). Then we have that* $\{(t,\tau)| \ \phi_{(t,\tau)} \in h(N_{(K,\lambda)})\} = K \times h(N_\lambda)$. *Furthermore, if* $\Delta(A)$ *is locally equicontinuos, then* $h(N_{(K,\lambda)})$ *and* $K \times h(N_\lambda)$ *are homeomorphic.*

Proof. See [6], Corollary 10.

Lemma 5. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra with the property (LC). Then the family $\{K \times h(N_\lambda) |\ K \in \mathcal{K}, \lambda \in \Lambda\}$ forms a locally compact cover of $X \times \Delta(A)$ and if $(t, \tau) \in K \times h(N_\lambda)$, then*

$$|\phi_{(t,\tau)}(f)|^{r_K r_\lambda} = |\tau(f(t))|^{r_K r_\lambda} \leq q_{(K,\lambda)}(f)$$

for all $f \in C(X, A)$.

Proof. This result follows from Lemmas 3 and 4.

We shall define now a locally m-pseudoconvex topology $T(\mathcal{K} \times \Lambda)$ on $C_\infty(X \times \Delta(A))$ by using the family $\widehat{Q}(\mathcal{K}, \Lambda) = \{\widehat{q}_{(K,\lambda)} |\ K \in \mathcal{K}, \lambda \in \Lambda\}$ of seminorms where

$$\widehat{q}_{(K,\lambda)}(\widehat{f}) = [\sup_{(t,\tau) \in K \times h(N_\lambda)} |g(t, \tau)|]^{r_K r_\lambda}$$

for each $g \in C_\infty(X \times \Delta(A))$. Since $C(X, A)^\frown \subset C_\infty(X \times \Delta(A))$ we can define the topology $T(\mathcal{K} \times \Lambda)$ also on $C(X, A)^\frown$ and the Gelfand mapping $f \mapsto \widehat{f}$, with $f \in C(X, A)$, is a continuous algebra homomorphism from $(C(X, A), T(\mathcal{K}, \Lambda))$ into $(C_\infty(X \times \Delta(A)), T(\mathcal{K} \times \Lambda))$ by Lemma 5. Thus we have

Theorem 5. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a locally m-pseudoconvex algebra with the property (LC). If the carrier space $\Delta(A)$ of A is locally equicontinuous, then the Gelfand mapping from $(C(X, A), T(\mathcal{K}, \Lambda))$ into $(C_\infty(X \times \Delta(A)), T(\mathcal{K} \times \Lambda))$ is continuous.*

Proof. For the proof see Theorem 3 of [10].

Corollary 3. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a locally pseudoconvex algebra which carrier space $\Delta(A)$ is locally equicontinuous. If $(A, T(\mathcal{Q}))$ is a square algebra, then algebras $(C(X, A), T(\mathcal{K}, \Lambda))$ and $(C(X, A)^\frown, T(\mathcal{K} \times \Lambda))$ are topologically isomorphic.*

Next we shall consider locally pseudoconvex star algebras. As it was noted earlier, each (commutative) star algebra $(A, T(\mathcal{Q}))$ is a square algebra and thus is automatically locally m-pseudoconvex and has the property (LC). We shall now consider conditions under which

$$C(X, A)^\frown = C_\infty(X \times \Delta(A)), \tag{4}$$

$$q_{(K,\lambda)}(f) = \widehat{q}_{(K,\lambda)}(\widehat{f}) \tag{5}$$

for all $f \in C(X, A)$, $K \in \mathcal{K}$ and $\lambda \in \Lambda$.

Theorem 7. *Let X be a completely regular space and $(A, T(\mathcal{Q}))$ be a full locally pseudoconvex star algebra with the locally equicontinuous carrier space $\Delta(A)$. Then the Gelfand transform of $(C(X, A), T(\mathcal{K}, \Lambda))$ has the properties* (4) *and* (5).

Proof. Since each commutative locally pseudoconvex star algebra is also a square algebra it follows that $(C(X,A),T(\mathcal{K},\Lambda))$ has the property (5). To prove (4) we first assume that A has unit. Let $g \in C(X\times\Delta(A))$ be arbitrary. For each fixed t_0 in X the function g_{t_0}, defined by $g_{t_0}(\tau) = g(t_0,\tau)$ for each $\tau \in \Delta(A)$, belongs to $C(\Delta(A)) = \widehat{A}$. Thus there is an element $x_{t_0} \in A$ such that $g_{t_0} = \widehat{x}_{t_0}$. Letting now t_0 vary thorough all the points of X we get a function f from X into A defined by $f(t) = x_t$ for each $t \in X$. It can be shown that f is continuous (for a detailed proof see [7], Theorem 3). Since $\widehat{f}(t,\tau) = \widehat{x}_t(\tau) = g(t,\tau)$ for all $(t,\tau) \in X \times \Delta(A)$ we see that $\widehat{f} = g$. Suppose now that A does not have unit. Let $g \in C_\infty(X \times \Delta(A))$ be arbitrary. Now g can be considered as a function of $k(X \times \{\tau_\infty\})$ (we must only define $g((t,\tau_\infty)) = 0$ for each $t \in X$). Since A is full it follows that $\widehat{A}_e = C(\Delta(A_e))$ (see [10], Theorem 5). Just like above we get a function $f(t) = (x_t,\alpha_t)$, for each $t \in X$, such that $f \in C(X,A_e)$ and $\widehat{f} = g$. Since $g(t,\tau_\infty) = 0$ for all $t \in X$, we see that $\alpha_t = 0$. Thus $f(t) = x_t \in A$ which proves (4).

Corollary 4. *Suppose that the hyphotheses of Theorem 7 are valid. Then* $C(X,A)^\frown = C_\infty(\Delta(C(X,A)))$.

In the particular case, when $(A,T(\mathcal{Q}))$ is a full locally pseudoconvex star algebra, the description of all closed ideals of $(C(X,A),T(\mathcal{K},\Lambda))$ are given just like in the locally convex case by using different kinds of slice ideals. This has been done in [7], pp. 388-391, for the case when A is a locally convex algebra with unit. However the same type of description for the locally pseudoconvex algebra $(C(X,A),T(\mathcal{K},\Lambda))$ with or without unit is also possible. Let now $(A,T(\mathcal{Q}))$ be a full locally pseudoconvex star algebra and J be a closed proper (not necessarily regular) ideal of $(C(X,A),T(\mathcal{K},\Lambda))$. Then $\widehat{J} = \{\widehat{f} |\ f \in J\}$ is a closed ideal of $(C_\infty(X \times \Delta(A)),T(\mathcal{K} \times \Lambda))$ and we can consider $h(\widehat{J})$ as a closed subset of $X \times \Delta(A)$. Now J can be studied also as a closed ideal of $(C(X,A_e),T_e(\mathcal{K},\Lambda))$. By Theorem 7 and Lemma 4 of [10] there is a closed subset F of $X \times \Delta(A_e)$ such that $\widehat{J} = \{g \in C(X \times \Delta(A_e))|\ g(t,\tau) = 0, (t,\tau) \in F\}$. As $F = h(\widehat{J}) \cup X \times \{\tau_\infty\}$, then $\widehat{J} = k(h(\widehat{J}))$ from which it follows that $k(h(J)) = J$. We have proved the following result.

Lemma 6. *Suppose that the hyphotheses of Theorem 7 are valid. If J is a closed ideal of $(C(X,A),T(\mathcal{K},\Lambda))$, then $k(h(J)) = J$.*

Theorem 8. *Let X be a completely regular space, $(A,T(\mathcal{Q}))$ be a full locally pseudoconvex star algebra with the nonempty locally equicontinuous carrier space $\Delta(A)$ and J be a closed ideal of $(C(X,A),T(\mathcal{K},\Lambda))$. Then there is a subset E of X and a family $I(t)$, $t \in E$ of closed ideals of $(A,T(\mathcal{Q}))$ such that $J = \cap_{t\in E} J_{(t,I(t))}$.*

Proof. Let $E = \{t \in X|\ \phi_{(t,\tau)} \in h(J)\}$. It can be shown that $h(J) = \{\phi_{(t,\tau)}|(t,\tau) \in h(\widehat{J})\}$ and $h(\widehat{J}) = \{(t,\tau)|\ \phi = \phi_{(t,\tau)} \in h(J)\}$ (for a proof see [7], Lemma 4.2). So we have $E = \{t \in X|\ (t,\tau) \in h(\widehat{J})\}$. Since $J \neq$

$C(X, A)$ we have $E \neq \emptyset$. Moreover, by Theorem 7 and Lemma 6 we have $k(h(J)) = J$, and thus also $k(h(\widehat{J})) = \widehat{J}$. Now for each $t \in E$ let $E_t = h(\widehat{J}) \cap \{t\} \times \Delta(A)\}$. It is clear that each $k(E_t)$, $t \in E$, is a proper closed ideal of $(A, T(Q))$ and $J = \cap_{t \in E} J_{(t, I(t))}$ (for a detailed proof see [5], pp. 53-54).

Corollary 5. *Suppose that the hyphotheses of Theorem 8 are valid. Then* $cl(\{f(t) |\ f \in J\}) = k(E_t)$ *for all* $t \in E$ *where* $E_t = h(\widehat{J}) \cap \{t\} \times \Delta(A)$, $t \in E$. *Moreover, if J is regular, then τ_∞ is an isolated point of E_t for all $t \in E$.*

Proof. The first claim is obvious and the second claim follows from Theorem 6 of [10].

The ideals $J_{(t, k(E_t))}$ in Theorem 8 are slice ideals of the first type defined in [7], p. 388. By using same kind of technics that was used in [5] (Theorem 4.2) and [7] (Theorem 4) it can be shown that under the hyphotheses of Theorem 8 for each closed ideal (not necessarily regular) there are a subset $E_0 \subset \Delta(A)$ and a family J_τ, $\tau \in E_0$, of slice ideals of the second type (for a definition see [5] or [7]) such that $J = \cap_{\tau \in E_0} J_\tau$. Furthermore, if A is an algebra without unit and J is regular, then τ_∞ is an isolated point of E_0 (this condition follows from Corollary 5). Note that in the proof of Theorem 8 in describing all (and not just regular) closed ideals we have used technics that differs from the technics for the corresponding results proved in [22] and [23]. The original technics used in those papers are due to Kaplansky (see [17]).

References

1. M. Abel, *Description of linear multiplicative functionals in algebras of continuous functions*, Tartu Ül. Toimetised **430** (1977), 14–21.
2. M. Abel, *Description of the space of continuous linear multiplicative functionals of the algebra of vector-valued funtions*, Abstracts of the conference "Theoretical and applied problems of mathematics", Tartu, 1980, pp. 111–113 (in Russian).
3. M. Abel, *Description of closed ideals in algebras of continuous vector-valued functions*, Math. Notes **30** (1981), 887–892.
4. M. Abel, *On dense subsets in some spaces of vector-valued functions*, Tartu Ül. Toimetised **770** (1987), 26–37.
5. J. Arhippainen, *On the ideal structure and approximation properties of algebras of continuous B^*-algebra-valued functions*, Acta Univ. Ouluensis Ser. A **187** (1987).
6. J. Arhippainen, *On the ideal structure of algebras of LMC-algebra valued functions*, Studia Math. **101** (1992), 311–318.
7. J. Arhippainen, *On the ideal structure of star-algebra valued functions*, Proc. Amer. Math. Soc. **123** (1995), 381–391.
8. J. Arhippainen, *On locally convex square algebras*, Funct. et Appr. **XXII** (1993), 57–63.

9. J. Arhippainen, *On some properties of algebras of LMC-algebra valued functions*, (to appear in Demonstr. Math.).
10. J. Arhippainen, *On functional representation of locally m-pseudoconvex algebras*, (submitted).
11. S.J. Bhatt and D.J. Karia, *Uniqueness of the uniform norm with an application to topological algebras*, Proc. Amer. Math. Soc. **116** (1992), 499–503.
12. W. Dietrich, *The maximal ideal space of the topological algebra $C(X, E)$*, Math. Ann. **183** (1969), 201–212.
13. W. Dietrich, *Function algebras on completely regular spaces*, Diss. Northwestern Univ. Evanston Ill., 1971.
14. W. Hery, *Rings of continuous Banach algebra valued functions*, Diss. Abstr., Polytech. Instit. of New York **45** (1974).
15. W. Hery, *Maximal ideals in algebras of continuous $C(S)$-valued functions*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat.Natur. (8) **58** (1975), 195–199.
16. W. Hery, *Maximal ideals in algebras of topological algebra valued functions*, Pacific J. Math **65** (1976), 365–373.
17. I. Kaplansky, *The structure of certain operator algebras*, Trans. Amer. Math. Soc. **70** (1951), 219–255.
18. A. Mallios, *Heredity of tensor products of topological algebras*, Math. Ann. **162** (1966), 246–257.
19. A. Mallios, *Topological algebras. Selected topics*, North-Holland, Amsterdam, 1986.
20. E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952).
21. J. Prolla, *Approximation of vector-valued functions*, North-Holland, Amsterdam, 1977.
22. J. Prolla, *On the spectra of non-Archimedean function algebras*, Lecture Notes in Math. **843** (1980), 547–560.
23. J. Prolla, *Topological algebra of vector-valued continuous functions*, Anal. and Appl. Part B Adv. in Math. Supplementary Studies **7** (1981), 727–740.
24. A. Shuchat, *Approximation of vector-valued continuous functions*, Proc. Amer. Math. Soc. **31** (1972), 97–103.
25. B. Yood, *Banach algebras of continuous functions*, Amer. J. Math. (1951), 30–48.
26. W. Żelazko, *Selected topics in topological algebras*, Lecture Notes Ser. Math. Aarhus Univ. **31** (1971).

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