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Dense subalgebras in noncommutative Jordan topological algebras*

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Wilansky conjectured in [12] that normed dense Q-algebras are full subalgebras of Banach algebras. Beddaa and Oudadess proved in [2] that Wilansky's conjecture was true. They showed that k-normed Q-algebras are full subalgebras of k-Banach algebras for each $k \in (0, 1]$ (the case of Banach algebras see [7], Proposition 5.10). Moreover, Pérez, Rico and Rodríguez showed in [8], Theorem 4, that this was true also in the case of noncommutative Jordan-Banach algebras. In the present paper this problem has been studied in more general case. It is proved that all dense Q-subalgebras of topological algebras and of noncommutative Jordan topological algebras with continuous multiplication are full subalgebras. Some equivalent conditions that a dense subalgebra would be a Q-algebra (in subspace topology) in Q-algebras and in nonassociative Jordan Q-algebras with continuous multiplication are given.

1. Introduction

A linear topological space A over the field of complex numbers $\mathbb{C}$ is called *a complex topological algebra* (shortly *a topological algebra*) if in A there has been defined a separately continuous (not necessarily associative) multiplication. It means that for each neighborhood of zero O of A and each $a \in A$ there exists a neighborhood of zero U such that $aU \subset O$ and $Ua \subset O$. In particular, if for each neighborhood of zero O of A there exists another neighborhood of zero U satisfying $U^2 \subset O$, then A is called *a topological algebra with continuous multiplication.*

A topological algebra A is called *an associative (a nonassociative) topological algebra* if its multiplication of elements is associative (respectively, not associative). We shall use the short term "topological algebra" instead of "associative topological algebra" in the following text. When a nonassociative topological algebra A satisfies identities $ab = ba$ and $a^2(ba) = (a^2b)a$ for each $a, b \in A$, then A is called *a Jordan topological algebra*, and when A satisfies identities $a(ba) = (ab)a$ and $a^2(ba) = (a^2b)a$ for each $a, b \in A$, then *a*

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noncommutative Jordan topological algebra. At this, if the given product ab of elements a and b in A is replaced by the so called *Jordan product* $a \cdot b = \frac{1}{2}(ab + ba)$ of elements a and b and preserves the topology of A, we have a new topological algebra A^+. It is easy to see that A^+ is a Jordan topological algebra for each noncommutative Jordan topological algebra A.

In particular, when A is a noncommutative Jordan algebra without unit, we replace A with $A_{\mathbb{C}} = A \times \mathbb{C}$. Defining algebraic operations in $A_{\mathbb{C}}$ as usual (that is, $(a, \lambda)+(a', \lambda') = (a + a', \lambda + \lambda')$, $(a, \lambda)(a', \lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda')$ and $\mu(a, \lambda) = (\mu a, \mu\lambda)$ for each $(a, \lambda), (a', \lambda') \in A_{\mathbb{C}}$ and $\mu \in \mathbb{C}$), then $A_{\mathbb{C}}$ is a noncommutative Jordan algebra with unit $(\theta_A, 1)$ (θ_A denotes the zero elemet in A), which is called *an algebra, obtained from A by adding the unit.*

Besides, when a nonassociative topological algebra satisfies identities $a^2b = a(ab)$ and $ba^2 = (ba)a$ for each $a, b \in A$, then A is called *an alternative topological algebra.* It is easy to see that every associative topological algebra is an alternative topological algebra, and every Jordan topological algebra and every alternative topological algebra are noncommutative Jordan topological algebras (see [9], p. 473).

An element a in a noncommutative Jordan algebra A with unit e_A is said to be *invertible in A*, if there exists an element $a^{-1} \in A$ (the inverse of a) such that

$$aa^{-1} = a^{-1}a = e_A \tag{1}$$

and

$$a^2a^{-1} = a^{-1}a^2 = a. \tag{2}$$

Moreover, an element a in a noncommutative Jordan algebra A is said to be *quasi-inverible in A*, if there exists an element $a_q^{-1} \in A$ (the quasi-inverse of a) such that

$$a + a_q^{-1} - aa_q^{-1} = a + a_q^{-1} - a_q^{-1}a = \theta_A \tag{3}$$

and

$$a + a_q^{-1} - 2aa_q^{-1} - a^2 + a^2a_q^{-1} = a + a_q^{-1} - 2a_q^{-1}a - a^2 + a_q^{-1}a^2 = \theta_A. \tag{4}$$

In particular, when A is an associative algebra, then $a \in A$ is invertible in A (in the case, when A has unit), if the equality (1) holds, and is quasi-invertible in A, if the equality (3) holds. The set of all invertible elements of A is denoted by Inv$A$ and the set of all quasi-invertible elements of A is denoted by Qinv$A$.

A subalgebra B of a noncommutative Jordan algebra A is called *a full subalgebra of A* if

$$\text{Qinv}B = B \cap \text{Qinv}A. \tag{5}$$

In particular, when A is a noncommutative Jordan algebra with unit e_A and $e_A \in B$, then B is a full subalgebra of A if and only if

$$\text{Inv}B = B \cap \text{Inv}A. \tag{6}$$

A noncommutative Jordan (patricularly an associative) topological algebra A is called *a Q-algebra*, if the set Qinv$A$ is open. When A has unit, then A is a Q-algebra if and only if the set Inv$A$ is open. In the case, when a subalgebra B of a (not necessarily associative) topological algebra A is a Q-algebra in subspace topology, then B is called *a Q-subalgebra of A*.

Let now A be a complex (not necessarily associative) algebra and $\mathrm{sp}_A(a)$ be the spectrum of $a \in A$ that is

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}a \notin \mathrm{Qinv}A\} \cup \{0\},$$

if A is an algebra without unit, and

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\},$$

if A is an algebra with unit e_A. In the both cases the spectral radius $r_A(a)$ of $a \in A$ is defined by

$$r_A(a) = \sup\{|\lambda| : \lambda \in \mathrm{sp}_A(a)\}.$$

2. Dense Q-subalgebras in noncommutative Jordan topological algebras

Let A be a noncommutative Jordan topological algebra, B be a dense Q-subalgebra of A, and $A_{\mathbb{C}}$ ($B_{\mathbb{C}}$) be the algebra obtained from A (respectively from B) by adding the unit. To show that B is a full subalgebra of A, we first have to prove some necessary results.

Proposition 1. *Let A be a noncommutative Jordan algebra without unit and B be a subalgebra of A. Then B is a full subalgebra of A if and only if $B_{\mathbb{C}}$ is a full subalgebra of $A_{\mathbb{C}}$.*

Proof. Let B be a full subalgebra of A and $(a, \lambda) \in B_{\mathbb{C}} \cap \mathrm{Inv}A_{\mathbb{C}}$. Then $-\lambda^{-1}a \in \mathrm{Qinv}A$ and $(a, \lambda)^{-1} = (-\lambda^{-1}(-\lambda^{-1}a)_q^{-1}, \lambda^{-1})$. Therefore $(a, \lambda) \in \mathrm{Inv}B_{\mathbb{C}}$ by the equality (5) (the identity $(\theta_A, 1)$ of $A_{\mathbb{C}}$ belongs to $B_{\mathbb{C}}$). Hence $B_{\mathbb{C}} \cap \mathrm{Inv}A_{\mathbb{C}} \subset \mathrm{Inv}B_{\mathbb{C}}$. Since the converse inclusion holds always, we have

$$\mathrm{Inv}B_{\mathbb{C}} = B_{\mathbb{C}} \cap \mathrm{Inv}A_{\mathbb{C}}. \tag{7}$$

It means that $B_{\mathbb{C}}$ is a full subalgebra of $A_{\mathbb{C}}$.

Let now $B_{\mathbb{C}}$ be a full subalgebra of $A_{\mathbb{C}}$ and $b \in B \cap \mathrm{Qinv}A$. Then $(-b, 1)^{-1} = (-b_q^{-1}, 1) \in A_{\mathbb{C}}$. Therefore from $(-b, 1) \in \mathrm{Inv}A_{\mathbb{C}}$ follows that $(-b, 1) \in \mathrm{Inv}B_{\mathbb{C}}$ by the equality (7). Hence $b \in \mathrm{Qinv}B$, because of which the equality (5) holds. Consequently, B is a full subalgebra of A.

Proposition 2. *Let A be a topological algebra or a noncommutative Jordan topological algebra with continuous multiplication. Then $A_{\mathbb{C}}$ is a Q-algebra (in product topology) if and only if A is a Q-algebra.*

Proof. Let A be a Q-algebra. Then Qinv$A$ is a neighborhood of zero in A (see [5], Lemma 6.4, and [1], Proposition 2). Whereas $(a, \lambda) \to \lambda a$ is a continuous map on $A_{\mathbb{C}}$, then there exists an open balanced neighborhood of zero U in A, and a neighborhood of unit V in $\mathbb{C}$ such that $VU \subset$ Qinv$A$. As the inversion of elements in $\mathbb{C}$ is continuous, then V defines an open neighborhood of unit W in $\mathbb{C}$ such that $\lambda^{-1} \in V$ for each $\lambda \in W$. Therefore $-\lambda^{-1}a \in$ Qinv$A$ for each $a \in U$ and $\lambda \in W$. Hence $(a, \lambda)^{-1} = (-\lambda^{-1}(-\lambda^{-1}a)_q^{-1}, \lambda^{-1}) \in A_{\mathbb{C}}$ for each $(a, \lambda) \in U \times W$. Since $U \times W$ is an open set in $A_{\mathbb{C}}$ and $(\theta_A, 1) \in U \times W \subset \text{Inv}A_{\mathbb{C}}$, then $(\theta_A, 1)$ is an interior point of Inv$A_{\mathbb{C}}$, because of which $A_{\mathbb{C}}$ is a Q-algebra (see [5], Lemma 6.4, and [1], Proposition 2).

Let now $A_{\mathbb{C}}$ be a Q-algebra. Then Inv$A_{\mathbb{C}}$ is an open set in $A_{\mathbb{C}}$. Therefore there exists a balanced neighborhood of zero U in A and a neighborhood of unit V in $\mathbb{C}$ such that $(\theta_A, 1) \in U \times V \subset \text{Inv}A_{\mathbb{C}}$. Since $(-a, 1) \in \text{Inv}A_{\mathbb{C}}$ for each $a \in U$, then $a \in$ Qinv$A$ for each $a \in U$. Hence θ_A is an interior point of Qinv$A$, because of which A is a Q-algebra.

Now we shall prove the main result of this paper.

Theorem 1. *Let A be a topological algebra or a noncommutative Jordan topological algebra with continuous multiplication. Then every dense Q-subalgebra of A is a full subalgebra.*

Proof. Let B be a Q-subalgebra of A. We first assume that A is a topological algebra with unit e_A and $e_A \in B$. Since the inclusion Inv$B \subset B \cap \text{Inv}A$ always holds, we shall prove only the opposite inclusion. For it let $b \in B \cap \text{Inv}A$. Then there exists an element $b' \in A$ such that $bb' = b'b = e_A$. By assumption, B is a dense subalgebra of A. Therefore there exists a net $(b_\lambda)_{\lambda \in \Lambda}$ of elements of B such that it converges to b' in topology of A. Then the nets $(bb_\lambda)_{\lambda \in \Lambda}$ and $(b_\lambda b)_{\lambda \in \Lambda}$ converge to e_A in topology of B. That is, b is a topologically invertible element in B. It is known (see [10], p. 1309) that the set of all topologically invertible elements of a Q-algebra coincides with the set Inv$B$. Consequently $b \in$ Inv$B$, because of which B is a full subalgebra of A.

Let now A be a noncommutative Jordan topological algebra with unit e_A, multiplication of which is continuous, and let $e_A \in B$. Let A^+ and B^+ be Jordan algebras with unit, which we get from A and B by replacing the multiplication of elements in these algebras with the Jordan multiplication of elements, preserving at this the topology of both algebras. It is known (see [6], p. 4) that Inv$A$ = InvA^+ for each noncommutative Jordan algebra A. Therefore B is a full subalgebra of A if and only if

$$\text{Inv}B^+ = B^+ \cap \text{Inv}A^+. \tag{8}$$

To show it, let $b \in B^+ \cap \text{Inv}A^+$ and U_b be a linear operator on A^+ defined by $U_b(a) = b \cdot (b \cdot a + a \cdot b) - b^2 \cdot a$ for each $a \in A^+$. Since $b \in \text{Inv}A^+$, then $e_A \in U_b(A^+)$, U_b^{-1} exists and $U_b{}^{-1} = U_{b^{-1}}$ (see [4], Theorem 13, or [3], Theorem 2.1). Thus U_b is a linear homeomorphism on A^+. By assumption B^+ is a Q-algebra. Therefore, by Proposition 3 from [1] (in the case of topological algebras see [11], Theorem 4.1), there exists a balanced neighborhood of zero V of B^+ such that $r_{B^+}(b) \leqslant q_V(b)$ for each

$b \in B^+$ (here q_V denotes the gauge (Minkowski functional) of V). Let now U be a neighborhood of zero in A^+ such that $V = U \cap B^+$. As U_b is a homeomorphism on A^+, then $U_b(A^+) = \mathrm{cl}_{A^+}U_b(B^+)$ (the closure of $U_b(B^+)$ in A^+). Since $e_A \in U_b(A^+)$, then $(e_A + U) \cap U_b(B^+)$ is not empty. Hence there exists an element $b_0 \in B^+$ such that $U_b(b_0) - e_A \in U \cap B^+ = V$. Taking it into account, we have $q_V(U_b(b_0) - e_A) < 1$. Thus from $r_{B^+}(U_b(b_0) - e_A) < 1$ follows that $U_b(b_0) = (U_b(b_0) - e_A) - (-1)e_A \in \mathrm{Inv}B^+$. Therefore $b \in \mathrm{Inv}B^+$ by Theorem 13 from [4]. Hence, the equality (8) holds and B is a full subalgebra of A.

Finally, let A be a topological algebra or a noncommutative Jordan topological algebra with continuous multiplication. If one of the algebras A or B does not have unit, then instead of algebras A and B we take under consideration the algebras $A_{\mathbb{C}}$ and $B_{\mathbb{C}}$. Since both algebras have unit and $B_{\mathbb{C}}$ is a Q-algebra in product topology (by Proposition 2) and dense in $A_{\mathbb{C}}$, then $B_{\mathbb{C}}$ is a full subalgebra of $A_{\mathbb{C}}$ by the first part of this proof. Consequently, B is a full subalgebra of A by Proposition 1.

Theorem 2. *Let A be (an associative) Q-algebra or a noncommutative Jordan Q-algebra with continuous multiplication and B be a dense subalgebra of A. Then the following statements are equivalent:*

(a) B is a full subalgebra of A;

(b) $sp_A(b) = sp_B(b)$ for each $b \in B$;

(c) B is a Q-algebra in subspace topology.

Proof. Let B be a full subalgebra of A and let $b \in B$. Then $\mathrm{sp}_A(b) \subset \mathrm{sp}_B(b)$ by the equality (5). If $\lambda \in \mathrm{sp}_B(b)$ then from $\lambda^{-1}b \notin \mathrm{Qinv}B$ follows that $\lambda^{-1}b \notin \mathrm{Qinv}A$ by the equality (5). Hence $\lambda \in \mathrm{sp}_A(b)$. It means that from (a) follows (b).

Let now $b \in \mathrm{Qinv}B$. Then $1 \notin \mathrm{sp}_A(b)$ by (b). Hence $b \in \mathrm{Qinv}A$. Since A is a Q-algebra, then there exists a neighborhood O of b such that $O \subset \mathrm{Qinv}A$. As $O \cap B \subset \mathrm{Qinv}B$ by (b), then B is a Q-algebra in subspace topology. That is , from (b) follows (c).

The implication (c) $\Rightarrow$ (a) holds by Theorem 1.

References

1. M. Abel, *Noncommutative Jordan locally pseudoconvex Q-algebras,* Algebras Groups Geom. (to appear).
2. A. Beddaa, M. Oudadess, *On a question of A. Wilansky in normed al- gebras,* Studia Math. **95** (1989), 175–177.
3. C. V. Devapakkiam, *Jordan algebras with continuous inverse,* Math. Ja- pon. **16** (1971), 115–125.
4. N. Jacobson, *Structure and representations of Jordan algebras,* Amer. Math. Soc. Colloq. Publ., vol. 39, Amer. Math. Soc., Providence, RI, 1968.
5. A. Mallios, *Topological algebras. Selected topics,* North-Holland, Ams- terdam, 1986.

6. K. McCrimmon, *Noncommutative Jordan rings,* Trans. Amer. Math. Soc. **158** (1971), 1–33.

7. T. W. Palmer, *Spectral algebras,* Rocky Mountain J. Math. **22** (1992), 293–328.

8. J. Pérez, L. Rico, A. Rodríguez, *Full subalgebras of Jordan-Banach al- gebras and algebra norms on JB^*-algebras,* Proc. Amer. Math. Soc. **121** (1994), 1133–1143.

9. R. D. Schafer, *Noncommutative Jordan algebras of characteristic 0,* Proc. Amer. Math. Soc. **6** (1955), 472–475.

10. A. D. Thatte, S. J. Bhatt, *On topolizing invertibility,* Indian J. Pure Appl. Math. **15** (1984), 1308–1312.

11. Y. Tsertos, *Representations and extensions of positive functionals on *-algebras,* Bulletino U.M.I. **(7) 8-B** (1994), 541–555.

12. A. Wilansky, *Letter to the editor,* Amer. Math. Monthly **91** (1984), 531.

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