ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 1, 1996

On the sequence space defined by a sequence of moduli and on the rate-spaces*

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A function $f:\ [0,\infty)\to[0,\infty)$ is called a modulus if

(i) $f(t)=0$ iff $t=0$,
(ii) $f(t+u)\le f(t)+f(u),\ u,t\ge 0$,
(iii) f is increasing,
(iv) f is continuous from the right of 0.

For a certain sequence space X of real or complex numbers Ruckle [6] and Maddox [4] considered a new sequence space

$$X(f)=\{x=(x_k)\mid (f(|\ x_k\ |))\in X\}.$$

The extension of this definition was given by Kolk [2], for a sequence space X, and a sequence of moduli $F=(f_k)$ he defined

$$X(F)=\{x=(x_k)\mid F(x)=(f_k(|\ x_k\ |))\in X\}.$$

The sequence space X is called normal if from $(y_k)\in X,\ |\ x_k\ |\le|\ y_k\ |$, $k=0,1,\ldots$ it follows that $(x_k)\in X$.

Theorem 1. *If X is a normal linear space then $X(F)$ is also a normal linear space.*

Theorem 2. *If f and g are moduli then Φ, where $\Phi(t)=g[f(t)]$, is also a modulus.*

We omit the standard proofs of Theorems 1 and 2.

The real function g on the linear space X is called a paranorm if

(i) $g(0)=0$,
(ii) $g(-x)=g(x)$,
(iii) $g(x+y)\le g(x)+g(y)$, for all $x,y\in X$,
(iv) if $t_n\to t,\ t_n,\ t\in\mathbb{R}$ and $g(x^n-x)\to 0$, then $g(t_nx^n-tx)\to 0,\ n\to\infty$.

* This research was supported by Estonian Science Foundation (grant no. 2416).

Theorem 3. *Let* X *be a paranormed space with a paranorm* g *and* $F = (f_n)$ *a sequence of moduli. If the conditions*

(i) $|x_k| \leq |y_k| \Rightarrow g(x) \leq g(y)$ for all $x, y \in X$,

(ii) there exists a function h such that $f_k(ut) \leq h(u)f_k(t)$ for all $0 \leq u < 1,\ t > 0$ and $\lim_{u\to 0+} h(u) = 0$

hold, then $X(F)$ *is a paranormed space with a paranorm* g_F, *where* $g_F(x) = g(F(x))$.

Proof. It follows immediately from the definitions of modulus and paranorm that $g_F(0) = 0,\ g_F(-x) = g_F(x)$. As

$$f_k(|x_k + y_k|) \leq f_k(|x_k|) + f_k(|y_k|),$$

then by the condition (i) of the present theorem we have

$$g_F(x+y) \leq g[(f_k(|x_k + y_k|))]$$

$$\leq g[(f_k(|x_k|))] + g[(f_k(|y_k|))] = g_F(x) + g_F(y).$$

Let now $t_n \to t,\ t_n,\ t \in \mathbb{R}$ and $g_F(x^n - x) \to 0$, where $x^n = (x_k^{(n)}),\ x = (x_k)$. The sequence (t_n) as a convergent scalar sequence is bounded and let

$$K = \max_n[|t_n|] + 1,$$

where $[t]$ denotes the integer part of t. Then we have

$$g_F(t_n x^n - tx) \leq g_F(t_n(x^n - x)) + g_F((t_n - t)x). \tag{1}$$

By the condition (i) of the present theorem and by the properties (ii) and (iii) of modulus we may write

$$g_F(t_n(x^n - x)) = g[(f_k(|t_n(x_k^{(n)} - x_k)|))]$$

$$\leq Kg[(f_k(|x_k^{(n)} - x_k|))] = Kg_F(x^n - x) = o(1).$$

Using the conditions (i) and (ii) of the present theorem we have

$$g_F[(t_n - t)x] = g[(f_k(|(t_n - t)x_k|))]$$

$$\leq g[(h(t_n - t)f_k(|x_k|))] = o(1).$$

Consequently, it follows from (1) that $g_F(t_n x^n - tx) \to 0,\ n \to \infty$. This completes the proof.

Example 1. (The space $m(p)$). Let $X = m$ (the space of bounded sequences with supremum norm) and $f_k(t) = t^{p_k},\ \ 0 < \alpha \leq p_k < 1$. Then the conditions of Theorems 1, 2 and 3 are fulfilled and hence $m(p)$ is a normal linear space with the paranorm $g_F(x) = \sup_k |x_n|^{p_k}$.

Example 2. (The space $[A,F,p]_0$.) Let $A = (a_{nk})$ be a matrix method with $a_{nk} \geq 0$; $p = (p_k)$ a sequence of positive real numbers and $F = (f_k)$ a sequence of moduli. We denote

$$[c_A]_0^p = \{x = (x_k) \mid \lim_n \sum_k a_{nk} \mid x_k \mid^{p_k} = 0\},$$

i.e. $[c_A]_0^p$ is the space of strongly A-summable to zero sequences with exponent p. Then it is known that in the case $\sup p_k < \infty$ the space $[c_A]_0^p$ is a normal linear topological space with the paranorm g where

$$g(x) = \sup_n (\sum_k a_{nk} \mid x_k \mid^{p_k})^{\frac{1}{M}},$$

and $M = \max\{1, \sup_k p_k\}$.

Let

$$[A,F,p]_0 = \{x = (x_k) \mid \lim_n \sum_k a_{nk}[f_k(\mid x_k \mid)]^{p_k} = 0\},$$

then for $X = [c_A]_0^M$ and $G = (g_k)$ where $g_k(t) = [f_k(t)]^{\frac{p_k}{M}}$ we may write $[A,F,p]_0 = X(G) = [c_A]_0^M(G)$. It is easy to see that the paranorm g satisfies the condition (i) of Theorem 3 and if the moduli f_k satisfie the condition (ii) of Theorem 3 and if $\inf_k p_k > 0$, then by Theorem 2 the condition (ii) of Theorem 3 is fulfilled for g_k, $k = 0, 1, \dots$. Consequently, if $0 < \inf_k p_k \leq \sup_k p_k < \infty$, then by Theorems 1 and 3 the space $[A,F,p]_0$ is a normal linear paranormed space with the paranorm $\tilde{g}$ where

$$\tilde{g}(x) = \sup_n (\sum_k a_{nk}[f_k(\mid x_k \mid)]^{p_k})^{\frac{1}{M}}.$$

For $F = (f)$ this result is proved by Bilgin [1] (without the restriction (ii) of Theorem 3 for f_k).

Let now $p = (p_k)$ be a real sequence with $p_k > 0$ and $\lambda = (\lambda_k)$ be a real sequence with $\lambda_k \neq 0$. We define

$$X(p,\lambda) = \{x = (x_k) \mid (\mid \lambda_k x_k \mid^{p_k}) \in X\},$$

$$X(p) = \{x = (x_k) \mid (\mid x_k \mid^{p_k}) \in X\}.$$

If $0 < p_k < 1$ then $X(p,\lambda) = X(F)$ where $f_k(t) = (\mid \lambda_k \mid t)^{p_k}$. Let $X = c_0, m, l$ and $\inf p_k > 0$, then the conditions of Theorem 1 are satisfied.

For a real sequence space X Sikk [7] introduced the rate-space

$$X_\lambda = \{x = (x_k) \mid (\lambda_k x_k) \in X\}.$$

Let a matrix method A be determined by a matrix $A = (a_{nk})$. If for every sequence $x = (x_k)$ the sequence $y = (\sum_k a_{nk} x_k) \in Y$, we write

$A \in (X,Y)$. Since $X(p,\lambda) = (X(p))_\lambda$, then it follows immediately from Theorem 1 of [7] that the next theorem is valid.

Theorem 4. *Let* $A = (a_{nk})$, $A(\lambda^{-1},\mu) = (a_{nk}\frac{\mu_n}{\lambda_k})$, $\lambda_k, \mu_k \neq 0$ *and* $p_k, q_k > 0$. *Then* $A \in (X(p,\lambda), Y(q,\mu))$ *iff* $A(\lambda^{-1},\mu) \in (X(p), Y(q))$.

The conditions for $A \in (X(p,\lambda), Y(q,\mu))$ have been found for several special cases (see [5]). But if we know conditions for $A \in (X(p), Y(q))$, then it is very easy to find conditions for $A \in (X(p,\lambda), Y(\mu,q))$ by using Theorem 4.

For example, let $X = c_0(p) = \{x = (x_k) \mid \lim_k \mid x_k \mid^{p_k} = 0\}$, then (see [3]) $A \in (c_o(p), c_0(q)$ iff

(i) $\lim_n \mid a_{nk} \mid^{q_n} = 0, \ k = 0,1,\ldots)$,

(ii) $\lim_N \limsup_n (\sum_n \mid a_{nk} \mid N^{-1/p_k})^{q_n} = 0$.

Then by Theorem 4 we get that $A \in (c_0(p,\lambda), c_0(q,\mu))$ iff

(i) $\lim_n \mid a_{nk}\frac{\mu_n}{\lambda_k} \mid^{q_n} = 0, \ k = 0,1,\ldots,$

(ii) $\lim_N \limsup_n (\sum_k \mid a_{nk}\frac{\mu_n}{\lambda_k} \mid N^{-1/p_k})^{q_n} = 0$.

This result is also proved in [5].

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Received November 27, 1995

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