ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 1, 1996

Spaces of strongly $\mathcal{A}$-summable sequences

Tunay Bilgin

1. Introduction

The class of sequences which are strongly summable with respect to a modulus was introduced by Maddox [6] and extended by Connor [3]. In [2,4,5,8] a further extension of these definitions was given by using a sequence of positive real numbers $p = (p_k)$ or a sequence of moduli $F = (f_k)$.

We first recall the notion of modulus.

Definition 1. *A function* $f : [0, \infty) \to [0, \infty)$ *is called a modulus if*

1) $f(t) = 0$ *if and only if* $t = 0$;
2) $f(t + s) \leq f(t) + f(s)$ *for all* $t \geq 0, s \geq 0$;
3) f *is increasing;*
4) f *is continuous from the right at* 0 .

The notion of strong A-summability with respect to a modulus was given in [1,2].

Let $A = (a_{nk})$ be an infinite matrix of nonnegative real numbers, $p = (p_k)$ be a sequence of positive real numbers and f be a modulus. A sequence $x = (x_k)$ is called strongly A-summable to L with respect to the modulus f if (see [2])

$$\lim_{n\to\infty} \sum_k a_{nk} f(| x_k - L |)^{p_k} = 0.$$

Here and henceforth we write $f(t)^{p_k}$ instead of $[f(t)]^{p_k}$.

Let $\mathcal{A}$ denote the sequences of infinite matrices $A^i = (a_{nk}(i))$ of nonnegative real numbers. A sequence $x = (x_k)$ is called strongly $\mathcal{A}$-summable to L with respect to the modulus f if

$$\lim_{n\to\infty} \sum_k a_{nk}(i) f(| x_k - L |)^{p_k} = 0 \text{ uniformly in } i$$

(notation $[\mathcal{A}, f, p]$-$\lim x = L$). The sets of strongly $\mathcal{A}$-summable sequences with respect to the modulus, and strongly $\mathcal{A}$-summable to zero sequences with respect to the modulus are denoted, respectively, by $[\mathcal{A}, f, p]$ and $[\mathcal{A}, f, p]_0$.

A sequence $x = (x_k)$ is called strongly $\mathcal{A}$-bounded with respect to the modulus f if

$$\sup_{n,i} \sum_k a_{nk}(i) f(\mid x_k \mid)^{p_k} < \infty.$$

The set of strongly $\mathcal{A}$-bounded sequences with respect to the modulus is denoted by $[\mathcal{A}, f, p]_\infty$. If $\mathcal{A} = (A), A = (a_{nk})$, then in the notations we write A instead of $\mathcal{A}$. If $f(t) = t$, then we omit f in the notations.

The various special cases of the spaces $[\mathcal{A}, f, p], [\mathcal{A}, f, p]_0$ and $[\mathcal{A}, f, p]_\infty$ are considered earlier by Bilgin [1,2] and Connor [3] (in the case $\mathcal{A} = (A)$), Soomer [9] (in the case $f(t) = t$) and Kolk [5] (in the case $\mathcal{A} = (A)$ and $p_k = p$ $(k \in \mathbb{N})$, where one modulus f is replaced with a sequence of moduli (f_k)).

In the present paper we examine some properties of the sequence spaces $[\mathcal{A}, f, p]_0$, $[\mathcal{A}, f, p]$ and $[\mathcal{A}, f, p]_\infty$.

2. Fundamental and inclusion theorems

The following theorem gives inclusion relations among the spaces $[\mathcal{A}, f, p]$, $[\mathcal{A}, f, p]_0$, and $[\mathcal{A}, f, p]_\infty$. This is a routine verification and therefore we omit the proof. We have

Theorem 1. $[\mathcal{A}, f, p]_0 \subset [\mathcal{A}, f, p]$, $[\mathcal{A}, f, p]_0 \subset [\mathcal{A}, f, p]_\infty$ *and* $[\mathcal{A}, f, p] \subset [\mathcal{A}, f, p]_\infty$ *if*

$$\| \mathcal{A} \| = \sup_{n,i} \sum_k a_{nk}(i) < \infty. \tag{1}$$

Theorem 2. *Let* $0 < p_k \leq \sup p_k = H < \infty$. *Then* $[\mathcal{A}, f, p]_0$ *is complete linear topological spaces paranormed by* h *defined by*

$$h(x) = \sup_{n,i} \Big(\sum_k a_{nk}(i) f(\mid x_k \mid)^{p_k} \Big)^{1/M}$$

where $M = \max\{1, H\}$. *If* (1) *holds and* $\inf p_k > 0$, *then* $[\mathcal{A}, f, p]$ *is paranormed with the same paranorm* h . *The space* $[\mathcal{A}, f, p]$ *is complete if*

$$\lim_n \sum_k a_{nk}(i) = 0 \textit{ uniformly in } i. \tag{2}$$

Proof. By using standard techniques we can prove that $[\mathcal{A}, f, p]_0$ and $[\mathcal{A}, f, p]$ (if (1) holds and $\inf p_k > 0$) have the paranorm h and that $[\mathcal{A}, f, p]_0$ is complete.

If $H = \sup p_k$ and $K = \max\{1, 2^{H-1}\}$, we have (see, Maddox [7])

$$| a_k + b_k |^{p_k} \leq K(| a_k |^{p_k} + | b_k |^{p_k}) \tag{3}$$

and for all $\lambda \in \mathbb{C}$,

$$| \lambda |^{p_k} \leq \max\{1, | \lambda |^H\}. \tag{4}$$

Now by the inequalities (3) and (4)

$$
\begin{aligned}
T_n^i(y) = \sum_k a_{nk}(i) f(| x_k |)^{p_k} &= \sum_k a_{nk}(i) f(| x_k - L + L |)^{p_k} \\
&\leq K \sum_k a_{nk}(i) f(| x_k - L |)^{p_k} \\
&\quad + K \max\{1, f(| L |)^H\} \sum_k a_{nk}(i).
\end{aligned}
$$

From this inequality, (2) and Theorem 1, it is easy to see that $[\mathcal{A}, f, p] = [\mathcal{A}, f, p]_0$ and therefore the completeness of $[\mathcal{A}, f, p]$ follows from the completeness of $[\mathcal{A}, f, p]_0$.

We now characterize the class of strongly regular methods $\mathcal{A}$. The summability method $\mathcal{A}$ is said to be strongly regular if $x_k \to L$ implies that $[\mathcal{A}, f, p]$-$\lim x_k = L$.

Let X, and Y be two nonempty subsets of the space w of all sequences. If $x \in X$ implies that $\left(\sum_k a_{nk} x_k\right) \in Y$, we say that A defines a matrix transformation from X into Y and we write $A : X \to Y$. The symbol (X, Y) denotes the class of matrices A such that $A : X \to Y$. It is known that $A \in (c_0, c_0)$ if and only if $\| A \| = \sup_n \sum_k | a_{nk} | < \infty$ and $\lim_n a_{nk} = 0$ for all k , where c_0 denotes the Banach spaces of null sequences $x = (x_k)$.

By $\mathcal{A} \in (c_0, c_0)$ we mean that for every $x \in c_0$,

$$\lim_{n \to \infty} \sum_k a_{nk}(i) x_k = 0 \text{ uniformly in } i.$$

Theorem 3. *Let* $0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty$ *and*

$$\lim \frac{f(t)}{t} = \beta > 0. \tag{5}$$

Then $\mathcal{A}$ *is strongly regular if and only if* $\mathcal{A} \in (c_0, c_0)$.

For $\mathcal{A} = (A)$ this result is proved by Bilgin in [1].

Theorem 4. *Suppose that* $\mathcal{A} \in (c_0, c_0)$ *and* $p = (p_k)$ *converges to a positive limit. Then* $x_k \to L$, $[\mathcal{A}, f, p]$-$\lim x = L$, $[\mathcal{A}, f, p]$-$\lim x = L'$ *imply* $L = L'$ *if and only if*

$$\lim_n \sum_k a_{nk}(i) \neq 0 \textit{ uniformly in } i. \tag{6}$$

Proof. Let $\mathcal{A} \in (c_0, c_0)$ and (p_k) be bounded. Suppose that $x_k \to L$ imply $[\mathcal{A}, f, p]$-$\lim x = L$ uniquely. By Definition 1 we get $[\mathcal{A}, f, p]$-$\lim e = 1$, where $e = (1, 1, 1, \ldots)$. Hence, we must have (6), for otherwise $[\mathcal{A}, f, p]$-$\lim e = 0$ which contradicts the uniqueness of L.

The rest of the claim can be proved by using the techniques similar to those used in Theorem 2 of Bilgin [1].

Using the same technique as in Theorem 1 in [1], it is easy to prove the following theorem.

Theorem 5. *Suppose that* $0 < p_k \leq q_k$ *(for all* k *),* (q_k/p_k) *is bounded and* (1) *holds. Then* $[\mathcal{A}, f, q] \subset [\mathcal{A}, f, p]$.

Theorem 6. *If* (1) *holds and* $0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty$, *then* $[\mathcal{A}, p]_0 \subset [\mathcal{A}, f, p]_0$ *and* $[\mathcal{A}, p] \subset [\mathcal{A}, f, p]$.

Proof. We consider $[\mathcal{A}, p]_0 \subset [\mathcal{A}, f, p]_0$ only. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. For a sequence $(x_k) \in [\mathcal{A}, p]_0$, let

$$T_n^i = \sum_k a_{nk}(i) \mid x_k \mid^{p_k},$$

so that $\lim_n T_n^i = 0$ uniformly in i. We split the sum $T_n^i(f)$ into two sums $\sum_1$ and $\sum_2$ over $\{k : \mid x_k \mid \leq \delta\}$ and $\{k : \mid x_k \mid > \delta\}$, respectively. Then

$$\sum\nolimits_1 < \max\{\epsilon, \epsilon^r\} \parallel \mathcal{A} \parallel . \tag{7}$$

Further, for $\mid x_k \mid > \delta$ we have by Definition 1 that $f(\mid x_k \mid) \leq \frac{2f(1)}{\delta} \mid x_k \mid$. Hence

$$\sum\nolimits_2 \leq \max\Big\{1, \Big(\frac{2f(1)}{\delta}\Big)^H\Big\} T_n^i,$$

which together with (7) yields $(x_k) \in [\mathcal{A}, f, p]_0$.

Corollary 7. *If* $\parallel A \parallel = \sup_n \sum_k a_{nk} < \infty$ *and* $0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty$, *then* $[A, p]_0 \subset [A, f, p]_0$ *and* $[A, p] \subset [A, f, p]$.

Öztürk and Bilgin ([8], Theorem 5) proved Corollary 7 in the case $A = (C, 1)$. Note that in this case if $p_k = 1$ for all k, Maddox ([6], Theorem 1) proved Corollary 7.

Theorem 8. *If* (1) *and* (5) *hold and* $0 < r = \inf p_k \leq p_k \leq \sup p_k = H < \infty$, *then* $[\mathcal{A}, p] = [\mathcal{A}, f, p]$.

Proof. In Theorem 6, it was shown that $[\mathcal{A}, f, p] \supset [\mathcal{A}, p]$. We must show that $[\mathcal{A}, f, p] \subset [\mathcal{A}, p]$. This inclusion can be proved by using the techniques similar to those used in Theorem 4 of Bilgin [2].

Let $\mathcal{B}$ denote the sequence of infinite matrices $B^i = (b_{nk}(i))$ of nonnegative real numbers. We write $[\mathcal{A}, f, p] \subset [\mathcal{B}, f, q](\text{reg})$ if $[\mathcal{A}, f, p] \subset [\mathcal{B}, f, q]$ and $[\mathcal{A}, f, p]\text{-}\lim x = [\mathcal{B}, f, q]\text{-}\lim x$ for every $x \in [\mathcal{A}, f, p]$.

We now establish an inclusion relation between the spaces $[\mathcal{A}, f, p]$ and $[\mathcal{B}, f, q]$.

Theorem 9. *Suppose that* $0 < q_k < p_k$, $r = \sup \frac{q_k}{p_k} < 1$, $\lambda = \inf \frac{q_k}{p_k} > 0$ *and* $b_{nk}(i) \neq 0$ *implies* $a_{nk}(i) \neq 0$. *If the conditions*

$$\sup_{n,i} \sum_k \left[b_{nk}(i)\right]^{1/1-r} \left[a_{nk}(i)\right]^{r/r-1} < \infty \tag{8}$$

and

$$\sup_{n,i} \sum_k \left[b_{nk}(i)\right]^{1/1-\lambda} \left[a_{nk}(i)\right]^{\lambda/\lambda-1} < \infty \tag{9}$$

are fulfilled, then $[\mathcal{A}, f, p] \subset [\mathcal{B}, f, q]$ (reg).

Proof. Let $x = (x_k) \in [\mathcal{A}, f, p]$ and $[\mathcal{A}, f, p]\text{-}\lim x = L$. We write $t_k = f(|\, x_k - L \,|)^{p_k}$ and $\lambda_k = \frac{q_k}{p_k}$, so that $0 < \lambda \leq \lambda_k < 1$, and

$$\lim_n \sum_k a_{nk}(i) t_k = 0 \text{ uniformly in } i. \tag{10}$$

Define

$$U_k = \begin{cases} t_k, & t_k \geq 1 \\ 0, & t_k < 1 \end{cases} \quad \text{and} \quad V_k = \begin{cases} t_k, & t_k \geq 1 \\ 0, & t_k < 1. \end{cases}$$

So $t_k = U_k + V_k$, $t_k^{\lambda_k} = U_k^{\lambda_k} + V_k^{\lambda_k}$, $U_k \leq t_k$, $V_k \leq t_k$, $U_k^{\lambda_k} \leq U_k^r$ and $V_k^{\lambda_k} \leq V_k^{\lambda}$. By Hölder's inequality we obtain

$$\begin{aligned}
\sum_k b_{nk}(i) f(|\, x_k - L \,|)^{q_k} &= \sum_k b_{nk}(i) t_k^{\lambda_k} \\
&= \sum_k b_{nk}(i) U_k^{\lambda_k} + \sum_k b_{nk}(i) V_k^{\lambda_k} \\
&\leq \sum_k b_{nk}(i) U_k^r + \sum_k b_{nk}(i) V_k^{\lambda} \\
&= \sum_k \left[a_{nk}(i) U_k\right]^r \frac{b_{nk}(i)}{a_{nk}(i)^r} \\
&\quad + \sum_k \left[a_{nk}(i) V_k\right]^{\lambda} \frac{b_{nk}(i)}{a_{nk}(i)^{\lambda}} \\
&\leq \left(\sum_k a_{nk}(i) t_k\right)^r \left(\sum_k b_{nk}(i)^{\frac{1}{1-r}} a_{nk}(i)^{\frac{r}{r-1}}\right)^{1-r} \\
&\quad + \left(\sum_k a_{nk}(i) t_k\right)^{\lambda} \left(\sum_k b_{nk}(i)^{\frac{1}{1-\lambda}} a_{nk}(i)^{\frac{\lambda}{\lambda-1}}\right)^{1-\lambda}.
\end{aligned}$$

The result follows from (8), (9) and (10).

It is essential to note that for A=B Theorem 9 follows from Theorem 5.

Soomer ([9], Theorem 1) proved Theorem 9 in the case $f(t) = t$.

Acknowledgement

The author wishes to thank Professor E. Kolk for valuable suggestions and the referee for some useful comments which improved the presentation of the paper.

References

1. T. Bilgin, *A note on some sequence spaces defined by a modulus* (to appear).
2. T. Bilgin, *On strong A-summability defined by a modulus*, Ch. J. Math. (to appear).
3. J. Connor, *On strong matrix summability with respect to a modulus and statistical convergence*, Canad. Math. Bull. **32** (2) (1989), 194–198.
4. E. Kolk, *Sequence spaces defined by a sequence of moduli*, *Problems of Pure and Applied Mathematics* (Abstracts of a conference held Sept. 21-22,1990), Tartu, 1990, pp. 131–134.
5. E. Kolk, *On strong boundedness and summability with respect to a sequence of moduli*, Tartu Ül. Toimetised **960** (1993), 41–50.
6. I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 161–166.
7. I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. (2) **18** (1967), 345–355.
8. E. Öztürk and T. Bilgin, *Strongly summable sequence spaces defined by a modulus*, Indian J. Pure Appl. Math. **25** (6) (1994), 621–625.
9. V. Soomer, *Inclusion theorems for strong summability*, Tartu Ül. Toimetised **960** (1993), 93–104.

Received May 8, 1995, revised December 14, 1995

Department of Mathematics
Yüzüncü Yil University
Van Turkey