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A note on Knopp's core of a vector valued sequence*

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The concept of the core of a sequence in space $\mathbf{C}$ of complex numbers is due to Knopp [2]. The definition of core by Knopp is equivalent to the following one:

The core $K^{\circ}(x)$ of the sequence $x = (x_n) \subset \mathbf{C}$ is the intersection of all closed convex sets in space $\mathbf{C}$ which contains all except finitely many of the points x_n .

Knopp's core of a sequence of numbers has two essential properties.

A. *A sequence is convergent if and only if its core is a singleton.*

This property establishes a simple connection between the theory of topological vector spaces and the theory of cores. It is now easy to use topological methods in sequence spaces for the study of cores.

B. *If a sequence is bounded, then its core is not empty.*

The sequence $x = (x_k)$ with empty core is written as $x_k \sim \infty$, and it is called *essentially divergent sequence.* It is vital to be aware of elements with empty cores in investigations of core inclusion problems like

$$K^{\circ}(Ax) \subset K^{\circ}(x) \quad \forall x \in E \tag{1}$$

where A is an operator between sequence spaces E and X. If the range of operator A has sequences with empty cores, the inclusion (1) may hold in trivial case. For example, let ω be the space of all sequences of complex numbers and $\mathbf{c}$ be the space of convergent sequences. Let $a = (1, 2, 3, \ldots)$ and let $A : \mathbf{c} \to \omega$ be defined as

$$Ax = a \cdot \lim x_n.$$

Then

$$K^{\circ}(Ax) \subset K^{\circ}(x) \quad \forall x \in \mathbf{c}.$$

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Following Knopp, N.R.Das and Ajanta Chowdhury have defined the core for a sequence in a topological vector space (see [1]). Their definition is as follows.

Let E be a topological vector space and let $\bar{x} = (x_n)$ be a sequence in E. Let

$$E_n(\bar{x}) := \{x_n,\ x_{n+1}, \ldots\}$$

and let $R_n(\bar{x})$ be the closure of the convex hull of $E_n(\bar{x})$ in space E, i.e.,

$$R_n(\bar{x}) := clco\ E_n(\bar{x}).$$

Definition 1. *The intersection*

$$K^{\circ}(\bar{x}) = \bigcap\nolimits_{n=1}^{\infty} R_n(\bar{x}) \tag{2}$$

is called core of the sequence $\bar{x} = (x_n)$.

We shall call this core Knopp's core for the vector valued sequences.

The purpose of the present note is to show that Knopp's core for a vector valued sequence does not have properties **A** and **B** .

Let E be a Hausdorff locally convex topological vector space and let E' be its topological dual.

Proposition 1. *A sequence* $\bar{x} = (x_n)$ *in* E *has the same Knopp's core in every topology which is compatible with the duality between* E *and* E' .

Proof. Any two locally convex topologies which are compatible with the duality between E and E' have the same closed convex sets. This means that the sets $R_n(\bar{x})$ are same for those topologies. The result follows from (2). ■

Proposition 2. *Knopp's core of a sequence* $\bar{x} = (x_n)$ *contains its weak cluster points.*

Proof. If y is a weak cluster point of $\bar{x}$, then for every n it belongs to the weak closure of $E_n(\bar{x})$. Therefore, it belongs to $R_n(\bar{x})$, since weak and original closures for the co $E_n(\bar{x})$ are the same. The result follows from (2). ■

Proposition 3. *If* E *is weakly sequentially non-complete, then there exists a bounded sequence which has empty Knopp's core.*

Proof. Let $\bar{x} = (x_1, x_2, ..., x_n, \ldots)$ be a sequence in E which is weakly Cauchy and divergent in the weak topology. This sequence is bounded. To prove the proposition, let us show that $K^{\circ}(\bar{x}) = \varnothing$. Suppose the contrary.

Let $y \in K^{\circ}(\bar{x})$. Then there exists $(y_{nk}) \subset E$ such that, for every k and n,

$$y_{nk} \in co\{x_n,\ x_{n+1}, \ldots\}$$

and

$$\lim_n \lim_k y_{nk} = y$$

in the topology of E.

Let f be an arbitrary element in E'. Then, for every k and n,

$$f(y_{nk}) \in co\{f(x_n), f(x_{n+1}), \ldots)\} \subset \mathbf{C}.$$

Moreover,

$$\lim_n \lim_k f(y_{nk}) = f(y)$$

and, therefore, $f(y)$ belongs to the Knopp's core of the sequence

$$f(\bar{x}) := (f(x_1), f(x_2), \ldots)$$

of complex numbers, i.e.,

$$f(y) \in K^{\circ}(f(\bar{x})) \subset C.$$

The sequence $f(\bar{x}) = (f(x_1), f(x_2), \ldots)$ is a Cauchy sequence in C and, therefore, its Knopp's core is a singleton, i.e.,

$$\{f(y)\} = K^{\circ}(f(\bar{x})) \quad \forall f \in E'.$$

It means that y is a weak limit on $\bar{x}$. By the proposition, $\bar{x}$ has no weak limits, therefore, $K^{\circ}(\bar{x})$ must be empty. ■

Proposition 4. *Let E be a space with a topology stronger than the weak topology. There exists a non-convergent sequence the Knopp's core of which is a singleton.*

Proof. Let $\bar{x} = (x_1, x_2, \ldots, x_n, \ldots)$ be a non-convergent sequence that is weakly convergent to y. By Proposition 2,

$$y \in K^{\circ}(\bar{x}).$$

If there exists another z such that

$$z \in K^{\circ}(\bar{x}),$$

then analogously to the proof of Proposition 3 we can show that z is the weak limit on $\bar{x}$. This means that $z = y$ and $K^{\circ}(\bar{x})$ is a singleton. ■

Let φ denote the subspace of sequences $x = (\xi_k)$ for which $\xi_k \neq 0$ at most finitely often. Let E be a K-space that is a linear space of sequences containing φ and having a locally convex Hausdorff topology with the property that the coordinate linear functionals $x \to \xi_k$ are continuous. Let $P(x) := (P_n(x))$, where $P_n(x) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, 0, \ldots)$. As $\varphi \subset E$, $P(x)$ is a sequence in E. Let

$$\begin{aligned} E_{AB} &:= \{x : \ P(x) \text{ is bounded in } E\}, \\ E_{AK} &:= \{x : \ P_n(x) \to x \text{ in } E\}, \\ E_{FAK} &:= \{x : \ (P_n(x)) \text{ is weakly Cauchy in } E\}, \\ E_{SAK} &:= \{x : \ (P_n(x)) \text{ is weakly convergent in } E\}. \end{aligned}$$

Proposition 5. *Let* $x \in E_{AB}$.

a) If $x \in E_{FAK} \setminus E_{SAK}$, *then* $K^\circ(P(x)) = \varnothing$.

b) If $x \in E_{SAK} \setminus E_{AK}$, *then* $K^\circ(P(x)) = \{x\}$.

Proof. If $x \in E_{FAK} \setminus E_{SAK}$, then $P(x)$ is a sequence in E which is weakly Cauchy and divergent in the weak topology. Thus the part a) follows from the proof of Proposition 3.

If $x \in E_{SAK} \setminus E_{AK}$, then $P(x)$ is a sequence which is non-convergent in the topology of E but convergent in the weak topology. Thus part b) follows from the proof of Proposition 4. ■

Example. Let E be the space $\mathbf{c}$ of convergent sequences with ordinary norm $\| x \| = \sup_k | \xi_k |$. It is well-known that

$$E' := \{(\alpha_k) \in \omega, \text{ where } \sum | \alpha_k | < \infty\}.$$

Let $x = (0, 1, 0, 1, 0, \ldots)$. As

$$f(P_n(x)) = \sum_{k=1}^{n} \alpha_k (1 - (-1)^k),$$

then for every $m > n$

$$| f(P_m(x) - P_n(x)) | \leq \sum_{k=n}^{m} | \alpha_k |,$$

and as

$$\sum | \alpha_k | < \infty$$

then $(f(P_n(x)))$ is a Cauchy sequence in C for every $f = (\alpha_k) \in E'$, i.e. $x \in E_{FAK}$. In addition, $x \notin E_{SAK}$ as x is not a convergent sequence, i.e., xE. Therefore,

$$K^\circ(P(x)) = \varnothing.$$

Let us consider the sequence $e = (1, 1, 1, \ldots)$. It is obvious that

$$e \in E_{SAK} \setminus E_{AK} \text{ and } w - \lim_n P_n(e) = e.$$

Therefore,

$$K^\circ(P(e)) = \{e\}.$$

References

1. N.R. Das and Ajanta Chowdhury, *On core of a vector valued sequence*, Bull. Calcutta Math. Soc. **86** (1994), 27–32.
2. K. Knopp, *Zur Theorie der Limitierungsverfahren*, Math. Z., **31** (1930), 97–127, 276–305.

Vektorväärtusega jadade tuumadest

Resümee

Vaadeldakse Dasi ja Chowdhury poolt defineeritud tuuma omadusi (vt. [1]). Osutub, et Knoppi tuuma mõiste formaalsel ülekandmisel suvalisse topoloogilisse ruumi lähevad kaduma kaks järgmist olulist omadust.

A. *Jada on koonduv parajasti siis, kui ta tuum on ühepunktiline.*

B. *Tõkestatud jada tuum pole tühi.*

Käesolevas töös näidatakse, et formaalselt üldistatud tuuma korral on ühepunktilise tuumaga ka mittekoonduvaid jadasid ja leidub tõkestatud jadasid, mille tuum on tühi.

Näited. Vaatleme koonduvate jadade ruumi **c**. Olgu $x = (\xi_k)$ suvaline arvjada. Olgu $P(x) = (P_n(x))$ tema lõigetest koosnev jada ruumis **c**, s.t. $P_n(x) = (\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots)$. Kui $x = (1, 0, 1, 0, 1, \ldots)$, siis $P(x) \subset \mathbf{c}$ ja $K^\circ(P(x)) = \varnothing$.

Jada $e = (1, 1, \ldots, 1, \ldots)$ korral $K^\circ(P(e)) = \{e\}$.

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