ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 1, 1996

Multipliers and L^1-convergence of cosine series

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1. We will consider the integrability and L^1-convergence of the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1}$$

where (a_k) is a sequence of real numbers. The problem of integrability of (1) is to decide $(a_k) \in \{(\hat{f}_k) : f \in L^1\} = \hat{L}^1$, where

$$\hat{f}_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx.$$

The other question investigated in this paper is the L^1-convergence of (1). It is known that $(a_k) \in \hat{L}^1$ alone does not guarantee the L^1-convergence of (1). Connecting the two problems we introduce the following concept. A class of sequences is said to be an integrability and L^1-convergence class if it is an integrability class and for each (a_n) of it the corresponding series converges in L^1-norm if and only if

$$\lim_{n\to\infty} a_n \ln n = 0.$$

Kolmogorov [6] showed that the set of quasiconvex null sequences,

$$\sum_{k=0}^{\infty} (k+1) \mid \Delta^2 a_k \mid < \infty, \ a_k \to 0 \ (k \to \infty),$$

form an L^1-convergence class.

In this paper we will show that the L^1-convergence class can be replaced by a class of multipliers.

2. Let ω denote the set of all sequences of real numbers. Let $T = (\tau_{nk})$ be a regular triangular matrix of real numbers:

$$\lim_{n\to\infty} \tau_{nk} = 1; \ \sup_n \sum_{k=0}^{n} \mid \tau_{nk} - \tau_{n,k+1} \mid < \infty. \tag{2}$$

Throughout, T will be a reversible matrix $(\tau_{nn} \neq 0)$. We denote the summability fields of T by

$$c_T = \{(x_k) \in \omega : \lim_{n\to\infty} \sum_{k=0}^{n} \tau_{nk} x_k \quad \text{exist}\},$$

the boundedness domain of T by

$$m_T = \{(x_k) \in \omega : \sup_n \mid \sum_{k=0}^{n} \tau_{nk} x_k \mid < \infty\}$$

and the class of multipliers by

$$(m_T, c_T) = \{(a_k) \in \omega : (a_k x_k) \in c_T \text{ for every } (x_k) \in m_T\}.$$

If $(a_k) \in (m_T, c_T)$, then $\lim_{k\to\infty} a_k = 0$ and

$$\sum_{n=0}^{\infty} \mid \sum_{k=n}^{\infty} \tau_{kn}^{-1} a_k \mid < \infty, \tag{3}$$

where (τ_{kn}^{-1}) denotes the matrix T^{-1}. (See [2], Theorem 3 and [10], 6.4., p. 92).

3. **Theorem.** *Let T be reversible regular triangular matrix with*

$$\sup_n \int_0^{\pi} \mid \frac{\tau_{n0}}{2} + \sum_{k=1}^{n} \tau_{nk} \cos kx \mid dx = K < \infty. \tag{4}$$

Then the cosine series (1) *converges, except possibly at $x = 0$, to an integrable function $f(x)$, is the Fourier series of $f(x)$, and the partial sums converge in L^1-norm to f if and only if $\lim_{n\to\infty} a_n \ln n = 0$.*

Proof. Cleary,

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \sum_{k=0}^{n}\Big(\sum_{j=k}^{n} \tau_{jk}^{-1} a_j\Big) K_k(x) =$$

$$= \sum_{k=0}^{n-1}\Big(\sum_{j=k}^{n-1} \tau_{jk}^{-1} a_j\Big) K_k(x) + a_n D_n(x), \tag{5}$$

where

$$K_k(x) = \frac{\tau_{k0}}{2} + \sum_{j=1}^{k} \tau_{kj} \cos jx = \sum_{j=0}^{k} t_{kj} D_j(x),$$

$$t_{kj} = \tau_{kj} - \tau_{k,j+1},$$

$$D_j(x) = \frac{1}{2} + \sum_{\nu=1}^{j} \cos \nu x = \frac{\sin(j+\frac{1}{2})x}{2\sin\frac{x}{2}}.$$

By (2) we have for some C and M

$$| K_k(x) | \le \sum_{j=0}^{k} | t_{kj} | D_j(x) | \le$$

$$\le \frac{C}{| x |} \sum_{j=0}^{k} | t_{kj} | \le \frac{CM}{| x |} \ (x \neq 0) \text{ for } k = 0, 1, 2, \dots .$$

Since $(a_k) \in (m_T, c_T)$, then by (3) we have the pointwise convergence of

$$\sum_{k=0}^{\infty} \Big(\sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \Big) K_k(x) = f(x) \ (x \neq 0)$$

and $\lim_{n\to\infty} S_n(x) = f(x)$ in $(0, \pi]$. Clearly $f(x)$ is an integrable function and by [4] and [8] series (1) is the Fourier series of $f(x)$. By (5) we have

$$\int_0^{\pi} | S_n(x) - f(x) | \, dx =$$

$$= \int_0^{\pi} | a_n D_n(x) - \sum_{k=n}^{\infty} \Big(\sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \Big) K_k(x) | \, dx \le$$

$$\le | a_n | \int_0^{\pi} | D_n(x) | \, dx + \sum_{k=n}^{\infty} | \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j | \int_0^{\pi} | K_k(x) | \, dx \le$$

$$\le | a_n | \int_0^{\pi} | D_n(x) | \, dx + K \sum_{k=n}^{\infty} | \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j |$$

and by

$$\int_0^{\pi} | a_n D_n(x) | \, dx - \int_0^{\pi} | \sum_{k=n}^{\infty} \Big(\sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j \Big) K_k(x) | \, dx \le$$

$$\le \int_0^{\pi} | S_n(x) - f(x) | \, dx,$$

then

$$| a_n | \int_0^{\pi} | D_n(x) | \, dx \le \int_0^{\pi} | S_n(x) - f(x) | \, dx + K \sum_{k=n}^{\infty} | \sum_{j=k}^{\infty} \tau_{jk}^{-1} a_j | .$$

If $(a_k) \in (m_T, c_T)$, then by (3) and (4)

$$\lim_{n\to\infty} \int_0^{\pi} | S_n(x) - f(x) | \, dx = 0$$

if and only if

$$\lim_{n\to\infty} \mid a_n \mid \int_0^\pi \mid D_n(x) \mid dx = 0,$$

it is (see [3], 5.1.1) if and only if $\lim a_n \ln n = 0$.

4. For the Cesàro matrix $T = C^\alpha$ $(\alpha > 0)$,

$$\tau_{nk} = \frac{A^\alpha_{n-k}}{A^\alpha_n}, \; A^\alpha_n = \frac{(n+\alpha)(n+\alpha-1)\ldots(\alpha+1)}{n!},$$

is by [1] and [7]

$$(m_{C^\alpha}, c_{C^\alpha}) = \{(a_k) \in \omega : \sum_{k=0}^{\infty} (k+1)^\alpha \mid \Delta^{\alpha+1} a_k \mid < \infty; \; \lim_{k\to\infty} a_k = 0\}.$$

It is well known ([5]) that the Riesz matrix P ,

$$\tau_{nk} = \begin{cases} 1 - \dfrac{P_{k-1}}{P_n}, & 0 \le k \le n \\ 0, & n < k, \end{cases}$$

$P_{-1} = 0; \; P_n = p_0 + p_1 + \ldots + p_n,$ is the regular matrix if and only if the following conditions hold:

$$\lim_{n\to\infty} \mid P_n \mid = \infty, \; \sup_n \frac{1}{\mid P_n \mid} \sum_{k=0}^{n} \mid p_k \mid < \infty.$$

Then by [5]

$$(m_P, c_P) =$$

$$= \{(a_k) \in \omega : \sum_{k=0}^{\infty} \mid P_k \Delta \frac{\Delta a_k}{p_k} \mid < \infty; \; \lim_{k\to\infty} \frac{P_k}{p_k} \Delta a_k = 0; \; \lim_{k\to\infty} a_k = 0\}.$$

For the case when $T = C^\alpha$ or $T = P$ this Theorem is proved in [9].

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Received December 18, 1995

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