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# Some notes on a convexity theorem for Cesàro-type families of summability methods

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In this paper the convexity theorem well known for the family of Cesàro methods (see[1] [4], Theorem 3.1) is transferred to the larger class of summability methods and applied to estimating the speed of summability.

## 1. Preliminaries

Let us consider sequences $x = (\xi_n)$ with $\xi_n \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$) for $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Let $A$ be a summability method given by sequence-to-sequence transformation of[2] $x \in \omega A$ into $Ax = (\eta_n)$ where $\eta_n \in \mathbb{K}$. The method $A$ can be, in particular, the matrix method $A = (a_{nk})$ with $a_{nk} \in \mathbb{K} \quad (n, k \in \mathbb{N})$.

Suppose $\lambda = (\lambda_n)$ is a non-decreasing positive sequence. In the sequel we need the following notations:

$m^\lambda = \{x = (\xi_n) \in c \mid (\beta_n) = (\lambda_n(\xi_n - \lim \xi_n)) \in m\}$,

$c^\lambda = \{x \in m^\lambda \mid (\beta_n) \in c\}$,

$c_*^\lambda = \{x \in c^\lambda \mid (\beta_n) \in c_0\}$.

The sequence $x$ is said to be summable by method $A$ with speed $\lambda$ (shortly $A^\lambda$-summable) if $Ax \in c^\lambda$. The sequence $x$ is said to be $A^\lambda$-bounded if $Ax \in m^\lambda$ (see [1], p. 252).

This paper is concerned with families $\{A_\alpha\}$ of summability methods $A_\alpha$ given by transformations of $x \in \omega A_\alpha$ into $A_\alpha x = (\eta_n^\alpha)$ where $\alpha$ is a continuous parameter with values $\alpha > \alpha_0$.

## 2. Cesàro-type families of summability methods

**2.1.** Let us consider first the family of generalized Nörlund methods defined in [9].

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[1] See also [6], Theorems 1 and 2.

[2] We denote by $\omega A$ the set of all sequences $x$ where the transformation $A$ is applied.

A matrix method $A = (a_{nk})$ is said to be a generalized Nörlund method (see [1], p. 138) $A = (N, p_n, q_n)$ if

$$a_{nk} = \begin{cases} p_{n-k}q_k/P_n & \text{if } k \le n, \\ 0 & \text{if } k > n \end{cases}$$

where $P_n = \sum_{k=0}^{n} p_{n-k}q_k \neq 0$ and $p_n,\ q_n \in \mathbb{K} \quad (n \in \mathbb{N})$.

Suppose further that the sequence[3] $(A_n^{\alpha\beta})$ is formally defined by the power series

$$f_{\alpha\beta}(\chi) = (1-\chi)^{-(\alpha+1)}(\log \frac{e}{1-\chi})^{\beta} = \sum_{n=0}^{\infty} A_n^{\alpha\beta}\chi^n$$

where $\alpha, \beta, \chi \in \mathbb{R}$ and $e$ is the base of Napierian logarithm.

Let $\{A_\alpha\}$ be the family of generalized Nörlund methods (see [9]) $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ where $p_n^{\alpha\beta_0} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1,\beta_0} p_k,\quad p_n, q_n \in \mathbb{K},\ \beta_0 \in \mathbb{R},\ \alpha > \alpha_0$ and $P_n^{\alpha\beta_0} = \sum_{k=0}^{n} p_{n-k}^{\alpha\beta_0} q_k \neq 0$.

**Remark 1.** 1) In particular, if $\beta_0 = 0$ then the methods $(N, p_n^{\alpha\beta_0}, q_n)$ become the generalized Nörlund methods $(N, p_n^{\alpha}, q_n)$ (see [8, 9]).

2) If besides the condition $\beta_0 = 0$ there is $q_n = 1$ $(n = 0, 1, 2, \ldots)$ then we get the Nörlund methods $(N, p_n^{\alpha})$ (see [2, 9]).

3) If $q_n = A_n^{\gamma_0\sigma_0}$ ($\gamma_0$ and $\sigma_0$ are fixed real numbers) and $p_n = A_n^{-1,0} = A_n^{-1}$ then the methods $(N, p_n^{\alpha\beta_0}, q_n)$ become the quasi-Cesàro methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ (see [3]).

4) If $\beta_0 = \sigma_0 = 0$ then the methods $(C, \alpha, \beta_0, \gamma_0, \sigma_0)$ become the generalized Cesàro methods $(C, \alpha, \gamma_0)$ with $\alpha > \alpha_0 = -\gamma_0 - 1$; if we add to the previous conditions the presumption $\gamma_0 = 0$ then we get the Cesàro methods $(C, \alpha)$ with $\alpha > -1$.

It was shown in [9] that the methods $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ and $A_\beta = (N, p_n^{\beta\beta_0}, q_n)$ for every $\alpha_0 < \alpha < \beta$ are connected by the relation

$$\eta_n^{\beta} = \frac{1}{P_n^{\beta\beta_0}} \sum_{k=0}^{n} A_{n-k}^{\beta-\alpha-1} P_k^{\alpha\beta_0} \eta_k^{\alpha} \quad (n \in \mathbb{N}), \tag{2.1}$$

i.e.

$$(N, p_n^{\beta\beta_0}, q_n) = (N, A_n^{\beta-\alpha-1}, P_n^{\alpha\beta_0}) \cdot (N, p_n^{\alpha\beta_0}, q_n).$$

**2.2.** The process of constructing new families of generalized Nörlund methods can be continued with the help of methods $(N, p_n^{\alpha\beta_0}, q_n)$.

[3] In particular, if $\beta = 0$ then $A_n^{\alpha,0} = A_n^{\alpha}$ are Cesàro numbers.

Let us have besides the sequences $(p_n)$ and $(q_n)$ also the sequence $(r_n)$ with $r_n \in \mathbb{K}$ and consider the family of generalized Nörlund methods $A_\alpha = (N, P_n^{\alpha\beta_0}, r_n)$.

It is easy to see that the last family satisfies the relation (2.1) with $Q_n^{\alpha\beta_0} = \sum_{k=0}^{n} P_{n-k}^{\alpha\beta_0} r_k$ and $Q_n^{\beta\beta_0}$ instead of $P_n^{\alpha\beta_0}$ and $P_n^{\beta\beta_0}$, respectively, i.e. the relation

$$(N, P_n^{\beta\beta_0}, r_n) = (N, A_n^{\beta-\alpha-1}, Q_n^{\alpha\beta_0}) \cdot (N, P_n^{\alpha\beta_0}, r_n)$$

holds for every $\alpha_0 < \alpha < \beta$.

The process of constructing new families of generalized Nörlund methods can be further continued with the help of the methods $(N, P_n^{\alpha\beta_0}, r_n)$ in the same way.

**2.3.** Let us generalize the above considered constructions for a family of summability methods $A_\alpha$ in general.

**Definition.** *A family $\{A_\alpha\}$ is said to be a Cesàro-type family if the methods $A_\alpha$ and $A_\beta$ for every $\alpha_0 < \alpha < \beta$ are connected by the relation*

$$\eta_n^\beta = \frac{1}{b_n^\beta} \sum_{k=0}^{n} A_{n-k}^{\beta-\alpha-1} b_k^\alpha \eta_k^\alpha \qquad (x \in \omega A_\alpha) \tag{2.2}$$

*where $b_n^\alpha \in \mathbb{K}$, $b_n^\alpha \neq 0$ $(n \in \mathbb{N})$.*

So a Cesàro-type family $\{A_\alpha\}$ is a family satisfying for every $\alpha_0 < \alpha < \beta$ and $x \in \omega A_\alpha$ the relation

$$A_\beta x = D_{\alpha\beta}(A_\alpha x) \tag{2.3}$$

where $D_{\alpha\beta} = (d_{nk}^{\alpha\beta})$ is a matrix defined by (2.2), i.e.

$$d_{nk}^{\alpha\beta} = \begin{cases} A_{n-k}^{\beta-\alpha-1} b_k^\alpha / b_n^\beta & \text{if } k \le n, \\ 0 & \text{if } k > n. \end{cases} \tag{2.4}$$

We notice that the families of generalised Nörlund methods $A_\alpha = (N, p_n^{\alpha\beta_0}, q_n)$ and $A_\alpha = (N, P_n^{\alpha\beta_0}, r_n)$ are Cesàro-type families satisfying for every $\alpha_0 < \alpha < \beta$ the condition (2.3) with $D_{\alpha\beta} = (N, A_{n-k}^{\beta-\alpha-1}, P_n^{\alpha\beta_0})$ and $D_{\alpha\beta} = (N, A_{n-k}^{\beta-\alpha-1}, Q_n^{\alpha\beta_0})$, respectively. Thus for those families we have (2.2) with $b_n^\alpha = P_n^{\alpha\beta_0}$ and $b_n^\alpha = Q_n^{\alpha\beta_0}$, respectively.

Further Cesàro-type families can be constructed with the help of following remark.

**Remark 2.** 1) Let $\{A_\alpha\}$ be a Cesàro-type family and $A$ be a summability method such that $Ax \in \omega A_\alpha$ for every $x \in \omega A$ and $\alpha > \alpha_0$. Then the family $\{B_\alpha\}$ with $B_\alpha = A_\alpha \cdot A$ is a Cesàro-type family also.

2) Let $D_{\alpha_0\beta} = (d_{nk}^{\alpha_0\beta})$ be defined by (2.4) for every $\beta > \alpha_0$ and $A$ be a summability method. Then $\{A_\beta\}$ where $A_\beta = D_{\alpha_0\beta} \cdot A$ is a Cesàro-type family with $\beta > \alpha_0$.

3) Let $\{A_\alpha\}$ be a Cesàro-type family and $(c_n^\alpha)$ be sequences with $c_n^\alpha \in \mathbb{K}$ and $c_n^\alpha \neq 0$. Then the family $\{B_\alpha\}$ where $B_\alpha x = (c_n^\alpha \eta_n^\alpha)$ and $(\eta_n^\alpha) = A_\alpha x$ is a Cesàro-type family also.

## 3. Notes on a convexity theorem

**3.1.** Let us denote

$$S_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1}\eta_k$$

where $\eta_n \in \mathbb{K}$ $(n \in \mathbb{N})$ and $\alpha > -1$. Let $(U_n)$ and $(V_n)$ be two positive sequences.

Our notes are concerned with the following theorem which is a slight generalization of Theorem 3.1 from [4].

**Theorem.** *Suppose $(U_n)$ and $(V_n)$ satisfy the conditions*

$$U_n \leq MU_{n+k} \quad (n, k \in \mathbb{N}) \tag{3.1}$$

*and*

$$V_n \leq NV_{n+k} \quad (n, k \in \mathbb{N}). \tag{3.2}$$

*Then the following statements hold for every $0 < \delta < \mu$ :*

$$1)\ \eta_n = O(U_n),\ S_n^\mu = O(V_n) \implies S_n^\delta = O(W_n^{\mu\delta}) \tag{3.3}$$

*where*

$$W_n^{\mu\delta} = (U_n)^{\frac{\mu-\delta}{\mu}}(V_n)^{\frac{\delta}{\mu}}.$$

2) *If* $\lim U_n = \infty$ *then*

$$\eta_n = o(U_n),\ S_n^\mu = O(V_n) \implies S_n^\delta = o(W_n^{\mu\delta}). \tag{3.4}$$

3) *If* $\lim V_n = \infty$ *then*

$$\eta_n = O(U_n),\ S_n^\mu = o(V_n) \implies S_n^\delta = o(W_n^{\mu\delta}). \tag{3.5}$$

The formulated theorem is an immediate consequence of the following lemma (see [4], Lemmas 3.1 and 3.2).

**Lemma.** *If $0 < \delta < \mu$ then there exists a number $C_{\mu\delta}$ such that*

$$\sup_{k\leq n}|S_k^\delta| \leq C_{\mu\delta}(\sup_{k\leq n}|\eta_k|)^{\frac{\mu-\delta}{\mu}}(\sup_{k\leq n}|S_k^\mu|)^{\frac{\delta}{\mu}} \quad (n \in \mathbb{N}).$$

In paper [4] the Theorem was given in assumptions $0 < U_n, V_n \uparrow \infty$. The statements 1) – 3) from Theorem for $0 < \delta < \mu$ were first proved by M. Riesz in 1922 (see [6]) for integral forms of Cesàro methods and non-decreasing functions $U(x)$ and $V(x)$ $(x \to \infty)$; in particular, the statement 1) was first proved by G. H. Hardy and J. E. Littlewood in 1912 for integer $0 < \delta < \mu$. The modifications of Theorem with some of implications (3.3) – (3.5) and different restrictions on $U_n$ and $V_n$ were also proved by other authors. For example, the theorems stating the implication (3.5) were proved by A. L. Dixon and W. L. Ferrar, K. Kanno, M. S. Rangachari and H. Sakata (see [5]).

In this paper we do not focus our attention on bettering the restrictions on $(U_n)$ and $(V_n)$ but on drawing conclusions from implications (3.3) – (3.5).

**3.2.** The Theorem together with Lemma can be transferred to a Cesàro-type family.

Let us further consider a Cesàro-type family $\{A_\alpha\}$ with $\alpha > \alpha_0$ and denote

$$T_n^\alpha = b_n^\alpha \eta_n^\alpha$$

where $(\eta_n^\alpha) = A_\alpha x$, $x \in \omega A_\alpha$ and $b_n^\alpha$ is defined by (2.2). Denote also

$$W_n^{\alpha\beta\gamma} = (U_n)^{\frac{\beta-\gamma}{\beta-\alpha}} (V_n)^{\frac{\gamma-\alpha}{\beta-\alpha}} \qquad (\alpha < \gamma < \beta,\ n \in \mathbb{N}). \tag{3.6}$$

**Proposition 1.** *Let $\{A_\alpha\}$ be a Cesàro-type family and $(U_n)$ and $(V_n)$ be positive sequences.*

*If the implication (3.3), (3.4) or (3.5) is true for every $0 < \delta < \mu$ and $\eta_n \in \mathbb{K}$ $(n = 0, 1, 2, \dots)$ then, respectively, the implication*

1) $T_n^\alpha = O(U_n)$, $T_n^\beta = O(V_n)$ $\implies$ $T_n^\gamma = O(W_n^{\alpha\beta\gamma})$,

2) $T_n^\alpha = o(U_n)$, $T_n^\beta = O(V_n)$ $\implies$ $T_n^\gamma = o(W_n^{\alpha\beta\gamma})$

*or*

3) $T_n^\alpha = O(U_n)$, $T_n^\beta = o(V_n)$ $\implies$ $T_n^\gamma = o(W_n^{\alpha\beta\gamma})$

*is true for every $\alpha_0 < \alpha < \gamma < \beta$ and $x \in \omega A_\alpha$.*

**Proposition 2.** *If $\{A_\alpha\}$ is a Cesàro-type family then for every $\alpha_0 < \alpha < \gamma < \beta$ there exists a number $C_{\alpha\beta\gamma}$ such that*

$$\sup_{k \le n} |T_k^\gamma| \le C_{\alpha\beta\gamma} (\sup_{k \le n} |T_k^\alpha|)^{\frac{\beta-\gamma}{\beta-\alpha}} (\sup_{k \le n} |T_k^\beta|)^{\frac{\gamma-\alpha}{\beta-\alpha}} \qquad (n \in \mathbb{N})$$

*for each $x \in \omega A_\alpha$.*

*Proofs of Propositions 1 and 2.* The truth of these propositions follows directly from implications (3.3) – (3.5) and Lemma, respectively, if we take $\eta_n = T_n^\alpha$, $\mu = \beta - \alpha$ and $\delta = \gamma - \alpha$ in them and realize with the help of (2.2) that $S_n^\mu = \sum_{k=0}^n A_{n-k}^{\beta-\alpha-1} T_k^\alpha = T_n^\beta$ and $S_n^\delta = T_n^\gamma$.

Thus we can reformulate the Theorem as a convexity theorem for a Cesàro-type family $\{A_\alpha\}$ by replacing the inequalities $0 < \delta < \mu$ and implications (3.3) – (3.5) by inequalities $\alpha_0 < \alpha < \gamma < \beta$ and implications 1) – 3) from Proposition 1, respectively[4].

Let us denote further:

$$\left.\begin{aligned} \lambda_n^\alpha &= |b_n^\alpha|/U_n, \\ \mu_n^\alpha &= |b_n^\alpha|/V_n, \\ \psi_n^{\alpha\beta\gamma} &= |b_n^\gamma|/W_n^{\alpha\beta\gamma} \quad (\alpha_0 < \alpha < \gamma < \beta,\ n \in \mathbb{N}) \end{aligned}\right\} \tag{3.7}$$

where $b_n^\alpha$ and $b_n^\gamma$ are defined by the Cesàro-type family $\{A_\alpha\}$ and $W_n^{\alpha\beta\gamma}$ by (3.6).

The next result is a slight generalization of Proposition 1.

**Proposition 3.** *Let $\{A_\alpha\}$ be a Cesàro-type family satisfying*

$$b_n^\beta = \sum_{k=0}^{n} A_{n-k}^{\beta-\alpha-1} b_k^\alpha \qquad (n \in \mathbb{N}) \tag{3.8}$$

*for every $\alpha_0 < \alpha < \beta$. Let $(U_n)$ and $(V_n)$ be positive sequences.*

*If the implication (3.3), (3.4) or (3.5) is true for every $0 < \delta < \mu$ and $\eta_n \in \mathbb{K}$ $(n \in \mathbb{N})$ then, respectively, the implication*

1) $\lambda_n^\alpha(\eta_n^\alpha - \eta) = O(1),\ \mu_n^\beta(\eta_n^\beta - \eta) = O(1) \implies \psi_n^{\alpha\beta\gamma}(\eta_n^\gamma - \eta) = O(1)$,

2) $\lambda_n^\alpha(\eta_n^\alpha - \eta) = o(1),\ \mu_n^\beta(\eta_n^\beta - \eta) = O(1) \implies \psi_n^{\alpha\beta\gamma}(\eta_n^\gamma - \eta) = o(1)$,

*or*

3) $\lambda_n^\alpha(\eta_n^\alpha - \eta) = O(1),\ \mu_n^\beta(\eta_n^\beta - \eta) = o(1) \implies \psi_n^{\alpha\beta\gamma}(\eta_n^\gamma - \eta) = o(1)$

*is true for every $\alpha_0 < \alpha < \gamma < \beta$, $\eta \in \mathbb{K}$ and $x \in \omega A_\alpha$.*

*Proof.* Let us take $\eta_n = b_n^\alpha(\eta_n^\alpha - \eta)$, $\delta = \gamma - \alpha$ and $\mu = \beta - \alpha$ in implications (3.3) – (3.5). Now we see with the help of (2.2) and (3.8) that $S_n^\delta = \sum_{k=0}^n A_{n-k}^{\gamma-\alpha-1} b_k^\alpha(\eta_k^\alpha - \eta) = b_n^\gamma(\eta_n^\gamma - \eta)$ and $S_n^\mu = b_n^\beta(\eta_n^\beta - \eta)$. This completes our proof.

The last proposition enables us to apply the Theorem to summability with speed.

**Proposition 4.** *Let $\{A_\alpha\}$ be a Cesàro-type family and the sequences $\lambda_\alpha = (\lambda_n^\alpha)$, $\mu_\beta = (\mu_n^\beta)$ and $\psi_{\alpha\beta\gamma} = (\psi_n^{\alpha\beta\gamma})$ defined by (3.7) be non-decreasing for some $\alpha < \gamma < \beta$.*

*If the implication 1), 2) or 3) from Proposition 3 is true for every $\eta \in \mathbb{K}$ and $x \in \omega A_\alpha$ then, respectively, the implication*

[4] For a corollary from this result for the family of methods $(N, p_n^\alpha, q_n)$ see [7], Theorem 1.

1) $\psi_n^{\alpha\beta\gamma} \nearrow \infty$, $A_\alpha x \in m^{\lambda_\alpha}$, $A_\beta x \in m^{\mu_\beta}$, $\lim \eta_n^\alpha = \lim \eta_n^\beta \implies A_\gamma x \in m^{\psi_{\alpha\beta\gamma}}$,

2) $A_\alpha x \in c_*^{\lambda_\alpha}$, $A_\beta x \in m^{\mu_\beta}$, $\lim \eta_n^\alpha = \lim \eta_n^\beta \implies A_\gamma x \in c_*^{\psi_{\alpha\beta\gamma}}$

*or*

3) $A_\alpha x \in m^{\lambda_\alpha}$, $A_\beta x \in c_*^{\mu_\beta}$, $\lim \eta_n^\alpha = \lim \eta_n^\beta \implies A_\gamma x \in c_*^{\psi_{\alpha\beta\gamma}}$.

*is true for every* $x \in \omega A_\alpha$.

*Proof.* It is sufficient to notice that

$$\psi_n^{\alpha\beta\gamma}(\eta_n^\gamma - \eta) = o(1) \implies \lim \eta_n^\gamma = \eta$$

and that the last implication is true also with $O(1)$ instead of $o(1)$ if $\psi_n^{\alpha\beta\gamma} \nearrow \infty \quad (n \to \infty)$.

**Remark 3.** The Theorem and further propositions remain true for the summability of sequences in a locally convex space $E$ over $\mathbb{K}$ with the set of continuous seminorms $\mathcal{P} = \{p\}$. To prove this it is sufficient to realize (with the help of the proofs of Lemmas 3.1 and 3.2 from [4]) that Lemma can be transferred to the sequences $(\eta_n)$ where $\eta_n \in E \quad (n \in \mathbb{N})$ by replacing the module $|\cdot|$ by every seminorm $p(\cdot)$.

## 4. The comparative estimates for speeds of summability in a Cesàro-type family

Suppose $\{A_\alpha\}$ is still a Cesàro-type family and $\lambda_\alpha = (\lambda_n^\alpha)$, $\mu_\beta = (\mu_n^\beta)$ and $\psi_{\alpha\beta\gamma} = (\psi_n^{\alpha\beta\gamma})$ are the sequences defined by (3.7). The comparative estimates for these sequences can be easily inferred from (3.7).

**Proposition 5.** *Let* $\{A_\alpha\}$ *be a Cesàro-type family and* $(U_n)$ *be a positive sequence satisfying (3.1). Then the implications*

$$T_n^\alpha = O(U_n) \implies T_n^\delta = O(U_n n^{\delta-\alpha}) \tag{4.1}$$

*and*

$$T_n^\alpha = o(U_n) \implies T_n^\delta = o(U_n n^{\delta-\alpha}) \tag{4.2}$$

*are true for every* $\alpha_0 < \alpha < \delta$ *and* $x \in \omega A_\alpha$.

*Proof.* By (2.2) we have that

$$\frac{T_n^\delta}{n^{\delta-\alpha}U_n} = \frac{1}{n^{\delta-\alpha}U_n} \sum_{k=0}^{n} A_{n-k}^{\delta-\alpha-1} U_k \frac{T_k^\alpha}{U_k}.$$

Our implications are true because the triangular matrix $C_{\alpha\delta} = (c_{nk}^{\alpha\delta})$ with $c_{nk}^{\alpha\delta} = A_{n-k}^{\delta-\alpha-1} U_k / U_n n^{\delta-\alpha} \quad (k \le n)$ is a $c_0 \to c_0$ matrix for every $\alpha_0 < \alpha < \delta$.

**Remark 4.** It follows from the last proposition that if $(U_n)$ satisfies (3.1) and $V_n \geq M n^{\beta-\alpha} U_n \quad (n \in \mathbb{N})$ then the implications 1) and 2) from Proposition 1 for $\alpha < \gamma < \beta$ are equivalent, respectively, to the implications (4.1) and (4.2) for every $\alpha < \delta$.

**Proposition 6.** *Suppose the condition*

$$M_{\alpha\beta} n^{\beta-\alpha} \leq |b_n^\beta / b_n^\alpha| \leq N_{\alpha\beta} n^{\beta-\alpha} \quad (n = 1, 2, \ldots) \tag{4.3}$$

*holds for every* $\alpha_0 < \alpha < \beta$.

*Then the sequences* $\lambda_\alpha$, $\mu_\beta$ *and* $\psi_{\alpha\beta\gamma}$ *satisfy for every* $\alpha_0 < \alpha < \gamma < \beta$ *the inequalities*

$$M_{\alpha\beta\gamma} (\lambda_n^\alpha)^{\frac{\beta-\gamma}{\beta-\alpha}} (\mu_n^\beta)^{\frac{\gamma-\alpha}{\beta-\alpha}} \leq \psi_n^{\alpha\beta\gamma} \leq N_{\alpha\beta\gamma} (\lambda_n^\alpha)^{\frac{\beta-\gamma}{\beta-\alpha}} (\mu_n^\beta)^{\frac{\gamma-\alpha}{\beta-\alpha}} \tag{4.4}$$

$(n \in \mathbb{N})$.

*Proof.* By the relations (3.7) we have the equality

$$\psi_n^{\alpha\beta\gamma} = (\lambda_n^\alpha)^{\frac{\beta-\gamma}{\beta-\alpha}} (\mu_n^\beta)^{\frac{\gamma-\alpha}{\beta-\alpha}} |b_n^\gamma| / (|b_n^\alpha|)^{\frac{\beta-\gamma}{\beta-\alpha}} (|b_n^\beta|)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

The condition (4.3) implies the inequalities

$$M_{\alpha\beta\gamma} \leq |b_n^\gamma| / (|b_n^\alpha|)^{\frac{\beta-\gamma}{\beta-\alpha}} (|b_n^\beta|)^{\frac{\gamma-\alpha}{\beta-\alpha}} \leq N_{\alpha\beta\gamma}$$

and therefore the condition (4.4) is satisfied for every $\alpha_0 < \alpha < \gamma < \beta$.

**Remark 5.** We note that the inequalities (4.3) are satisfied, for example, for the family of methods $A_\alpha = (C, \alpha, \beta_0, \gamma_0, \sigma_0)$ with $-\gamma_0 - 1 < \alpha < \beta$ because here $b_n^\alpha = A_n^{\alpha+\gamma_0, \beta_0+\sigma_0}$ and (see [3], (1.7))

$$A_n^{\alpha+\gamma_0, \beta_0+\sigma_0} \sim \frac{n^{\alpha+\gamma_0}}{\Gamma(\alpha + \gamma_0 + 1)} (\log n)^{\beta_0+\delta_0} \quad (n \to \infty). \tag{4.5}$$

In particular, the inequalities (4.3) hold for $A_\alpha = (C, \alpha, \gamma_0)$ and for $A_\alpha = (C, \alpha)$ with $\alpha > -1$.

The inequalities (4.3) are satisfied also for the family of methods $A_\alpha = (N, p_n^\alpha)$ if $p_0 > 0$, $p_n \geq 0$ and the sequence $(p_n)$ is non-increasing (see [2], p. 359).

**Proposition 7.** *Suppose the inequality*

$$|b_n^\beta / b_n^\alpha| \leq N_{\alpha\beta} n^{\beta-\alpha} \quad (n = 1, 2, \ldots) \tag{4.6}$$

*is satisfied for every* $\alpha_0 < \alpha < \beta$ *and there exists a number* $\delta > 0$ *such that*

$$V_n \geq M U_n n^\delta \quad (n = 1, 2, \ldots). \tag{4.7}$$

*Then the sequences $\lambda_\alpha$, $\mu_\beta$ and $\psi_{\alpha\beta\gamma}$ satisfy the condition*

$$M_{\alpha\beta\gamma}\mu_n^\beta \le \psi_n^{\alpha\beta\gamma} \le N_{\alpha\beta\gamma}\lambda_n^\alpha \quad (n \in \mathbb{N}) \tag{4.8}$$

*for every $\alpha_0 < \alpha < \gamma < \beta = \alpha + \delta$.*

*The formulated statement remains true if we replace inequalities (4.6) – (4.8) by their contrary inequalities.*

*Proof.* By the relations (3.7) we have

$$\psi_n^{\alpha\beta\gamma} = \lambda_n^\alpha |b_n^\gamma / b_n^\alpha| (U_n/V_n)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

With the help of conditions (4.6) and (4.7) we involve from the upper equality the right-hand of inequalities (4.8). The left-hand equality follows with the help of the same conditions drectly from the equality

$$\psi_n^{\alpha\beta\gamma} = \mu_n^\beta |b_n^\gamma / b_n^\beta| (V_n/U_n)^{\frac{\beta-\gamma}{\beta-\alpha}}.$$

The statement with contrary inequalities can be proved analogously.

**Remark 6.** The Cesàro-type family $\{A_\alpha\}$ satisfies (4.6) for every $\alpha_0 < \alpha < \beta$ if the conditions (3.8) and $|b_n^\alpha| \le M_\alpha |b_{n+k}^\alpha|$ $(n, k \in \mathbb{N})$ hold for every $\alpha_0 < \alpha < \beta$.

So the inequality (4.6) is satisfied, for example, for the family of methods $A_\alpha = (N, p_n^\alpha, q_n)$ for every $1 < \alpha < \beta$ if $p_0 > 0$, $p_n \ge 0$ and $q_n > 0$ (see [9], Corollary 3.1).

**Remark 7.** Propositions 3, 4, 6 and 7 remain true if we replace the sequences $\lambda_\alpha$, $\mu_\alpha$ and $\psi_{\alpha\beta\gamma}$ defined by (3.7) by those which are determined by the inequalities

$$M_\alpha \frac{|b_n^\alpha|}{U_n} \le \lambda_n^\alpha \le N_\alpha \frac{|b_n^\alpha|}{U_n}, \tag{4.9}$$

$$P_\alpha \frac{|b_n^\alpha|}{V_n} \le \mu_n^\alpha \le R_\alpha \frac{|b_n^\alpha|}{V_n}, \tag{4.10}$$

and

$$S_{\alpha\beta\gamma} \frac{|b_n^\gamma|}{W_n^{\alpha\beta\gamma}} \le \psi_n^{\alpha\beta\gamma} \le T_{\alpha\beta\gamma} \frac{|b_n^\gamma|}{W_n^{\alpha\beta\gamma}} \tag{4.11}$$

$(\alpha_0 < \alpha < \gamma < \beta,\ n \in \mathbb{N})$.

To complete our paper we give the following numerical example as a simple illustration for our results.

**Example.** Let $\{A_\alpha\}$ be the family of methods $A_\alpha = (C, \alpha, 1, 2, 3)$ with $\alpha > -3$. Let us consider, for example, two speeds $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$

with $\lambda_n = n^6$ and $\mu_n = n^6 \log n$ (for $n \geq 2$) and the methods $A_5$ and $A_{10}$. Suppose that

$$A_5 x \in m^\lambda, \quad A_{10} x \in c_*^\mu \tag{4.12}$$

for some $x$.

Let us estimate the speed of summability of sequence $x$ by method $A_7$. We know that $\{A_\alpha\}$ is the Cesàro-type family with $b_n^\alpha = A_n^{\alpha+2,4}$, the condition (3.8) is satisfied for every $-3 < \alpha < \beta$ and the methods $A_\alpha$ are pairwise consistent ([3], Theorem 1). Let us denote $\lambda_5 = \lambda$ and $\mu_{10} = \mu$. In order to satisfy the inequalities (4.9) – (4.10) we can take $U_n = n \log^4 n$ and $V_n = n^6 \log^3 n$ (for $n \geq 2$) because due to (4.5) we have the relations

$$\lambda_n^5 = n^6 = \frac{n^7 \log^4 n}{n \log^4 n} \sim \frac{\Gamma(8)|A_n^{7,4}|}{n \log^4 n} = \Gamma(8)\frac{|b_n^5|}{U_n}$$

and

$$\mu_n^{10} = n^6 \log n = \frac{n^{12} \log^4 n}{n^6 \log^3 n} \sim \frac{\Gamma(13)|A_n^{12,4}|}{n^6 \log^3 n} = \Gamma(13)\frac{|b_n^{10}|}{V_n} \quad (n \to \infty).$$

The sequences $(U_n)$ and $(V_n)$ satisfy the all restrictions put in Theorem. Thus the implications 1) – 3) from Proposition 3 are true for every $-3 < \alpha < \gamma < \beta$ with

$$\lambda_n^\alpha = \frac{n^{\alpha+2} \log^4 n}{n \log^4 n} = n^{\alpha+1}, \quad \mu_n^\beta = \frac{n^{\beta+2} \log^4 n}{n^6 \log^3 n} = n^{\beta-4} \log n,$$

and

$$\psi_n^{\alpha\beta\gamma} = \frac{n^{\gamma+2} \log^4 n}{W_n^{\alpha\beta\gamma}}.$$

In particular, we have $\psi_n^{5,10,7} = n^6 \log^{2/5} n$. Due to the implication 3) from Proposition 4 the relations (4.12) imply the relation $A_7 x \in c_*^{\psi_{5,10,7}}$ which says that $x$ is summable by $A_7$ with speed $\psi_{5,10,7}$. It remains to notice that this example illustrates also the Propositions 6 and 7.

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