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A global stability result for a certain system of fifth order nonlinear differential equations

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ABSTRACT. This paper establishes sufficient conditions which ensure the uniform global asymptotic stability of the zero solution of (1.1).

1. Introduction and statement of the result

We consider the real non-linear autonomous vector differential equation of fifth order

$$X^{(5)} + F(\ddot{X}, \dddot{X})X^{(4)} + \Phi(\ddot{X}, \dddot{X}) + G(\ddot{X}) + H(\dot{X}) + \Psi(X) = 0 \qquad (1.1)$$

in which $X \in \mathbb{R}^n$, $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space, F is a $n \times n$ matrix function, $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$.

The non-linear functions F, Φ, G, H and Ψ are continuous and so constructed such that the uniqueness theorem is valid. The equation (1.1) represents a system of real fifth-order differential equations of the form

$$\begin{aligned} &x_i^{(5)} + \sum_{k=1}^{n} f_{ik}(\ddot{x}_1, \ldots, \ddot{x}_n; \dddot{x}_1, \ldots, \dddot{x}_n)x_k^{(4)} \\ &+ \phi_i(\ddot{x}_1, \ldots, \ddot{x}_n; \dddot{x}_1, \ldots, \dddot{x}_n) + g_i(\ddot{x}_1, \ldots, \ddot{x}_n) \\ &+ h_i(\dot{x}_1, \ldots, \dot{x}_n) + \psi_i(x_1, \ldots, x_n) = 0 \quad (i = 1, 2, \ldots, n). \qquad (1.2) \end{aligned}$$

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For convenience, we fix some notations. Given any matrix M, its eigenvalues will be denoted simply by $\lambda_i(M) \quad (i = 1, 2, \ldots, n)$. Next, given any pair of vectors $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$, we use $\langle X, Y\rangle$ to denote their scalar product $\sum_{i=1}^{n} x_i y_i$. Thus, in particular, $\langle X, X\rangle = \|X\|^2$. Also the Jacobian matrices $J(\Phi(Z,W)|Z)$, $J(\Phi(Z,W)|W)$, $J_G(Z)$, $J_H(Y)$, and $J_\Psi(X)$ are given by

$$J(\Phi(Z,W)|Z) = \left(\frac{\partial \phi_i}{\partial z_j}\right), \quad J(\Phi(Z,W)|W) = \left(\frac{\partial \phi_i}{\partial w_j}\right),$$

$$J_G(Z) = \left(\frac{\partial g_i}{\partial z_j}\right), \quad J_H(Y) = \left(\frac{\partial h_i}{\partial y_j}\right), \quad J_\Psi(X) = \left(\frac{\partial \psi_i}{\partial x_j}\right) \quad (i, j = 1, 2, \ldots, n).$$

Moreover, let the Jacobian matrices $J(\Phi(Z,W)|Z)$, $J(\Phi(Z,W)|W)$, $J_G(Z)$, $J_H(Y)$ and $J_\Psi(X)$ exist and be continuous.

The problem in this paper, in the case $n = 1$, has been investigated to quite a considerable extent.

Chukwu [4] established sufficient conditions for the asymptotic stability in the large of the zero solution of the equation

$$x^{(5)} + ax^{(4)} + f_2(\dddot{x}) + c\ddot{x} + f_4(\dot{x}) + f_5(x) = 0.$$

In [2], Abou-El-Ela and Sadek derived similar results for the problem

$$x^{(5)} + f_1(\dddot{x})x^{(4)} + f_2(\dddot{x}) + f_3(\ddot{x}) + f_4(\dot{x}) + f_5(x) = 0.$$

Yuanhong [9] and Tunç [7, 8] investigated the differential equations

$$x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b\dddot{x} + h(\ddot{x}) + g(\dot{x}) + f(x) = 0,$$
$$x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + b\dddot{x} + h(\ddot{x}) + g(x, \dot{x}) + f(x) = 0$$

and

$$x^{(5)} + \varphi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})x^{(4)} + \psi(\ddot{x}, \dddot{x}) + h(\ddot{x}) + g(\dot{x}) + f(x) = 0,$$

respectively and proved stability in the large under certain conditions.

Recently, Abou-El-Ela and Sadek [3] and Sadek [6] also presented sufficient conditions for the asymptotic stability in the large of zero solution of the equations

$$X^{(5)} + AX^{(4)} + \Phi(\dddot{X}) + G(\ddot{X}) + H(\dot{X}) + BX = 0$$

and

$$X^{(5)} + F(\ddot{X})X^{(4)} + \Phi(\dddot{X}) + G(\ddot{X}) + H(\dot{X}) + \Psi(X) = 0,$$

respectively.

This paper for $n = 1$ includes the results of Abou-El-Ela and Sadek [2] and Chukwu [4] and also extends the correspondent results of Abou-El-Ela and Sadek [3] and Sadek [6].

Using $Y = \dot{X}$, $Z = \dot{Y}$, $W = \dot{Z}$ and $U = \dot{W}$ the differential equation (1.1) will be transformed to the equivalent system

$$\begin{aligned} &\dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W,\ \dot{W} = U, \\ &\dot{U} = -F(Z,W)U - \Phi(Z,W) - G(Z) - H(Y) - \Psi(X). \end{aligned} \tag{1.3}$$

The purpose of the paper is to prove the following

Theorem. *In addition to the fundamental assumptions on F, Φ, G, H and Ψ, we suppose the existence of arbitrary positive constants a_1, a_2, a_3, a_4, a_5 and of sufficiently small positive constants ε_0, ε, ε_1, ε_2, ε_3, ε_4, ε_5 such that*

(i) *There is*

$$\begin{aligned} &a_1a_2 - a_3 > 0,\ (a_1a_2 - a_3)a_3 - (a_1a_4 - a_5)a_1 > 0, \\ &\delta_0 = (a_3a_4 - a_2a_5)(a_1a_2 - a_3) - (a_1a_4 - a_5)^2 > 0 \end{aligned} \tag{1.4}$$

and for all $Y \in \mathbb{R}^n$,

$$\begin{aligned} \Delta_1 = &\frac{(a_3a_4 - a_2a_5)(a_1a_2 - a_3)}{a_1a_4 - a_5} \\ &- [a_1\|J_H(Y)\| - a_5] > 2\varepsilon a_2, \end{aligned} \tag{1.5}$$

$$\Delta_2 = \frac{a_3a_4 - a_2a_5}{a_1a_4 - a_5} - \frac{a_1a_4 - a_5}{a_4(a_1a_2 - a_3)}\Gamma(Y) - \frac{\varepsilon}{a_1} > 0, \tag{1.6}$$

where

$$\Gamma(Y) = \int_0^1 J_H(\sigma Y)d\sigma; \tag{1.7}$$

(ii) *$F(Z,W)$ is symmetric and for all Z, $W \in \mathbb{R}^n$,*

$$\varepsilon_0 \le \lambda_i\left(F(Z,W) - a_1I\right) \le \varepsilon_1 \quad (i = 1,2,\ldots,n);$$

(iii) *$\Phi(Z,0) = 0$, $J(\Phi(Z,W)|Z)$ is negative-definite, $J(\Phi(Z,W)|W)$ is symmetric and for all Z, $W \in \mathbb{R}^n$,*

$$0 \le \lambda_i\left(\int_0^1 [J(\Phi(Z,\sigma W)|\sigma W) - a_2I]\,d\sigma\right) \le \varepsilon_2 \quad (i = 1,2,\ldots,n);$$

(iv) $G(0)=0,\ J_G(Z)$ *is symmetric and for all* $Z \in \mathbb{R}^n$,

$$0 \le \lambda_i \left(\int_0^1 [J_G(\sigma Z) - a_3 I]\, d\sigma \right) \le \varepsilon_3 \quad (i = 1, 2, \ldots, n);$$

(v) $H(0)=0,\ J_H(Y)$ *is symmetric and for all* $Y \in \mathbb{R}^n$ *and* $1 \le i \le n$,

$$\lambda_i \left(\int_0^1 J_H(\sigma Y) d\sigma \right) \ge a_4, \ \|a_4 I - J_H(Y)\| \le \varepsilon_4,$$
$$\lambda_i \left(J_H(Y) - \int_0^1 J_H(\sigma Y) d\sigma \right) \le \frac{a_5 \delta_0}{a_4^2(a_1 a_2 - a_3)};$$

(vi) $\Psi(0)=0,\ J_\Psi(X)$ *is symmetric and for all* $X \in \mathbb{R}^n$,

$$0 \le \lambda_i (a_5 I - J_\Psi(X)) \le \varepsilon_5 \quad (i = 1, 2, \ldots, n);$$

(vii) $J_\Psi(X)$ *commutes with* $J_\Psi(X')$ *for all* $X, X' \in \mathbb{R}^n$ *and*

$$\lambda_i \left(\int_0^1 J_\Psi(\sigma X) d\sigma \right) \ge a_5' > 0 \quad (i = 1, 2, \ldots, n)$$

for all $X \in \mathbb{R}^n$.

Then every solution $X(t)$ *of* (1.1) *satisfies*

$$\|X(t)\| \to 0, \ \|\dot{X}(t)\| \to 0, \ \|\ddot{X}(t)\| \to 0, \ \|\dddot{X}(t)\| \to 0, \ \|X^{(4)}(t)\| \to 0$$

as $t \to \infty$.

In the subsequent discussion we require the following lemmas.

Lemma 1.1[6]. *Let* M *be a real symmetric* $n \times n$ *matrix and*

$$a' \ge \lambda_i(M) \ge a > 0 \quad (i = 1, 2, \ldots, n).$$

Then

$$a'\|X\|^2 \ge \langle MX, X \rangle \ge a\|X\|^2,$$
$$a'^2\|X\|^2 \ge \langle MX, MX \rangle \ge a^2\|X\|^2.$$

Lemma 1.2. *The following statements hold:*

(I) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle \Phi(Z,\sigma W), W\rangle d\sigma \le \langle \Phi(Z,W), U\rangle;$

(II) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle G(\sigma Z), Z\rangle d\sigma = \langle G(Z), W\rangle;$

(III) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma = \langle H(Y), Z\rangle;$

(IV) $\dfrac{d}{dt}\displaystyle\int_0^1 \langle \Psi(\sigma X), X\rangle d\sigma = \langle \Psi(X), Y\rangle.$

Proof. (I). We have

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle \Phi(Z,\sigma W), W\rangle d\sigma &= \int_0^1 \langle \Phi(Z,\sigma W), U\rangle d\sigma \\ &+ \int_0^1 \langle J(\Phi(Z,\sigma W)|Z)W, W\rangle d\sigma \\ &+ \int_0^1 \sigma\langle J(\Phi(Z,\sigma W)|\sigma W)U, W\rangle d\sigma. \qquad (1.8)\end{aligned}$$

Since $J(\Phi|W)$ is symmetric from condition (iii) we have

$$\begin{aligned}\int_0^1 \sigma\langle J(\Phi(Z,\sigma W)|\sigma W)U, W\rangle d\sigma &= \int_0^1 \sigma\langle J(\Phi(Z,\sigma W)|\sigma W)W, U\rangle d\sigma \\ &= \int_0^1 \sigma \frac{\partial}{\partial\sigma}\langle \Phi(Z,\sigma W), U\rangle d\sigma \\ &= \langle \Phi(Z,W), U\rangle - \int_0^1 \langle \Phi(Z,\sigma W), U\rangle d\sigma. \qquad (1.9)\end{aligned}$$

Then we get

$$\begin{aligned}\frac{d}{dt}\int_0^1 \langle \Phi(Z,\sigma W), W\rangle d\sigma &= \langle \Phi(Z,W), U\rangle + \int_0^1 \langle J(\Phi(Z,\sigma W)|Z)W, W\rangle d\sigma \\ &\le \langle \Phi(Z,W), U\rangle,\end{aligned}$$

since $J(\Phi|Z)$ is negative-definite from assumption (iii).

Statements (II), (III) and (IV) can be proved similarly as (I). □

2. The Lyapunov function $V(X,Y,Z,W,U)$

The main tool in the proof of Theorem is the function $V(X,Y,Z,W,U)$ defined for arbitrary X, Y, Z, W and U in $\mathbb{R}^n$ by

$$\begin{aligned}
2V =& \langle U,U\rangle + 2a_1\langle U,W\rangle + \frac{2a_4(a_1a_2-a_3)}{a_1a_4-a_5}\langle U,Z\rangle + 2\delta\langle Y,U\rangle \\
&+ 2\int_0^1 \langle \Phi(Z,\sigma W),W\rangle d\sigma + \left[a_1^2 - \frac{a_4(a_1a_2-a_3)}{a_1a_4-a_5}\right]\langle W,W\rangle \\
&+ 2\left[a_3 + \frac{a_1a_4(a_1a_2-a_3)}{a_1a_4-a_4} - \delta\right]\langle W,Z\rangle + 2\delta a_1\langle W,Y\rangle \\
&+ 2\langle \Psi(X),W\rangle + 2\langle W,H(Y)\rangle + 2a_1\int_0^1 \langle G(\sigma Z),Z\rangle d\sigma \\
&+ \left[\frac{a_2a_4(a_1a_2-a_3)}{a_1a_4-a_5} - a_4 - a_1\delta\right]\langle Z,Z\rangle + 2a_2\delta\langle Y,Z\rangle \\
&+ 2a_1\langle Z,H(Y)\rangle - 2a_5\langle Y,Z\rangle + 2a_1\langle \Psi(X),Z\rangle \\
&+ (\delta a_3 - a_1a_5)\langle Y,Y\rangle + \frac{2a_4(a_1a_2-a_3)}{a_1a_4-a_5}\int_0^1 \langle H(\sigma Y),Y\rangle d\sigma \\
&+ \frac{2a_4(a_1a_2-a_3)}{a_1a_4-a_5}\langle \Psi(X),Y\rangle + 2\delta\int_0^1 \langle \Psi(\sigma X),X\rangle d\sigma, \qquad (2.1)
\end{aligned}$$

where

$$\delta = \frac{a_5(a_1a_2-a_3)}{a_1a_4-a_4} + \varepsilon. \qquad (2.2)$$

The following two lemmas are essential for the actual proof of the Theorem.

Lemma 2.1. *Suppose that the conditions of the Theorem hold. Then the function V satisfies*

$$V(X,Y,Z,W,U) = 0, \quad \text{at } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 = 0, \qquad (2.3)$$

$$V(X,Y,Z,W,U) > 0, \quad \text{if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0, \qquad (2.4)$$

$$V(X,Y,Z,W,U) \to \infty, \quad \text{as } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 \to \infty. \qquad (2.5)$$

Proof. $V(0,0,0,0,0) = 0$, since $\Phi(Z,0) = H(0) = G(0) = \Psi(0) = 0$. By virtue of (1.7) the matrices Γ are symmetric, because $J_H(Y)$ is symmetric. The eigenvalues of Γ are positive because of (v). Consequently the square root $\Gamma^{1/2}$ exists, and this is again symmetric and nonsingular for all $Y \in \mathbb{R}^n$. Therefore the function $V(X,Y,Z,W,U)$ can be rearrangered as follows:

$$\begin{aligned}2V =& \left\|U + a_1 W + \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5} Z + \delta Y\right\|^2 + \frac{a_4\delta_0}{(a_1a_4 - a_5)^2}\left\|Z + \frac{a_5}{a_4}Y\right\|^2 \\ &+ \frac{a_4(a_1a_4 - a_5)}{a_1a_2 - a_3}\left\|\frac{a_1a_2 - a_3}{a_1a_4 - a_5}\Gamma^{-1/2}\Psi(X) + \frac{a_1a_2 - a_3}{a_1a_4 - a_5}\Gamma^{1/2}Y\right. \\ &\left. + \frac{a_1}{a_4}\Gamma^{1/2}Z + \frac{1}{a_4}\Gamma^{1/2}W\right\|^2 + \Delta_2\|W + a_1 Z\|^2 + \sum_{i=1}^{5} V_i \qquad (2.6)\end{aligned}$$

where

$$\begin{aligned}V_1 &= 2\delta\int_0^1 \langle\Psi(\sigma X), X\rangle d\sigma - \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\left\|\Gamma^{-1/2}\Psi(X)\right\|^2, \\ V_2 &= \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\left[2\int_0^1 \langle H(\sigma Y), Y\rangle d\sigma - \langle H(Y), Y\rangle\right] \\ &\quad + \left[\delta a_3 - a_1a_5 - \frac{a_5^2\delta_0}{a_4(a_1a_4 - a_5)} - \delta^2\right]\|Y\|^2, \\ V_3 &= \frac{\varepsilon}{a_1}\|W\|^2 + 2\int_0^1 \langle\Phi(Z, \sigma W), W\rangle d\sigma - a_2\|W\|^2, \\ V_4 &= 2a_1\int_0^1 \langle G(\sigma Z), Z\rangle d\sigma - a_1a_2\|Z\|^2, \\ V_5 &= \frac{2\varepsilon(a_3a_4 - a_2a_5)}{a_1a_4 - a_5}\langle Y, Z\rangle.\end{aligned}$$

The functions V_1, V_2 and V_4 can be estimated as in [6]. In fact the estimates there show that

$$V_1 \geq \varepsilon a_5'\|X\|^2, \quad V_2 \geq \frac{a_5\delta_0}{4a_4(a_1a_4 - a_5)}\|Y\|^2, V_4 \geq 0. \qquad (2.7)$$

Since

$$\frac{\partial}{\partial\sigma}\Phi(Z, \sigma W) = J(\Phi(Z, \sigma W)|\sigma W)W,$$

then

$$\Phi(Z, W) = \int_0^1 J(\Phi(Z, \sigma W)|\sigma W)W d\sigma.$$

Therefore

$$\begin{aligned}V_3 &= \frac{\varepsilon}{a_1}\|W\|^2 + 2\int_0^1 \langle\Phi(Z, \sigma W), W\rangle d\sigma - a_2\|W\|^2 \\ &= \frac{\varepsilon}{a_1}\|W\|^2 + 2\int_0^1\int_0^1 \langle\{J[\Phi(Z, \sigma_1\sigma_2 W)|\sigma_1\sigma_2 W] - a_2 I\}\sigma_2 W, W\rangle d\sigma_1 d\sigma_2 \\ &\geq \frac{\varepsilon}{a_1}\|W\|^2 \qquad (2.8)\end{aligned}$$

by (iii) and Lemma 1.1.

Combining inequalities (2.7) and (2.8) in (2.6) we obtain

$$
\begin{aligned}
2V \geq & \left\| U + a_1 W + \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5} Z + \delta Y \right\|^2 + \frac{a_4\delta_0}{(a_1a_4 - a_5)^2} \left\| Z + \frac{a_5}{a_4} Y \right\|^2 \\
& + \Delta_2 \|W + a_1 Z\|^2 + \varepsilon a_5' \|X\|^2 + \frac{a_5\delta_0}{4a_4(a_1a_4 - a_5)} \|Y\|^2 \\
& + \frac{\varepsilon}{a_1} \|W\|^2 + \frac{2\varepsilon(a_3a_4 - a_2a_5)}{a_1a_4 - a_5} \langle Y, Z \rangle. \qquad (2.9)
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
2V \geq & D_1\|X\|^2 + 2D_2\|Y\|^2 + 2D_3\|Z\|^2 + D_4\|W\|^2 + D_5\|U\|^2 \\
& + \frac{2\varepsilon(a_3a_4 - a_2a_5)}{a_1a_4 - a_5} \langle Y, Z \rangle, \qquad (2.10)
\end{aligned}
$$

for some sufficiently small positive constants $D_i \quad (i = 1, 2, 3, 4, 5)$. Let

$$
V_6 = D_2\|Y\|^2 + \frac{2\varepsilon(a_3a_4 - a_2a_5)}{a_1a_4 - a_5} \langle Y, Z \rangle + D_3\|Z\|^2.
$$

Since by Schwarz's inequality

$$
|\langle Y, Z \rangle| \leq \|Y\|\|Z\| \leq (\|Y\|^2 + \|Z\|^2)/2,
$$

then we get

$$
\begin{aligned}
V_6 & \geq D_2\|Y\|^2 - \frac{\varepsilon(a_3a_4 - a_2a_5)}{a_1a_4 - a_5}(\|Y\|^2 + \|Z\|^2) + D_3\|Z\|^2 \\
& \geq D_6(\|Y\|^2 + \|Z\|^2),
\end{aligned}
$$

for some $D_6 > 0, \quad D_6 = (1/2)\min\{D_2, \ D_3\}$, if

$$
\varepsilon \leq \frac{a_1a_4 - a_5}{2(a_3a_4 - a_2a_5)} \min\{D_2, \ D_3\}.
$$

Consequently

$$
2V \geq D_1\|X\|^2 + (D_2 + D_6)\|Y\|^2 + (D_3 + D_6)\|Z\|^2 + D_4\|W\|^2 + D_5\|U\|^2,
$$

which proves the lemma.

Lemma 2.2. *Let $(X(t), Y(t), Z(t), W(t), U(t))$ be any solution of (1.3). We define $\vartheta(t) = V(X(t), Y(t), Z(t), W(t), U(t))$. Then*

$$\dot{\vartheta}(t) \leq 0 \quad \textit{for all} \;\; t \leq 0. \tag{2.11}$$

In particular

$$\frac{d}{dt}\vartheta(t) < 0, \quad \textit{whenever}\; \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0. \tag{2.12}$$

Proof. Starting from (2.1) we obtain by applying Lemma 1.2

$$\begin{aligned}
\frac{d}{dt}\vartheta(t) \leq & - [\langle F(Z,W)U, U\rangle - a_1\langle U, U\rangle] \\
& - a_1\langle \Phi(Z,W), W\rangle - \left[a_3 + \frac{a_1a_4(a_1a_2 - a_3)}{a_1a_4 - a_5} - \delta\right]\langle W, W\rangle \\
& - \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\langle G(Z), Z\rangle \\
& + [a_2\delta\langle Z, Z\rangle + a_1\langle J_H(Y)Z, Z\rangle - a_5\langle Z, Z\rangle] \\
& - \left[\delta\langle Y, H(Y)\rangle - \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\langle J_\Psi(X)Y, Y\rangle\right] - a_1\langle F(Z,W)U, W\rangle \\
& + a_1^2\langle U, W\rangle - [\langle G(Z), U\rangle - a_3\langle Z, U\rangle] \\
& - \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\langle F(Z,W)U, Z\rangle \\
& + \frac{a_1a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}\langle U, Z\rangle - \delta\langle F(Z,W)Y, U\rangle + \delta a_1\langle Y, U\rangle \\
& - \frac{a_4(a_1a_2 - a_3)}{a_1a_4 - a_5}[\langle \Phi(Z,W), Z\rangle - a_2\langle W, Z\rangle] \\
& - [a_4\langle W, Z\rangle - \langle W, J_H(Y)Z\rangle] \\
& - a_5\langle Y, W\rangle + \langle J_\Psi(X)Y, W\rangle - \delta[\langle G(Z), Y\rangle - a_3\langle Y, Z\rangle] \\
& - \delta[\Phi(Z,W), Y\rangle - a_2\langle W, Y\rangle] \\
& - a_1a_5\langle Y, Z\rangle + a_1\langle J_\Psi(X)Y, Z\rangle.
\end{aligned} \tag{2.13}$$

By using (ii) and Lemma 1.1 we find

$$\langle F(Z,W)U, U\rangle - a_1\langle U, U\rangle \geq \varepsilon_0\|U\|^2. \tag{2.14}$$

We get also from (2.2), (iii) and Lemma 1.1

$$\begin{aligned}
&a_1\langle \Phi(Z,W),W\rangle - \left[a_3 + \frac{a_1a_4(a_1a_2-a_3)}{a_1a_4-a_5} - \delta\right]\langle W,W\rangle \\
&= a_1\int_0^1 \langle [J(\Phi(Z,\sigma W)|\sigma W) - a_2 I]\, W, W\rangle d\sigma \\
&\quad + \left[a_1a_2 - a_3 + \delta - \frac{a_1a_4(a_1a_2-a_3)}{a_1a_4-a_5}\right]\langle W,W\rangle \\
&\geq \varepsilon\|W\|^2. \qquad (2.15)
\end{aligned}$$

By using techniques similar to those used by Sadek [6] and (2.13)- (2.15) it can be seen that $\dot{\vartheta}(t) \leq 0$ for all $t \geq 0$ and in particular

$$\frac{d}{dt}\vartheta(t) < 0 \quad \text{whenever } \|X\|^2 + \|Y\|^2 + \|Z\|^2 + \|W\|^2 + \|U\|^2 > 0$$

which proves the lemma.

3. Completion of the proof

The usual Barabashin-type arguments, Theorem 1.5 in [4], applied to (2.3)–(2.5), (2.11) and (2.12) would then show that for any solution $(X(t), Y(t), Z(t), W(t), U(t))$ of (1.3) we have

$$\|X(t)\| \to 0,\ \|Y(t)\| \to 0,\ \|Z(t)\| \to 0,\ \|W(t)\| \to 0,\ \|U(t)\| \to 0$$

as $t \to \infty$, which are equivalent to

$$\|X(t)\| \to 0,\ \|\dot{X}(t)\| \to 0,\ \|\ddot{X}(t)\| \to 0,\ \|\dddot{X}(t)\| \to 0,\ \|X^{(4)}(t)\| \to 0$$

as $t \to \infty$. This completes the proof of the Theorem.

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