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A unified approach to some Tauberian theorems of Hardy and Littlewood

HUBERT TIETZ AND KARL ZELLER

1. Introduction

Let $a_1 + a_2 + \ldots$ be a series with real terms and the partial sums $s_n := a_1 + \ldots + a_n$. If

$$a(x) := \sum a_n x^n = (1-x) \sum s_n x^n =: (1-x)s(x) \quad (0 \le x < 1)$$

converges and $a(x) = (1-x)s(x) = o(1)$ as $x \to 1-$, then we write A-$\sum a_n = 0$ or A-lim $s_n = 0$. If $s_1 + \ldots + s_n = o(n)$ as $n \to \infty$, then we write C_1-lim $s_n = 0$. The following two Tauberian theorems for (the *Abel method*) A are due to Hardy and Littlewood [2, Theorems 11 and 9], their shortest proofs were found by Wielandt [7] respective Karamata [3].

Theorem HL1. A-$\sum a_n = 0 \wedge na_n \le 1 \Longrightarrow \sum a_n = 0$.

Theorem HL2. *A-lim* $s_n = 0 \wedge s_n \le 1 \Longrightarrow C_1$*-lim* $s_n = 0$.

As to the extensive literature on these results see [1], [4], [8] and [6]. In Section 3 of this paper we modify Wielandt's proof of Theorem HL1, and in Section 4 we prove Theorem HL2 in a very similar way. We also use the Weierstrass Approximation Theorem, but we avoid integrals, so our proofs are a little more direct.

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2. Preliminaries

Let $\alpha := 1/2$. We write $\sum$, lim and limsup instead of

$$\sum_{n=1}^{\infty}, \lim_{x\to 1-} \text{ and } \limsup_{x\to 1-}.$$

Let $\mathcal{F}$ be the set of all functions $f : [0,1] \to \mathbb{R}$. Members of $\mathcal{F}$ which are most important for us are the *identity* j, the *step* φ with $\varphi(x) := 0$ for $0 \le x < \alpha$ and $\varphi(x) := 1$ for $\alpha \le x \le 1$, and, with a fixed $\varepsilon \in (0, \alpha)$, the *ramps* $r_1, r_2 \in C[0,1]$ with

$$r_1 \begin{cases} := \varphi \text{ on } [0,\alpha) \cup [\alpha+\varepsilon, 1] \\ \text{linear on } [\alpha, \alpha+\varepsilon) \end{cases}, \quad r_2 \begin{cases} := \varphi \text{ on } [0,\alpha-\varepsilon) \cup [\alpha, 1] \\ \text{linear on } [\alpha-\varepsilon, \alpha). \end{cases}$$

We have $r_1 \le \varphi \le r_2$.

Lemma 2.1. *Let* $0 \le \beta < \gamma \le 1$ *and* $0 \le g \in \mathcal{F}$.

a) *If* $g \le j$ *on* $[\beta, \gamma)$ *and* $g := 0$ *otherwise, then*

$$(1-x)\sum g(x^n) \le \gamma - \beta x \quad (0 < x < 1).$$

b) *If* $g \le j(1-j)$ *on* $[\beta,\gamma)$ *and* $g := 0$ *otherwise, then*

$$\sum n^{-1} g(x^n) \le \gamma - \beta x \quad (0 < x < 1).$$

Proof. Let $x \in (0,1)$ be fixed. If $\beta > 0$, then we define $k, l \in \mathbb{N}$ by $x^k < \gamma \le x^{k-1}$ and $x^l < \beta \le x^{l-1}$. It follows

a) $(1-x)\sum g(x^n) \le (1-x)\sum_{n=k}^{l-1} x^n = x^k - x^l < \gamma - \beta x$,

b) $\sum n^{-1} g(x^n) \le \sum_{n=k}^{l-1} n^{-1} x^n (1-x^n) \le (1-x)\sum_{n=k}^{l-1} x^n < \gamma - \beta x$.

If $\beta = 0$, then we obtain the same inequalities with k as above and $l := \infty$.

3. Proof of Theorem HL1

Assume $A\text{-}\sum a_n = 0$ and $na_n \le 1$, and let $\mathcal{W}$ be the set of all functions $f \in \mathcal{F}$ such that $\sum a_n f(x^n)$ converges for $0 \le x < 1$ and $\lim \sum a_n f(x^n) = 0$. To prove Theorem HL1 it suffices to show $\varphi \in \mathcal{W}$. Since $A\text{-}\sum a_n = 0$, it follows $jp \in \mathcal{W}$ for each polynomial p. Next we show that

$$h \in C[0,1] \Longrightarrow f := j(1-j)h \in \mathcal{W}.$$

Let $\varepsilon \in (0,\alpha)$ and choose polynomials p_1, p_2 such that $p_1 \leq h \leq p_2$ and $p_2 - p_1 \leq \varepsilon$. Then $0 \leq f - j(1-j)p_1 \leq \varepsilon j(1-j)$and $0 \leq j(1-j)p_2 - f \leq \varepsilon j(1-j)$. Using Lemma 2.1.b) with $\beta = 0$ and $\gamma = 1$, we see that

$$\sum a_n(f - j(1-j)p_1)(x^n) \;\leq\; \sum n^{-1}(f - j(1-j)p_1)(x^n) \;\leq\; \varepsilon,$$
$$\sum a_n(j(1-j)p_2 - f)(x^n) \;\leq\; \sum n^{-1}(j(1-j)p_2 - f)(x^n) \;\leq\; \varepsilon.$$

From this and $j(1-j)p_1, j(1-j)p_2 \in \mathcal{W}$ we get $\limsup |\sum a_n f(x^n)| \leq \varepsilon$, and so $f \in \mathcal{W}$. For example $r_1 - j, r_2 - j \in \mathcal{W}$, hence $r_1, r_2 \in \mathcal{W}$. We further have

$$\varphi - r_1 \begin{cases} \leq 4j(1-j) \text{ on } [\alpha, \alpha+\varepsilon) \\ = 0 \text{ otherwise} \end{cases}, \quad r_2 - \varphi \begin{cases} \leq 4j(1-j) \text{ on } [\alpha-\varepsilon, \alpha), \\ = 0 \text{ otherwise} \end{cases}$$

and thus, again by Lemma 2.1.b), we get

$$\sum a_n(\varphi - r_1)(x^n) \leq 4(\alpha + \varepsilon - \alpha x), \quad \sum a_n(r_2 - \varphi)(x^n) \leq 4(\alpha - (\alpha - \varepsilon)x).$$

From this and $r_1, r_2 \in \mathcal{W}$ we obtain $\limsup \mid \sum a_n \varphi(x^n) \mid \leq 4\varepsilon$, and so $\varphi \in \mathcal{W}$.

4. Proof of Theorem HL2

Assume A-$\lim s_n = 0$ and $s_n \leq 1$, and let $\mathcal{K}$ be the set of all functions $f \in \mathcal{F}$ such that $\sum s_n f(x^n)$ converges for $0 \leq x < 1$ and $\lim(1-x) \cdot \cdot \sum s_n f(x^n) = 0$. To prove Theorem HL2 it suffices to show $\varphi \in \mathcal{K}$ because $\lim(1-x)/(-\ln x) = 1$. Since A-$\lim s_n = 0$, it follows $jp \in \mathcal{K}$ for each polynomial p. Next we show that

$$h \in C[0,1] \Longrightarrow f := jh \in \mathcal{K}.$$

Let $\varepsilon \in (0,\alpha)$ and choose polynomials p_1, p_2 such that $p_1 \leq h \leq p_2$ and $p_2 - p_1 \leq \varepsilon$. Then $0 \leq f - jp_1 \leq \varepsilon j$ and $0 \leq jp_2 - f \leq \varepsilon j$. Using Lemma 2.1.a) with $\beta = 0$ and $\gamma = 1$, we see that

$$(1-x)\sum s_n(f - jp_1)(x^n) \;\leq\; (1-x)\sum (f - jp_1)(x^n) \;\leq\; \varepsilon,$$
$$(1-x)\sum s_n(jp_2 - f)(x^n) \;\leq\; (1-x)\sum (jp_2 - f)(x^n) \;\leq\; \varepsilon.$$

From this and $jp_1, jp_2 \in \mathcal{K}$ we get $\limsup(1-x)|\sum s_n f(x^n)| \leq \varepsilon$, and so $f \in \mathcal{K}$. For example $r_1, r_2 \in \mathcal{K}$. We further have

$$\varphi - r_1 \begin{cases} \leq 2j \text{ on } [\alpha, \alpha+\varepsilon) \\ = 0 \text{ otherwise} \end{cases}, \quad r_2 - \varphi \begin{cases} \leq 2j \text{ on } [\alpha-\varepsilon, \alpha) \\ = 0 \text{ otherwise} \end{cases}$$

and thus, again by Lemma 2.1.a), we get

$$(1-x)\sum s_n(\varphi - r_1)(x^n) \ \le \ 2(\alpha + \varepsilon - \alpha x),$$
$$(1-x)\sum s_n(r_2 - \varphi)(x^n) \ \le \ 2(\alpha - (\alpha - \varepsilon)x).$$

From this and $r_1, r_2 \in \mathcal{K}$ we obtain $\limsup (1-x) \mid \sum s_n \varphi(x^n)| \le 2\varepsilon$, and so $\varphi \in \mathcal{K}$.

5. Concluding Remarks

The following theorems of Littlewood [5] for series with complex terms are easy consequences of Theorems HL1 and HL2.

Theorem L1. $A\text{-}\sum a_n = 0 \wedge n|a_n| \le 1 \Longrightarrow \sum a_n = 0.$

Theorem L2. $A\text{-}\lim s_n = 0 \wedge |s_n| \le 1 \Longrightarrow C_1\text{-}\lim s_n = 0.$

These theorems can be proved along the same lines as Theorems HL1 and HL2, but their proofs are shorter because one needs only one of the ramps r_1, r_2.

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Mathematisches Institut A, Universität Stuttgart, D-70569 Stuttgart, Germany
E-mail address: tietz@mathematik.uni-stuttgart.de

Sonnenstrasse 11, D-72076 Tübingen, Germany