ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 2, 1998

On strong almost convergence and uniform statistical convergence

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ABSTRACT. The definition of strong almost convergence is extended to a definition of strong almost convergence with respect to an Orlicz function. It is shown that every bounded uniformly statistically null sequence is also strongly almost convergent to zero with respect to any Orlicz function.

1. Introduction.

By ℓ_∞ we denote the Banach space of all bounded sequences of complex numbers with the usual supremum norm. The notion of almost convergence was introduced by Lorentz [4] and it has been studied e.g. in [2, 5, 10]. By c, f, and $[f]$ we denote the subspaces of ℓ_∞ consisting of convergent, almost convergent, and strongly almost convergent sequences, respectively.

Recall that x is almost convergent to a number s if and only if

$$\frac{1}{n+1}\sum_{i=0}^{n} x_{i+k} \to s \quad (n \to \infty, \text{uniformly in } k),$$

and $x \in [f]$ if and only if there exists a number s such that

$$\frac{1}{n+1}\sum_{i=0}^{n} |x_{i+k} - s| \to 0 \quad (n \to \infty, \text{uniformly in } k).$$

Received November 23, 1997; revised May 30, 1998.

1991 *Mathematics Subject Classification.* 40A05, 40F05,40G99.

Key words and phrases. Strong almost convergence, Orlicz function, uniform statistical null convergence.

This research was supported by the Scientific and Technical Research Council of Turkey (TBAG-AY64).

By $[f_0]$ we denote the set of sequences which are strongly almost convergent to zero. It is obvious that

$$[f_0] \subset [f] \subset f \subset \ell_\infty.$$

Fast [1] introduced the idea of statistical convergence, which is closely related to the concept of natural density or asymtotic density of subsets of $\mathbb{N} = \{0, 1, 2, ...\}$. Zygmund [10, p. 181] investigated a relation between this concept and strong summability. In [3], [8], and [9] these ideas were studied. If A is a subset of $\mathbb{N}$, let A_n denote the set $\{i \in A : i \leq n\}$ and let $|A_n|$ denote the cardinality of A_n. The *natural density* of A is given by $\delta(A) = \lim_n (n+1)^{-1}|A_n|$. A sequence x is statistically convergent to s provided that for every $\varepsilon > 0$ the set $A(\varepsilon) = \{i \in \mathbb{N} : |x_i - s| \geq \varepsilon\}$ has natural density zero. The set of all statistically convergent sequences is denoted by S. By S_0 we denote the set of all sequences which are statistically convergent to zero.

We now give the notion of uniformly statistically null sequences [8]: *A number sequence x is uniformly statistically convergent to 0 provided that for each $\varepsilon > 0$,*

$$\lim_{n\to\infty} (n+1)^{-1} \max_{k\geq 1} |\{0 \leq i \leq n : |x_{i+k}| \geq \varepsilon\}| = 0.$$

The set of all uniformly statistically null sequences is denoted by S_{u_o}.

It is easy to see that $S_{u_o} \subset S_0$. Note that this inclusion is strict. Indeed, define x by $x_i = 1$ if $2^n \leq i \leq 2^n + n - 1$ for $n = 1, 2, ...$ and $x_i = 0$ otherwise. Then we have $x \in S_0$ but $x \notin S_{u_o}$.

Recently Parashar and Choudhary [6] introduced a generalization of the classical spaces of strongly summably sequences by defining certain sequence spaces using an Orlicz function.

We recall that an Orlicz function M is a function from $[0, \infty)$ to $[0, \infty)$ such that it is continuous and convex with $M(0) = 0, M(t) > 0$ for $t > 0$, and $M(t) \to \infty$ as $t \to \infty$. Every Orlicz function M can be represented by the integral form

$$M(t) = \int_0^t p(s)ds$$

where p is the right-derivative of M. Recall that the function p is non-negative and non-decreasing.

2. Strong almost convergence with respect to an Orlicz function

We now introduce generalizations of the classical spaces of strongly almost summable sequences with the help of an Orlicz function M.

Let M be an Orlicz function. Then we define

$$[f_0(M)] = \{x = (x_i) : \lim_{n\to\infty} t_{n,k}(\frac{|x|}{\rho}) = 0 \text{ uniformly in } k, \text{ for some } \rho > 0\},$$

$$[f(M)] = \{x = (x_i) : \ x - se \in [f_0(M)] \text{ for some } s \in \mathbb{C}\},$$

where $e = (1, 1...)$ and $t_{n,k}(\frac{|x|}{\rho}) = (n+1)^{-1}\sum_{i=0}^{n} M(\frac{|x_{i+k}|}{\rho})$.

For comparison, let us recall that the Orlicz sequence space l_M is defined as follows:

$$l_M = \{x = (x_i) : \sum_{i=0}^{\infty} M(\frac{|x_i|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

It is easily verified that $[f_0(M)]$ and $f(M)]$ are linear spaces. If $M(t)$ is equivalent to t, then $[f_0(M)] = [f_0]$ and $[f(M)] = [f]$. Recall that then $\ell_M = \ell_1$. This is the case when $p(0) > 0$, because

$$p(0) = \lim_{t\to 0+} \frac{M(t)}{t} > 0$$

implies that $M(t)$ is equivalent to t.

The relation between the space of sequences almost strongly convergent to zero with respect to an Orlicz function and S_{u_0} is given in the following

Theorem 1. *Let M be an Orlicz function. Then $[f_0(M)] \subset S_{u_0}$.*

Proof. Let $x \in [f_0(M)]$ and $\varepsilon > 0$. Choose $\varepsilon' = \varepsilon\rho$. Then we have for every k,

$$t_{n,k}(\frac{|x|}{\rho}) = (n+1)^{-1}\sum_{i=0}^{n} M(\frac{|x_{i+k}|}{\rho}) \geq (n+1)^{-1}M(\varepsilon)|A(\varepsilon')|,$$

where $|A(\varepsilon')|$ denotes the cardinality of the set $A(\varepsilon') = \{0 \leq i \leq n : |x_{i+k}| \geq \varepsilon'\}$. Thus we have $x \in S_{u_0}$. $\square$

Theorem 2. *Let M be an Orlicz function. Then $S_{u_0} \cap \ell_\infty \subset [f_0(M)]$.*

Proof. Suppose that $x \in S_{u_0} \cap \ell_\infty$. Since x is bounded, we have for every i and k, $|x_{i+k}| \leq \| x \| = \sup_i |x_i|$. For $\varepsilon > 0$, we get

$$t_{n,k}(|x|) = (n+1)^{-1} \sum_{\substack{i=0 \\ |x_{i+k}|\geq\varepsilon}}^{n} M(|x_{i+k}|) + (n+1)^{-1} \sum_{\substack{i=0 \\ |x_{i+k}|<\varepsilon}}^{n} M(|x_{i+k}|)$$

$$\leq (n+1)^{-1} M(\| x \|) \max_{k \geq 0} |\{0 \geq i \geq n : \ |x_{i+k}| \geq \varepsilon\}| + M(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we conclude that $x \in [f_0(M)]$. □

We remark that the factor ℓ_∞ cannot be omitted in Theorem 2. To see this, consider the sequence x defined by $x_i = n$ if $i = n^2$ for $n \in \mathbb{N}$ and $x_i = 0$ otherwise. Then x is not bounded. On the other hand, we have

$$n^{-1} \max_{k \geq 1} |\{0 \leq i \leq n : \ |x_{i+k}| \geq \varepsilon\}| \leq \frac{\sqrt{n}}{n}$$

as $n \to \infty$. Hence $x \in S_{u_0}$. If we define an Orlicz function with $M(t) = t$ then $x \notin [f_0(M)]$.

Combining Theorem 1 and Theorem 2, we have already proved the following

Corollary 3. *Let M be an Orlicz function. Then $S_{u_0} \cap \ell_\infty = [f_0(M)] \cap \ell_\infty$.*

Acknowledgement

The author would like to express her thanks to the referee for several valuable comments and suggestions.

References

1. H. Fast, *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.
2. A. R. Freedman, J. J. Sember and M. Raphael, *Some Cesàro-type summability spaces*, Proc. London Math. Soc. **37**(3) (1978), 508–520.
3. J. A. Fridy, *On statistical convergence*, Analysis **5** (1985), 301–313.
4. G. G. Lorentz, *A contribution to the theory of divergent sequences*, Acta Math. **80** (1948), 167–190.
5. I. J. Maddox, *A new type of convergence*, Math Proc. Cambridge Philos. Soc. **83** (1978), 61–64.
6. S. D. Parashar and B. Choudhary, *Sequence spaces defined by Orlicz functions*, Indian J. Pure Appl. Math. **25**(4) (1994), 419–428.
7. S. Pehlivan and B. Fisher, *Some sequence spaces defined by a modulus*, Math. Slovaca **45**(3) (1995), 275-280.
8. T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca **30**(2) (1980), 139–150.
9. I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.
10. A. Zygmund, *Trigonometric Series*, Vol. II, Cambridge University Press, 1968.

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