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Factors for absolute Riesz summability methods

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ABSTRACT. In this paper we prove a theorem on $|\overline{N}, p_n; \delta|_k$ summability factors, which extends a theorem of Bor [3] on $|\overline{N}, p_n|_k$ summability factors.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n) and let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty \quad (P_{-i} = p_{-i} = 0, \ i \geq 1).$$

The sequence-to-sequence transformation

$$u_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequences (u_n) of the Riesz means or simply the $(\overline{N}, p_n)$ means, of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [4]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{k-1} |u_n - u_{n-1}|^k < \infty$$

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and it is said to be summable $|\overline{N}, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [2])

$$\sum_{n=1}^{\infty} (\frac{P_n}{p_n})^{\delta k+k-1} |u_n - u_{n-1}|^k < \infty.$$

In this special case when $\delta = 0$ (respectively, $k = 1$ and $\delta = 0$) $|\overline{N}, p_n; \delta|_k$ summability is the same as $|\overline{N}, p_n|_k$ (respectively $|\overline{N}, p_n|$) summability. Also if we take $p_n = 1$ for all values of n, $|\overline{N}, p_n; \delta|_k$ summability reduces to $|C, 1; \delta|_k$ summability.

If we write

$$X_n = \sum_{v=0}^{n} \frac{p_v}{P_v},$$

then $X_n \to \infty$ as $n \to \infty$.

Quite recently Bor [3] proved the following theorem.

Theorem A. *Let (p_n) be a sequence of positive numbers such that $P_n = O(np_n)$. Let $t_n = 1/(n+1) \sum_{v=1}^{n} v a_v$. If $\lambda_n \to 0$ as $n \to \infty$,*

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \text{ as } m \to \infty \tag{1.1}$$

and

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \text{ as } m \to \infty, \tag{1.2}$$

then the series $\sum a_n \lambda_n$ is summable $|\overline{N}, p_n|_k$, $k \geq 1$, where $\Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1}$ and $\Delta \lambda_n = \lambda_n - \lambda_{n+1}$.

2. The main result

The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n; \delta|_k$ summability in the form of the following theorem.

Theorem. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let the sequences (p_n) and (λ_n) such that conditions of Theorem A are satisfied with the condition* (1.2) *replaced by*

$$\sum_{v=1}^{n} (\frac{P_v}{p_v})^{\delta k-1} |t_v|^k = O(X_n) \text{ as } n \to \infty. \tag{2.1}$$

If

$$\sum_{n=v+1}^{\infty} (\frac{P_n}{p_n})^{\delta k-1} \frac{1}{P_{n-1}} = O\left\{ (\frac{P_v}{p_v})^{\delta k} \frac{1}{P_v} \right\}, \tag{2.2}$$

then the series $\sum a_n\lambda_n$ *is summable* $|\overline{N},p_n;\delta|_k$.

If we take $\delta=0$ in this theorem then we obtain Theorem A.

Remark. It should be noted that if we take $\delta = 0$ in our theorem, then condition (2.2) is superfluous. Because in this case the condition (2.2) reduces to

$$\sum_{n=v+1}^{\infty}\frac{p_n}{P_nP_{n-1}}=O(\frac{1}{P_v}),$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma ([3]). *If the condition* (1.1) *is satisfied, then*

$$nX_n|\Delta\lambda_n|=O(1) \text{ as } n\to\infty,$$

$$\sum_{n=1}^{\infty}X_n|\Delta\lambda_n|<\infty, \tag{2.3}$$

$$X_n|\lambda_n|=O(1) \text{ as } n\to\infty. \tag{2.4}$$

3. Proof of the Theorem

Let (T_n) be the $(\overline{N},p_n)$ means of the series $\sum a_n\lambda_n$. Then by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{r=0}^{v}a_r\lambda_r=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v.$$

Hence, for $n\geq 1$, we get

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v.$$

Applying Abel's transformation, we have

$$\begin{aligned}T_n-T_{n-1}&=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\Delta(\frac{P_{v-1}\lambda_v}{v})\sum_{r=1}^{v}ra_r+\frac{p_n\lambda_n}{nP_n}\sum_{v=1}^{n}va_v\\&=-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_v\lambda_v\frac{v+1}{v}t_v+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_v\frac{v+1}{v}t_v\\&+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\lambda_{v+1}\frac{1}{v}t_v+\frac{(n+1)p_n\lambda_nt_n}{nP_n}\\&=T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}.\end{aligned}$$

Since

$$|T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}|^k \le 4^k(|T_{n,1}|^k+|T_{n,2}|^k+|T_{n,3}|^k+|T_{n,4}|^k),$$

to complete the proof of the Theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty}(\frac{P_n}{p_n})^{\delta k+k-1}|T_{n,r}|^k<\infty, \text{ for } r=1,\ 2,\ 3,\ 4.$$

We shall prove this only for $r=1$, the proof for $r=2,\ 3,\ 4$ is similar.

Using Hölder's inequality we have that

$$\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k+k-1}|T_{n,1}|^k \le \sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}p_v|\lambda_v|\frac{v+1}{v}|t_v|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v|\lambda_v||t_v|^k\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}$$

$$=O(1)\sum_{v=1}^{m}p_v|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}$$

$$=O(1)\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k-1}|t_v|^k|\lambda_v|$$

$$=O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k-1}|t_r|^k+O(1)|\lambda_m|\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k-1}|t_v|^k$$

$$=O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m$$

$$=O(1)$$

as $m\to\infty$, by (2.1) - (2.4).

If we take $p_n=1$ for all values of n in Theorem, then we get a result related to $|C,1;\delta|_k$ summability factors.

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