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A proof of the Simons inequality

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Let S be a set and let $\ell_\infty(S)$ denote the metric space of all bounded real functions on S. For a sequence of functions $x_n = x_n(s)$, $s \in S$, its convex hull is denoted by $\operatorname{conv}\{x_n\}_{n=1}^\infty$, that is

$$\operatorname{conv}\{x_n\}_{n=1}^\infty = \{\sum_{n=1}^m \lambda_n x_n \colon m \in \mathbb{N},\ \lambda_n \geq 0,\ \sum_{n=1}^m \lambda_n = 1\}.$$

The following result of S. Simons [2, Lemma 2] (cf. also [1, p. 49]) is important in real analysis and geometry of Banach spaces (see e.g. [1, Chapter 3], [2], [3]).

Simons Inequality. *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in $\ell_\infty(S)$. Let $T \subset S$ be such that, for every $\lambda_n \geq 0$ with $\sum_{n=1}^\infty \lambda_n = 1$, there exists $t \in T$ satisfying*

$$\sum_{n=1}^\infty \lambda_n x_n(t) = \sup_{s \in S} \sum_{n=1}^\infty \lambda_n x_n(s).$$

Then

$$\inf\{\sup_{s\in S} x(s) \colon x \in \operatorname{conv}\{x_n\}_{n=1}^\infty\} \leq \sup_{t \in T} \limsup_n x_n(t).$$

In this note, we shall give a simple direct proof of the Simons inequality. In fact, the main formula which will be used in the proof below is

$$2^n = \sum_{k=0}^{n-1} 2^k + 1.$$

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Proof of the Simons inequality. Denote $\sigma(x) = \sup_{s\in S} x(s)$, $x \in \ell_\infty(S)$, and $C_k = \{\sum_{n=k}^\infty \lambda_n x_n \colon \lambda_n \geq 0,\ \sum_{n=k}^\infty \lambda_n = 1\}$, $k \in \mathbb{N}$. As $\inf\{\sigma(x)\colon x \in A\} = \inf\{\sigma(x)\colon x \in \overline{A}\}$ for any set $A \subset \ell_\infty(S)$ (where $\overline{A}$ denotes the closure of A), it is equivalent to prove that

$$\inf_{x\in C_1} \sigma(x) \leq \sup_{t\in T} \limsup_n x_n(t) =: \sigma_T. \tag{1}$$

To show (1), it clearly suffices to prove that, for any $\varepsilon > 0$, there exist $v \in C_1$, $y_m \in C_{m+1}$ (for $m \in \mathbb{N}$), and $t \in T$ so that

$$\sigma(v) - \varepsilon \leq y_m(t) \quad \forall m \in \mathbb{N}. \tag{2}$$

[In fact, by (2),

$$\inf_{x\in C_1} \sigma(x) - \varepsilon \leq \sigma(v) - \varepsilon \leq \limsup_m y_m(t) \leq \limsup_n x_n(t) \leq \sigma_T,$$

and inequality (1) follows because $\varepsilon > 0$ is arbitrary.]

Let $\varepsilon > 0$. Since C_k is a bounded set,

$$\inf_{z\in C_k} \sigma(x+z) > -\infty \quad \forall x \in \ell_\infty(S), \quad \forall k \in \mathbb{N}.$$

Choose inductively $z_1 \in C_1$, $z_2 \in C_2, \ldots$ so that, for $k = 0, 1, \ldots$,

$$\sigma(2^k v_k + z_{k+1}) \leq \inf_{z\in C_{k+1}} \sigma(2^k v_k + z) + \frac{\varepsilon}{2^{k+1}}$$

where $v_0 = 0$ and $v_k = \sum_{n=1}^k z_n/2^n$. Then put $v = \sum_{n=1}^\infty z_n/2^n$. Since $2^k v_k + z_{k+1} = 2^{k+1} v_{k+1} - 2^k v_k$ (because $v_{k+1} - v_k = z_{k+1}/2^{k+1}$) and $y_k := 2^k v - 2^k v_k = \sum_{n=k+1}^\infty 2^k z_n/2^n \in C_{k+1}$, we have, for $k = 0, 1, \ldots$,

$$\sigma(2^{k+1} v_{k+1} - 2^k v_k) \leq \sigma(2^k v) + \frac{\varepsilon}{2^{k+1}} = 2^k \sigma(v) + \frac{\varepsilon}{2^{k+1}}. \tag{3}$$

Since $v \in C_1$, there exists $t \in T$ satisfying $v(t) = \sigma(v)$. From (3) (note that $\sum_{k=0}^{m-1} 2^k = 2^m - 1$), we immediately get that, for any $m \in \mathbb{N}$,

$$2^m v_m(t) = \sum_{k=0}^{m-1} (2^{k+1} v_{k+1} - 2^k v_k)(t) \leq (2^m - 1)\sigma(v) + \varepsilon = 2^m v(t) - \sigma(v) + \varepsilon.$$

This means that (2) holds. □

References

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