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On sequence spaces defined by a sequence of moduli and an extension of Kuttner's theorem

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ABSTRACT. Let $A = (a_{nk})$ be an infinite matrix with $\lim_n \sum_k a_{nk} \neq 0$, and let $p = (p_k)$ be a sequence of positive real numbers. If $1 \leq p_k < H < \infty$, then the strong A-summability field (with the exponent p) is included in the summability field of A. But the result known as Kuttner's theorem asserts that if $0 < p_k = \tilde{p} < 1$ and A is regular, then there is a sequence which is strongly $(C,1)$-summable but which is not A-summable. This result was extended by Thorpe (cf. Theorem 6) and Maddox (cf. [8] and [9]). The purpose of the present paper is to extend the results of Thorpe and Maddox to a lacunary strong summability with respect to a sequence of modulus functions.

1. Introduction

A function $f: [0,\infty) \to [0,\infty)$ is called a modulus if

(i) $f(t) = 0 \Leftrightarrow t = 0$,
(ii) $f(t+u) \leq f(t) + f(u)$ for all $t \geq 0, u \geq 0$,
(iii) f is increasing,
(iv) f is continuous from the right of 0.

For a sequence space X and for a modulus f, Ruckle [12] and Maddox [9] considered a new sequence space:

$$X(f) = \{x = (x_k) |\ (f(|x_k|)) \in X\}.$$

An extension of this definition was given by Kolk [3]. For a sequence of moduli $F = (f_k)$, he defined:

$$X(F) = \{x = (x_k) |\ F(x) = (f_k(|x_k|)) \in X\}.$$

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A sequence space X is called solid (or normal) if from $(y_k) \in X$ and $|x_k| < |y_k|$, it follows that $(x_k) \in X$.

A real function g on the linear space X is called an F-norm if

(*i*) $g(0) = 0$,

(*ii*) $|\alpha| \leq 1$ $(\alpha \in \mathbb{K}) \Rightarrow g(\alpha x) \leq g(x)$,

(*iii*) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in X$,

(*iv*) $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$, $x \in X \Rightarrow \lim_n g(\alpha_n x) = 0$.

An F-space is defined as a complete F-normed space. If a sequence space X is an F-space on which the coordinate functionals $\pi_k(x) = x_k$ are continuous, then X is called an FK-space. An FK-space with normable topology is called a BK-space.

Let ϕ be the space of all finite sequences. An F-space X containing ϕ is called an AK-space if $\lim_n \sum_{k=1}^{n} x_k e_k = x$ for all $x \in X$, where $e_k = (\delta_{ki})_{i \in \mathbb{N}}$.

An F-norm g in a sequence space X is called absolutely monotone if $|x_k| \leq |y_k|$ implies $g(x) \leq g(y)$ for all $x = (x_k)$, $y = (y_k)$ in X.

A topologization of $X(F)$ is given by Kolk:

Theorem 1 (cf. [4]). *If X is a solid AK-FK-space with an absolutely monotone F-norm g, then $F(X)$ is a solid AK-FK-space with the absolutely monotone F-norm g_F, where*

$$g_F(x) = g(F(x)).$$

Define for $p = (p_k)$, $p_k > 0$,

$$w_o(p) = \{x = (x_k) | \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |x_k|^{p_k} = 0\},$$

i.e. $w_0(p)$ is the space of strongly (C,1)-summable (with exponent p) to zero sequences. If $p_k = 1$, $k \in \mathbb{N}$, we denote $w_0(p) = w_0$.

For a sequence of moduli $F = (f_k)$, define:

$$w_o(F) = \{x = (x_k) | \lim_n \frac{1}{n+1} \sum_{k=0}^{n} f_k(|x_k| = 0\}.$$

In the special case where $f_k(x) = x^{p_k}$, $0 < p_k < 1$, we have that $w_0(F) = w_0(p)$. In [2], it was shown that the strong (C,1)-summability is equivalent to a summability determined by some lacunary sequence.

An increasing sequence $\Theta = (k_r)$ of non-negative integers is called a lacunary sequence if $k_0 = 0$ and $\lim_r (k_{r+1} - k_r) = \infty$. We denote

$$h_r = (k_{r+1} - k_r), \quad q_r = \frac{k_{r+1}}{k_r}, \quad \sum_{(r)} = \sum_{k=k_r}^{k_{r+1}-1}.$$

For any lacunary sequence Θ, the space N_Θ^0 is defined as follows (cf. [2]):

$$N_\Theta^0 = \{x = (x_k) | \lim_r \frac{1}{h_r} \sum_{(r)} |x_k| = 0\}.$$

It is routine to show that the next statement is valid.

Theorem 2 (cf. [2]). *The space N_Θ^o is a solid AK-FK-space with the norm*

$$\| x \|_\Theta = \sup_r h_r^{-1} \sum_{(r)} |x_k|.$$

In the special case $\Theta = (2^r)$, we have that $N_\Theta^o = w_0$ and that the norm $\| x \|_\Theta$ is equivalent to the usual norm $\| x \| = \sup_n (n+1)^{-1} \sum_{k=0}^n |x_k|$ in w_0.

For a sequence of moduli $F = (f_k)$ and for a lacunary sequence Θ, define:

$$N_\Theta^0(F) = \{x = (x_k) | \lim_r \frac{1}{h_r} \sum_{(r)} f_k(|x_k|) = 0\},$$

hence $N_\Theta^0(F) = X(F)$ for $X = N_\Theta^0$. In particular, for $\Theta = (2^r)$, $f_k(x) = x^{p_k}$, $0 < p_k < 1$, we obtain that $N_\Theta^0(F) = w_0(p)$.

By Theorems 1 and 2, we can formulate

Theorem 3. *The space $N_\Theta^0(F)$ is a solid AK-FK-space with the F-norm*

$$g_F(x) = \sup_r h_r^{-1} \sum_{(r)} f_k(|x_k|).$$

2. The Köthe-Toeplitz duals of $N_\Theta^0(F)$.

For a sequence space X, we denote by X^α and X^β the Köthe-Toeplitz duals of X, i.e.

$$X^\alpha = \{\alpha = (\alpha_k) | \sum_k |\alpha_k x_k| < \infty \text{ for all } (x_k) \in X\}$$

and

$$X^\beta = \{\alpha = (\alpha_k) | \sum_k \alpha_k x_k \text{ converges for all } (x_k) \in X\},$$

and for a F-normed space X, we denote by X' the continuous dual of X, and in the case $\phi \subset X$, we denote

$$X^\varphi = \{(\varphi(e_k)) | \varphi \in X'\}.$$

Remark 1. Since for every solid AK-FK-space X,

$$X^\alpha = X^\beta = X^\varphi,$$

then, by Theorem 3, it holds for $X = N_\Theta^0(F)$.

By some restrictions for $F = (f_k)$, we will describe the Köthe-Toeplitz duals of $N_\Theta^0(F)$.

Let f_k be strictly increasing modulus functions and let f_k^{-1} be the inverse functions of modulus f_k.

For a sequence of moduli $F = (f_k)$ and for a lacunary sequence $\Theta = (k_r)$, we define

$$M_\Theta(F) = \left\{\alpha = (\alpha_k) \mid \sum_{r=0}^{\infty} \max_{(r)} \frac{|\alpha_k|}{f_k^{-1}(B/h_r)} < \infty \right. \\ \left. \text{for some integer } B > 1\right\}, \tag{1}$$

where $\max_{(r)} = \max_{k_r \le k < k_{r+1}-1}$.

Further, we will use the following characteristics for a sequence of moduli $F = (f_k)$:

(F1) *there exists $C > 0$ such that*

$$f_k(t) f_k(1/t) \le C$$

for all $k \in \mathbb{N}$ and $t > 0$,

(F2) *there exists a continuous function $\Phi : [0, \infty) \to [0, \infty)$ such that*

$$\sum_{(r)} f_k^{-1}(B/h_r)|x_k| \le \Phi(\sum_{(r)} B/h_r f_k(|x_k|))$$

for all $(x_k) \in N_\Theta^0(F)$ and for some $B \in \mathbb{N}$.

Theorem 4. *If condition* (F1) *is fulfilled, then*

$$(N_\Theta^0(F))^\alpha \subset M_\Theta(F).$$

Proof. If $\alpha = (\alpha_k)$ is not in $M_\Theta(F)$, then the series in (1) is divergent for each integer $B > 1$, and therefore, there exists a sequence $B_r \to \infty$ such that

$$\sum_{r=0}^{\infty} \max_{(r)} \frac{|\alpha_k|}{f_k^{-1}(B_r/h_r)} = \infty.$$

Let

$$A_{rk} = \frac{|\alpha_k|}{f_k^{-1}(B_r/h_r)}$$

and $A_{r\tilde{k}_r} = \max\limits_{(r)} A_{rk}$. Define $\tilde{x} = (\tilde{x}_k)$ by

$$\tilde{x}_k = \begin{cases} \dfrac{1}{f_k^{-1}(B_r/h_r)} & \text{for } k = \tilde{k}_r, \\ 0 & \text{for } k \neq \tilde{k}_r. \end{cases}$$

Then, by condition (F1), we have

$$\frac{1}{h_r}\sum_{(r)} f_k(|\tilde{x}_k|) = \frac{1}{h_r} f_{\tilde{k}_r}\left(\frac{1}{f_{\tilde{k}_r}^{-1}(B_r/h_r)}\right) \leq \frac{C}{B_r}$$

and, therefore, $\tilde{x} \in N_\Theta^0(F)$.

But

$$\sum_{(r)} |\alpha_k \tilde{x}_k| = \frac{|\alpha_{\tilde{k}_r}|}{f_{\tilde{k}_r}^{-1}(B_r/h_r)} = \max\nolimits_r \frac{|\alpha_k|}{f_k^{-1}(B_r/h_r)}$$

and hence $\sum_{k=0}^{\infty} |\alpha_k \tilde{x}_k| = \infty$. This completes the proof.

Theorem 5. *If condition* (F2) *is fulfilled, then*

$$M_\Theta(F) \subset (N_\Theta^0(F))^\alpha.$$

Proof. Let $x = (x_k) \in N_\Theta^0(F)$ and $\alpha = (\alpha_k) \in M_\Theta(F)$. Then, by condition (F2), we have

$$\begin{aligned}\sum_{(r)} |\alpha_k x_k| &= \sum_{(r)} A_{rk} f_k^{-1}(B/h_r)|x_k| \leq \\ &\leq \max_{(r)} A_{rk} \sum_{(r)} f_k^{-1}(B/h_r)|x_k| \leq \\ &\leq \max_{(r)} A_{rk} \Phi(B/h_r \sum_{(r)} f_k(|x_k|)) = \\ &= \max_{(r)} A_{rk} \Phi(B g_F(x)).\end{aligned}$$

Hence we have $\sum_{r=0}^{\infty} |\alpha_r x_r| < \infty$ and the proof is completed.

Remark 2. If we use the same technique as in [6], Theorem 4, we may estimate that the inclusion $M_\Theta(F) \subset (N_\Theta^0(F))^\alpha$ holds for the sequence of moduli $F = (f_k)$, where

(F3) $f_k^{-1}(uv) \geq K\, f_k^{-1}(u) f_k^{-1}(v)$ for some $K > 0$ ($k \in \mathbb{N}$; $u, v \geq 0$),

(F4) there exists $M > 0$ such that

$$f_k^{-1}(u) \leq Mu \quad \text{for all} \quad 0 \leq u \leq 1.$$

In the case $f_k(t) = t^{p_k}, 0 < p_k < 1$, and $\Theta = (2^r)$, conditions (F1) and (F2) (and also (F3)) hold and we have a result of the same kind as Theorem 4 in [6].

Condition (F2) follows from conditions (F3) and (F4).
Indeed, for each $x = (|x_k|) \in N_\Theta^0(F)$, we may state that there exists an integer $R > 0$ such that

$$Bh_r^{-1} f_k(|x_k|) < 1$$

for some $B > 1$ and $r > R$. Then, by conditions (F3) and (F4), we have

$$\sum_{(r)} f_k^{-1}(Bh_r^{-1})|x_k| \leq \frac{1}{K} \sum_{(r)} f_k^{-1}(Bh_r^{-1} f_k(|x_k|)) \leq$$
$$\leq \frac{M}{K} Bh_r^{-1} \sum_{(r)} f_k(|x_k|)$$

and, therefore, condition (F2) holds.

3. Some extensions of Kuttner's theorem

Let l_∞ denote the space of bounded sequences. We formulate:

Theorem 6 (cf. [14]). *Let X be a locally convex FK-space, and let $p = (p_k)$, where $0 < p_k = \tilde{p} < 1$ for each $k \in \mathbb{N}$. Then $X \supset w_0(p)$ implies $X \supset l_\infty$.*

In 1946, Kuttner [5] proved the statement of Theorem 6 for $X = c_A$, where c_A is a summability field of any regular matrix A. In 1968, Maddox [7] extended this result to coregular A, and in 1981, Thorpe [14] proved Theorem 6.

A generalization of Theorem 6 for non-constant $p = (p_k)$ is given by Maddox [8].

Further, we will use the following two propositions.

Proposition 1. *Let*

$$B_\Theta(F) = \bigcap_{B \subset \mathbb{N}} \{x = (x_k) | \lim_r \sum_{(r)} f_k^{-1}(B/h_r)|x_k| = 0\}.$$

Then

(i) *$B_\Theta(F)$ is a locally convex AK-FK-space with the norms*

$$\| x \|_B = \sup_r \sum_{(r)} f_k^{-1}(B/h_r)|x_k|, \ B \in \mathbb{N},$$

(ii) $(N_\Theta^0(F))^\varphi \subset (B_\Theta(F))^\varphi$ *for each $F = (f_k)$ which satisfies condition* (F1).

Proof. (*i.*) The space $B_\Theta(F)$ is a countable intersection of strong null summability fields of positive matrices without zero columns. By [1], we have that every such summability field is a solid AK-BK-space with respect to the norm $\| x \|_B$. Then $B_\Theta(F)$ is an locally convex FK-space by the well-known result of Zeller [14], and it is routine to show that $B_\Theta(F)$ is an AK-space.

(*ii*). By Remark 1 (let $X = B_\Theta(F)$) and by Theorem 4, it is sufficient to show that $M_\Theta(F) \subset (B_\Theta(F))^\alpha$. Let $\alpha = (\alpha_k) \in M_\Theta(F)$. Then there exists an integer $B > 1$ such that

$$H = \sum_{r=0}^{\infty} \max_{(r)} \frac{|\alpha_k|}{f_k^{-1}(B/h_r)} < \infty.$$

Now, for each $x = (x_k) \in B_\Theta(F)$, we have

$$\begin{aligned}\sum_{r=0}^{\infty} |\alpha_r x_r| &= \sum_{r=0}^{\infty} \sum_{(r)} |\alpha_k x_k| = \\ &= \sum_{r=0}^{\infty} \sum_{(r)} \frac{|\alpha_k|}{f_k^{-1}(B/h_r)} f_k^{-1}(B/h_r)|x_k| \le \\ &\le H \| x \|_B,\end{aligned}$$

which implies that $(\alpha_k) \in (B_\Theta(F))^\alpha$.

The FK-space X is called an AD-space if $\overline{\phi} = X$.

Proposition 2 (cf. [13]). *Let E be an AD-space, and let X be an locally convex FK-space such that $X \supset \phi$. Then $X^\varphi \subset E^\varphi$ implies $X \supset E$.*

The next two theorems are certain generalizations of Theorem 6 for some restrictions for the sequence of moduli.

We denote $e = (1, 1, ...)$.

Theorem 7. *Let X be a locally convex FK-space, and let the sequence of moduli $F = (f_k)$ satisfy condition* (F1). *Then the condition $e \in B_\Theta(F)$ is sufficient for the implication*

$$X \supset N_\Theta^o(F) \Rightarrow X \supset l_\infty.$$

Proof. Suppose that $X \supset N_\Theta^0(F)$. Then we have that $X^\varphi \subset (N_\Theta^0(F))^\varphi$, and it follows, by Proposition 1, that $X^\varphi \subset (B_\Theta(F))^\varphi$. Since $B_\Theta(F)$ is an AK-space, then it is also an AD-space and, by Proposition 2, we have that $X \supset B_\Theta(F)$. It is routine to show that $e \in B_\Theta(F)$ is equivalent to $l_\infty \subset B_\Theta(F)$. Therefore, we have that $X \supset l_\infty$, and the proof is completed.

Theorem 8. *Let X be a locally convex FK-space, and let the sequence of moduli $F = (f_k)$ satisfy condition* (F2). *Then the condition $e \in B_\Theta(F)$ is necessary for the implication*

$$X \supset N_\Theta^0(F) \Rightarrow X \supset l_\infty.$$

Proof. Suppose that $e \notin B_\Theta(F)$. Then there exists an integer $\tilde{B} \geq 1$ such that

$$\sum_{(r)} f_k^{-1}(\tilde{B}/h_r) \not\to 0 \text{ if } r \to \infty. \tag{2}$$

Define now a matrix method $A = (a_{rk})$ by

$$a_{rk} = \begin{cases} f_k^{-1}(\tilde{B}/h_r), & \text{for } k_r \leq k < k_{r+1} \\ 0, & \text{otherwise,} \end{cases}$$

and let $[c_A]_0$ be a strong null summability field of the matrix method A. Then condition (2) implies that $l_\infty \not\subset [c_A]_0$. But by condition (F2), we have

$$\sum_{(r)} a_{rk}|x_k| = \sum_{(r)} f_k^{-1}(\tilde{B}/h_r))|x_k| \leq$$

$$\leq \Phi(\tilde{B}\frac{1}{h_r}\sum_{(r)} f_k(|x_k|)).$$

Therefore, by the continuity of Φ, we have that $[c_A]_0 \supset N_\Theta^0(F)$. Hence,$[c_A]_0$ is a locally convex FK-space (more exactly, it is a BK-space, cf. [1]) such that $[c_A]_0 \supset N_\Theta^0(F)$ but $[c_A]_0 \not\supset l_\infty$. This completes the proof.

Example 1. *Let $0 < p < 1$ and $f_k(t) = t^p$, $0 < p < 1$, for all $t \geq 0, k \in \mathbb{N}$. Then it is evident that condition (F1) holds, and from the inequality*

$$(\sum_k |a_k|)^p \leq \sum_k |a_k|^p, \quad 0 < p < 1,$$

it follows that (F2) holds.

Example 2. *In the case, where $f_k(t) = |\lambda_k t|^{p_k}$, $\lambda_k \neq 0$, $0 < p_k < 1$, we may show that (F1) holds if $|\lambda_k|^{p_k} = 0(1)$, and (F2) holds if $1/\lambda_k = 0(1)$.*

Example 3. *In general, condition (F1) is fullfilled if $f_k(tu) \geq C_1 f_k(t) f_k(u)$ and $f_k(1) \leq C_2$ for some $C_1, C_2 > 0$.*

We note that $f(t) = t^p$, $0 < p < 1$, is not the unique modulus function such that $f(tu) \geq C_1 f(t) f(u)$ (see [9]).

Let now $F = (f_k)$ be a sequence of moduli such that the next conditions hold:

(F5) $\sup_k f_k(t) < \infty$ for each $t > 0$,

(F6) $\lim_{t \to 0+} \sup_k f_k(t) = 0$.

Kolk [3] proved that the function

$$f(t) = \sup_k f_k(t)$$

which satisfies (F5) is a modulus if and only if condition (F6) holds.

Theorem 9. *Let X be a locally convex FK-space. If the sequence of moduli $F = (f_k)$ satisfies conditions* (F5), (F6), *and the lacunary sequence $\Theta = (k_r)$ satisfies the condition* $\liminf q_r > 1$, *then the condition*

$$\lim_{t\to\infty} \frac{f(t)}{t} = 0 \tag{3}$$

is sufficient for the implication

$$X \supset N_\Theta^0(F) \Rightarrow X \supset l_\infty.$$

Proof. It is easy to show that, for all solid sequence spaces E_1 and E_2 with $E_1 \subset E_2$, the inclusion $E_1(f) \subset E_2(F)$ is valid. The condition $\liminf q_r > 1$ guarantees that $w_0 \subset N_\Theta^0$ (see [2], Lemma 2.1). Therefore, in the case $E_1 = w_0$, $E_2 = N_\Theta^0$, we have that $w_0(f) \subset N_\Theta^0(F)$. Maddox (see [10], Theorem 6) proved that $X \supset w_0(f)$ implies $X \supset l_\infty$ if and only if condition (3) holds (for an arbitrary modulus f). This completes the proof.

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