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# Inclusion relations between the statistical convergence and strong summability

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ABSTRACT. For a sequential method of summability $\mathcal{B}$ we define $\mathcal{B}$-density and $\mathcal{B}$-statistical convergence in a Banach space $X$, and investigate inclusion relations between the space of $\mathcal{B}$-statistically convergent sequences and the space of strongly $\mathcal{B}$-summable sequences with respect to a sequence of modulus functions $\mathcal{F} = (f_k)$. As an application, two theorems of Pehlivan and Fisher [27] are corrected.

## 1. Introduction

The notion of statistical convergence was introduced by Fast [7] and Schoenberg [31]. A real or complex sequence $x = (x_k) = (x_k)_{k=1}^{\infty}$ is called statistically convergent to a number $l$ if for each $\varepsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \le n : |x_k - l| \ge \varepsilon\}| = 0,$$

where $|S|$ denotes the cardinality of the set $S$. By *st* we denote the space of all statistically convergent sequences.

A subadditive and increasing function $f\colon [0,\infty) \to [0,\infty)$ is called a modulus function if $f$ is continuous from the right at 0 and $f(t) = 0$. Maddox [21] introduced a generalization of the classical notion of strong summability

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[12, 11] in the following way. A sequence $x = (x_k)$ is called strongly Cesàro summable to $l$ with respect to a modulus function $f$ if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f(|x_k - l|) = 0.$$

In [22] Maddox proved that $w(f) \subset st$ for every modulus function $f$ and $st \subset w(f)$ if and only if $f$ is bounded, where $w(f)$ denotes the space of sequences which are strongly Cesàro summable with respect to $f$. Some generalizations of Maddox's theorems may be found in [5, 13, 25, 26, 3, 27]. We extend these results to more general spaces of $X$-valued sequences $st(\mathcal{B}, X)$ and $w^p(\mathcal{B}, \mathcal{F}, X)$, where $X$ is a Banach space, $\mathcal{B} = (B_i)$ is a regular sequential matrix method of summability and $\mathcal{F} = (f_k)$ is a sequence of modulus functions $f_k$. As an application, two theorems of Pehlivan and Fisher [27] are corrected.

## 2. Definitions and preliminary results

In the classical theory of summability the matrix methods play an essential role. Let $A = (a_{nk})$ be an infinite matrix of real or complex numbers. A number sequence $x = (x_k)$ is called $A$-summable to a number $l$ if the series

$$A_n x = \sum_k a_{nk} x_k = \sum_{k=1}^{\infty} a_{nk} x_k$$

converge for all $n \in \mathbb{N} = \{1, 2, ...\}$ and $\lim_n A_n x = l$. A matrix method $A$ (or a matrix $A$) is called regular if all convergent sequences $x = (x_k)$ are $A$-summable and $\lim_n A_n x = \lim_k x_k$. It is known that $A$ is regular if and only if (see, for example, [34], Theorem 1.3.9)

(T1) $\lim_n a_{nk} = 0 \quad (k \in \mathbb{N})$,
(T2) $\lim_n \sum_k a_{nk} = 1$,
(T3) $\sup_n \sum_k |a_{nk}| < \infty$.

The set of all regular matrices $A = (a_{nk})$ with $a_{nk} \geq 0$ we denote by $\mathcal{T}^+$.

For example, Cesàro method $C_1 = (c_{nk})$, where

$$c_{nk} = \begin{cases} 1/n & \text{if } k \leq n \\ 0 & \text{otherwise,} \end{cases}$$

is non-negative and regular, i.e. $C_1 \in \mathcal{T}^+$. A similar summability method is given in

**Example 2.1.** *Lacunary convergence* [9]. An increasing sequence of positive integers $\theta = (k_r)$ with $k_0 = 0$ is called a lacunary sequence if $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The sequence $\theta$ determines the intervals $I_r = (k_{r-1}, k_r]$. A sequence $x = (x_k)$ is called lacunary convergent to $l$ if

$$\lim_r \frac{1}{h_r} \sum_{i \in I_r} x_i = l.$$

So, if $A_\theta = (a^\theta_{ri})$ is the matrix, where

$$a^\theta_{ri} = \begin{cases} 1/h_r & \text{if } i \in I_r \\ 0 & \text{otherwise,} \end{cases}$$

the $A_\theta$-summability is precisely the lacunary convergence. It is clear that $A_\theta \in \mathcal{T}^+$.

A well-known example of non-matrix method of summability is almost convergence, originally defined by Banach limits. Lorentz [17] proved that a sequence $x = (x_k)$ is almost convergent to $l$ if and only if

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} x_{k+i} = l \quad \text{uniformly in } i. \tag{2.1}$$

Introducing the matrices $B^1_i = (b^1_{nk}(i))$ by

$$b^1_{nk}(i) = \begin{cases} 1/n & \text{if } 1 + i \leq k \leq n + i \\ 0 & \text{otherwise,} \end{cases}$$

we may write (2.1) in the form

$$\lim_n \sum_k b^1_{nk}(i) x_k = l \quad \text{uniformly in } i.$$

In general, for an arbitrary sequence of infinite matrices $\mathcal{B} = (B_i)$, $B_i = (b_{nk}(i))$, a sequence $x = (x_k)$ is called $\mathcal{B}$-summable to $l$, briefly $\mathcal{B}$-$\lim x = l$, if [32]

$$\lim_n \sum_k b_{nk}(i) x_k = l \quad \text{uniformly in } i.$$

Such a method $\mathcal{B}$ is sometimes called a sequential method of summability. In our notations, the almost convergence coincides with the $\mathcal{B}_1$-summability, where $\mathcal{B}_1 = (B^1_i)$.

The almost convergence may be generalized using invariant means [30]. Let $\sigma\colon \mathbb{N}\to\mathbb{N}$ an one-to-one mapping such that $\sigma^k(n)\neq n$ for all $n,k\in\mathbb{N}$, where $\sigma^k(n)$ denotes the iterate of order $k$ of the mapping $\sigma$ at $n$. For a given matrix $A=(a_{nk})$ let $A_i^\sigma=(a_{nk}^\sigma(i))$, where for every $i\in\mathbb{N}$,

$$a_{nk}^\sigma(i)=\begin{cases} a_{nj} & \text{if } k=\sigma^j(i)\\ 0 & \text{otherwise.}\end{cases}$$

The sequential method $\mathcal{B}_A^\sigma=(A_i^\sigma)$ defines so-called $A$-invariant convergence. An interesting special case here is determined by $\sigma=\tau$, where $\tau(n)=n+1$. It is clear that $\mathcal{B}_A^\tau=\mathcal{B}_1$ if $A=C_1$.

Analogously to the matrix methods of summability, a sequential method of summability $\mathcal{B}$ is called regular if every convergent sequence $x=(x_k)$ is $\mathcal{B}$-summable and $\mathcal{B}\text{-}\lim x=\lim_k x_k$. Method $\mathcal{B}=(B_i)$ with $B_i=(b_{nk}(i))$ is regular if and only if [32, 2]

(R1) $\lim_n b_{nk}(i)=0$ for all $k\in\mathbb{N}$, uniformly in $i$,
(R2) $\lim_n\sum_k b_{nk}(i)=1$ uniformly in $i$,
(R3) $\sum_k|b_{nk}(i)|<\infty$ $(n,i\in\mathbb{N})$, $\exists N$ $\sup_{i\in\mathbb{N},n>N}\sum_k|b_{nk}(i)|<\infty$.

The set of all regular sequential methods $\mathcal{B}$ with $b_{nk}(i)\geq 0$ we denote by the symbol $\mathcal{R}^+$. Since for a constant sequence $\mathcal{B}=(A)$ the method $\mathcal{B}$ reduces to the method $A$, we may write $\mathcal{T}^+\subset\mathcal{R}^+$.

By an index set we mean a set $K=\{k_i\}\subset\mathbb{N}$, where $k_i<k_{i+1}$ for all $i$. For a sequential method $\mathcal{B}\in\mathcal{R}^+$ we define a density function $\delta_{\mathcal{B}}$ as follows.

**Definition 2.2.** An index set $K$ is said to have *$\mathcal{B}$-density* $\delta_{\mathcal{B}(K)}$ equal to $d$, if the characteristic sequence of $K$ is $\mathcal{B}$-summable to $d$, i.e.

$$\lim_n\sum_{k\in K} b_{nk}(i)=d \quad \text{uniformly in } i.$$

In particular case $\mathcal{B}=(C_1)$ the density $\delta_{\mathcal{B}}$ is called the asymptotic density. For $\mathcal{B}=\mathcal{B}_1$ the density $\delta_{\mathcal{B}}$ reduces to the uniform density [8]. In the case $\mathcal{B}=(A), A\in\mathcal{T}^+$, the $\mathcal{B}$-density is the $A$-density $\delta_A$, where [8, 5, 13]

$$\delta_A(K)=\lim_n\sum_{k\in K}a_{nk}.$$

Every density determines the corresponding statistical convergence [6]. So, using $\mathcal{B}$-density, we can introduce $\mathcal{B}$-statistical convergence in a Banach space $X$ over the field $\mathbb{K}$ of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$ and with the norm $\|\cdot\|$.

**Definition 2.3.** Let $\mathcal{B} \in \mathcal{R}^+$. A $X$-valued sequence $x = (x_k)$ is called *$\mathcal{B}$-statistically convergent* to an element $l \in X$, briefly $st(\mathcal{B}, X)\text{-}\lim x = l$, if for each $\varepsilon > 0$,

$$\delta_{\mathcal{B}}(\{k : \|x_k - l\| \geq \varepsilon\}) = 0.$$

By the symbol $st(\mathcal{B}, X)$ we denote also the space of all $\mathcal{B}$-statistically convergent $X$-valued sequences. The space of sequences which converge $\mathcal{B}$-statistically to zero in $X$ will be denoted by $st_0(\mathcal{B}, X)$. In the case $\mathcal{B} = (A)$ (with $A \in \mathcal{T}^+$) we write $st(A, X)$ and $st_0(A, X)$ instead of $st(\mathcal{B}, X)$ and $st_0(\mathcal{B}, X)$, respectively.

It is not difficult to see that Definition 2.3 gives the usual statistical convergence [7, 22] if $\mathcal{B} = (C_1)$, $A$-statistical convergence [5, 13] for $\mathcal{B} = (A)$, lacunary statistical convergence [10, 27] for $\mathcal{B} = (A_\theta)$, uniform statistical convergence [26] for $\mathcal{B} = \mathcal{B}_1$, $A$-invariant statistical convergence [25, 29] for $\mathcal{B} = \mathcal{B}_A^\sigma$ and lacunary $\sigma$-statistical convergence [29] if $\mathcal{B} = \mathcal{B}_{A_\theta}^\sigma$.

Since in view of (R1) any finite index set has $\mathcal{B}$-density 0, every convergent sequence in $X$ is $\mathcal{B}$-statistically convergent (to the same limit), i.e.

$$c(X) \subset st(\mathcal{B}, X),$$

where $c(X)$ denotes the space of all $X$-valued convergent sequences. It is necessary to know when this inclusion is strict. We can show that

$$c(X) \subsetneqq st(A, X) \tag{2.2}$$

if $A$ is so-called uniformly regular non-negative matrix, i.e. $A \in \mathcal{T}^+$ and

(T4) $\lim_n \sup_k |a_{nk}| = 0$.

The set of all such matrices we denote by $\mathcal{UT}^+$.

**Lemma 2.4.** *Let $A \in \mathcal{UT}^+$. Every infinite index set $K$ contains an infinite subset $K'$ with $\delta_A(K') = 0$. In addition,* (2.2) *holds for an arbitrary Banach space $X$.*

*Proof.* Agnew [1] has proved the following theorem: If $A = (a_{nk})$ satisfies (T4) and $\sum_k |a_{nk}| < \infty$ for all $n \in \mathbb{N}$, then there exists a divergent sequence of 0's and 1's which is $A$-summable to 0. If $A \in \mathcal{UT}^+$ and $K = \{k_i\}$ is an infinite index set, then the submatrix $(a_{n,k_i})$ obviously satisfies the assumptions of Agnew's theorem. Hence there is a divergent sequence $(\alpha_i)$, $\alpha_i = 0$ or $\alpha_i = 1$, such that

$$\lim_n \sum_i a_{n,k_i} \alpha_i = 0.$$

Thus the set $K' = \{k_i : \alpha_i = 1\}$ is the infinite subset of $K$ with $\delta_A(K') = 0$.

Further, for a fixed element $z_0 \in X$ with $\|z_0\| = 1$ the sequence $z = (\alpha_i z_0)$ diverges in $X$, but if $K_\varepsilon = \{i : \|\alpha_i z_0\| \geq \varepsilon\}$, then by $K_\varepsilon = K'$ if $0 < \varepsilon \leq 1$ and $K_\varepsilon = \emptyset$ otherwise we have $st(A, X)\text{-}\lim z = 0$. Consequently, (2.2) holds. □

We recall that a function $f: [0, \infty) \to (0, \infty)$ is called a modulus function if

(a) $f(t) = 0$ if and only if $t = 0$,
(b) $f(t + u) \leq f(t) + f(u)$ for all $t \geq 0, u \geq 0$,
(c) $f$ is increasing,
(d) $f$ is continuous from the right at 0.

It immediately follows from (b) and (d) that $f$ is continuous everywhere on $[0, \infty)$. A modulus function may be unbounded or bounded. For example, $f(t) = t^p \quad (0 < p \leq 1)$ is unbounded but $f(t) = t/(1 + t)$ is bounded.

Ruckle [28], Maddox [21] and other authors used modulus function to construct new sequence spaces. In [13, 14, 15, 27] some new sequence spaces are defined by means of a sequence of modulus functions $\mathcal{F} = (f_k)$. In this connection are important the following properties of $\mathcal{F}$:

(M1) $\inf_k f_k(t) > 0 \qquad (t > 0)$;
(M2) $\lim_{t \to 0+} \sup_k f_k(t) = 0$;
(M3) $\sup_t \sup_k f_k(t) = M < \infty$.

For a Banach space $X$ let $\omega(X)$ be the space of all $X$-valued sequences, $c_0(X)$ the space of all convergent to zero sequences in $X$ and

$$c_0(\mathcal{F}, X) = \{x = (x_k) \in \omega(X) : \lim_k f_k(\|x_k\|) = 0\}.$$

In the case $X = \mathbb{K}$ we write $c_0$ and $c_0(\mathcal{F})$ instead of $c_0(X)$ and $c_0(\mathcal{F}, X)$, respectively. In [14] it was proved that the inclusion $c_0(X) \subset c_0(\mathcal{F}, X)$ holds if and only if (M2) is satisfied. The next lemma gives a necessary and sufficient condition for the inverse inclusion.

**Lemma 2.5.** *The inclusion* $c_0(\mathcal{F}, X) \subset c_0(X)$ *is true if and only if* (M1) *holds.*

*Proof.* Since $(x_k) \in c_0(\mathcal{F}, X)$ and $(x_k) \in c_0(X)$ are equivalent to $(\|x_k\|) \in c_0(\mathcal{F})$ and $(\|x_k\|) \in c_0$, respectively, our statement immediately follows from Theorem 5 [15]. □

As a generalization of the classical notion of strong summability [12, 11, 18] and strong almost convergence [24, 33, 19], Maddox [20] and Mursaleen

[23] introduced strong $\mathcal{B}$-summability. A number sequence $x = (x_k)$ is called strongly $\mathcal{B}$-summable to a number $l$ if

$$\lim_n \sum_k b_{nk}(i)|x_k - l| = 0 \quad \text{uniformly in } i.$$

In [16] it was given following generalization of strong $\mathcal{B}$-summability by means of a sequence of modulus functions $\mathcal{F} = (f_k)$.

**Definition 2.6.** Let $p > 0$, $X$ be a Banach space, $\mathcal{F} = (f_k)$ be a sequence of modulus functions and $\mathcal{B}$ be a sequential method of summability with $b_{nk}(i) \geq 0$. A sequence $x = (x_k) \in \omega(X)$ is called *strongly* $(\mathcal{B}, p, \mathcal{F})$*-summable* to $l \in X$, briefly $w^p(\mathcal{B}, \mathcal{F}, X)$-$\lim x = l$, if

$$\lim_n \sum_k b_{nk}(i)[f_k(\|x_k - l\|)]^p = 0 \quad \text{uniformly in } i.$$

By the symbol $w^p(\mathcal{B}, \mathcal{F}, X)$ we denote also the space of all strongly $(\mathcal{B}, p, \mathcal{F})$-summable sequences.

A remarkable special case of Definition 2.6 is contained in the following example.

**Example 2.7.** *The space* $w^{\mathbf{p}}(\mathcal{B}, f, X)$. Let $X$ be a Banach space, $\mathcal{B}$ be a sequential method of summability and $\mathbf{p} = (p_k)$ be a positive sequence with $\sup_k p_k = H < \infty$. For a modulus function $f$ by $w^{\mathbf{p}}(\mathcal{B}, f, X)$ we denote the space of sequences $x = (x_k) \in \omega(X)$ such that for some $l \in X$,

$$\lim_n \sum_k b_{nk}(i)[f(\|x_k - l\|)]^{p_k} = 0 \quad \text{uniformly in } i.$$

In this case we write $w^{\mathbf{p}}(\mathcal{B}, f, X)$-$\lim x = l$.

If $q = \max\{1, H\}$ then $p_k/q \leq 1$ and the equality

$$f_k^{\mathbf{p}}(t) = [f(t)]^{p_k/q}$$

clearly defines a modulus function for each $k \in \mathbb{N}$. So for $\mathcal{F}^{\mathbf{p}} = (f_k^{\mathbf{p}})$,

$$w^{\mathbf{p}}(\mathcal{B}, f, X) = w^q(\mathcal{B}, \mathcal{F}^{\mathbf{p}}, X). \tag{2.3}$$

Bilgin [4] considered the space $w^{\mathbf{p}}(\mathcal{B}, f, X)$ for $X = \mathbb{K}$.

## 3. The inclusion $w^p(\mathcal{B},\mathcal{F},X)\subset st(\mathcal{B},X)$

Let $X$ be a Banach space. For a $X$-valued sequence $y=(y_k)$ and a scalar sequence $a=(a_k)$ let $a\cdot y=(a_k y_k)$. If $K$ is an index set, then by $y^{[K]}$ we denote the sequence $\chi(K)\cdot y$, where $\chi(K)$ is the characteristic sequence of $K$. Thus $y^{[K]}=(y_k^{[K]})$ with

$$y_k^{[K]}=\begin{cases} y_k & \text{if } k\in K\\ 0 & \text{otherwise.}\end{cases}$$

**Theorem 3.1.** *Let $\mathcal{B}\in\mathcal{R}^+$ and let $\mathcal{F}=(f_k)$ be a sequence of modulus functions. If for an infinite index set $K$,*

$$\inf_{k\in K} f_k(t)>0 \qquad (t>0), \tag{3.1}$$

*then $w^p(\mathcal{B},\mathcal{F},X)$-$\lim y^{[K]}=0$ implies $st(\mathcal{B},X)$-$\lim y^{[K]}=0$.*

*Proof.* Let $\varepsilon>0$. By (3.1) there exists a number $s>0$ such that $f_k(\varepsilon)>s \quad (k\in K)$. Denoting

$$L_\varepsilon=\{k:\|y_k^{[K]}\|\ge\varepsilon\}=\{k\in K:\|y_k\|\ge\epsilon\},$$

for all $i\in\mathbb{N}$ we have

$$\sigma_n(i)=\sum_k b_{nk}(i)[f_k(\|y_k^{[K]}\|)]^p\ge s^p\sum_{k\in L_\varepsilon} b_{nk}(i),$$

which gives

$$\sum_{k\in L_\varepsilon} b_{nk}(i)\le s^{-p}\sigma_n(i) \qquad (i,n\in\mathbb{N}).$$

Therefore, if $w^p(\mathcal{B},\mathcal{F},X)$-$\lim y^{[K]}=0$, i.e. $\lim_n\sigma_n(i)=0$ uniformly in $i$, then

$$\delta_A(L_\varepsilon)=\lim_n\sup_i\sum_{k\in L_\varepsilon} b_{nk}(i)=0.$$

Hence $st(\mathcal{B},X)$-$\lim y^{[K]}=0$. □

For $K=\mathbb{N}$ the condition (3.1) is equivalent to (M1). Thus, taking $y=(x_k-l)$ in Theorem 3.1, we can formulate

**Theorem 3.2.** *If $\mathcal{B}\in\mathcal{R}^+$ and $\mathcal{F}=(f_k)$ satisfies* (M1) *then*

$$w^p(\mathcal{B},\mathcal{F},X)\text{-}\lim x=l \implies st(\mathcal{B},X)\text{-}\lim x=l. \tag{3.2}$$

We have various special cases by concrete definitions of sequences $\mathcal{B}$ and $\mathcal{F}$. If $f$ is a modulus function and $\mathbf{p}=(p_k)$ is a bounded sequence of positive numbers, then the sequence of modulus functions $\mathcal{F}^{\mathbf{p}}$ (Example 2.7) obviously satisfies (M1). So by (2.3) we get the following generalization of Theorem 1 of Bilgin [3].

**Corollary 3.3.** *Let $\mathcal{B} \in \mathcal{R}^+$ and $0 < p_k \le \sup_k p_k = H < \infty$. For any modulus function $f$,*

$$w^p(\mathcal{B}, f, X)\text{-}\lim x = l \implies st(\mathcal{B}, X)\text{-}\lim x = l.$$

In the case $p_k = 1$ Corollary 3.3 was earlier proved by Nuray and Savas [25] ($\mathcal{B} = \mathcal{B}_A^\sigma$), Pehlivan [26] ($\mathcal{B} = \mathcal{B}_1$), Connor [5] ($\mathcal{B} = (A)$) and Maddox [22] ($\mathcal{B} = (C_1)$).

Taking $\mathcal{B} = (A)$ with $A \in \mathcal{T}^+$ in Theorem 3.2, we get a known result ([13], sufficiency in Theorem 3.1).

**Corollary 3.4.** *Let $A \in \mathcal{T}^+$. If $\mathcal{F}$ satisfies* (M1) *then*

$$w^p(A, \mathcal{F}, X)\text{-}\lim x = l \implies st(A, X)\text{-}\lim x = l. \tag{3.3}$$

For $A = A_\theta$ (Example 2.1) Corollary 3.4 generalizes a result of Pehlivan and Fisher ([27], sufficiency in Theorem 3.3).

From (3.3) it follows that $w_0^p(A, \mathcal{F}, X) \subset st_0(A, X)$. If $A$ is the unit matrix $I$, this inclusion reduces to $c_0(\mathcal{F}, X) \subset c_0(X)$ and so (M1) holds because of Lemma 2.5. Thus by $I \in \mathcal{T}^+$ we have reproved a known result about the matrix class $\mathcal{T}^+$ ([13], Theorem 3.1).

**Corollary 3.5.** *The implication* (3.3) *holds for all $A \in \mathcal{T}^+$ if and only if* (M1) *is satisfied.*

Corollaries 3.4 and 3.5 induce the natural question: Is the condition (M1) necessary in order that (3.3) holds for an individual matrix $A \in \mathcal{T}^+$? We prove that the answer is negative if $A$ is uniformly regular.

**Theorem 3.6.** *Let $A \in \mathcal{UT}^+$. There exists a sequence of modulus functions $\mathcal{F} = (f_k)$ with $\inf_k f_k(t) = 0 \quad (t > 0)$ such that* (3.3) *is true.*

*Proof.* By Lemma 2.4 there is an infinite index set $K = \{k_i\}$ with $\delta_A(K) = 0$. Defining, for example, $f_{k_i}(t) = i^{-1}t$ if $i \in K$ and $f_k(t) = t$ if $k \in \mathbb{N}\backslash K$, we have

$$\inf_k f_k(t) = \lim_i f_{k_i}(t) = \lim_i i^{-1}t = 0.$$

If $w^p(A, \mathcal{F}, X)$-$\lim x = l$ then $y = (x_k - l) \in w_0^p(A, \mathcal{F}, X)$. By $\delta_A(K) = 0$ we have $st(A, X)$-$\lim y^{[K]} = 0$. Since

$$\inf_{k \in \mathbb{N}\backslash K} f_k(t) > 0 \qquad (t > 0)$$

and $y^{[\mathbb{N}\setminus K]}$ clearly belongs to $w_0^p(A,\mathcal{F},X)$, from Theorem 3.1 it follows that $st(A,X)\text{-}\lim y^{[\mathbb{N}\setminus K]} = 0$. Thus

$$y = y^{[K]} + y^{[\mathbb{N}\setminus K]} \in st_0(A,X)$$

which implies $st(A,X)\text{-}\lim x = l$. □

**Remark 3.7.** Theorem 3.3 of Pehlivan and Fisher [27] asserts that (M1) is necessary and sufficient for the implication (3.3) in the case $A = A_\theta$, $p = 1$. Since $A_\theta \in \mathcal{UT}^+$, Theorem 3.6 shows that this theorem is not true in part. The condition (M1) is not necessary for (3.3) if $A = A_\theta$.

A necessary and sufficient condition for the implication (3.3) is contained in the following theorem.

**Theorem 3.8.** *Let $A \in \mathcal{UT}^+$ and suppose that $\mathcal{F} = (f_k)$ is pointwise convergent. The implication* (3.3) *is true if and only if*

$$\lim_k f_k(t) > 0 \qquad (t > 0). \tag{3.4}$$

*Proof.* Let $\varepsilon > 0$. If (3.4) is valid, then we can find numbers $s > 0$ and $r \in \mathbb{N}$ such that $f_k(\varepsilon) \geq s \quad (k \geq r)$. As in the proof of Theorem 3.1, we get

$$\sum_{k\in L_\varepsilon, k\geq r} a_{nk} \leq s^{-p} \sum_{k\geq r} a_{nk}[f_k(\|x_k - l\|)]^p, \tag{3.5}$$

where $L_\varepsilon = \{k : \|x_k - l\| \geq \varepsilon\}$. If $w^p(A,\mathcal{F},X)\text{-}\lim x = l$, then by (T1) the inequality (3.5) implies for $n \to \infty$ that $\delta_A(L_\varepsilon) = 0$, i.e. $st(A,X)\text{-}\lim x = l$.

Conversely, if (3.4) is not true, we have $\lim_k f_k(t_0) = 0$ for some $t_0 > 0$. Since $A \in \mathcal{UT}^+$, by Lemma 2.4 there exists an infinite index set $K = (k_i)$ with $\delta_A(K) = 0$. We define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0 & \text{if } k \in K \\ t_0 z & \text{otherwise,} \end{cases}$$

where $z \in X$ with $\|z\| = 1$. Then $\lim_k [f_k(\|x_k\|)]^p = 0$ and by regularity of $A$ we have $w^p(A,\mathcal{F},X)\text{-}\lim x = 0$. But for $0 < \varepsilon \leq t_0$,

$$\delta_A(\{k : \|x_k\| \geq \varepsilon\}) = \lim_n \sum_k a_{nk} - \delta_A(K) = 1$$

by (T2) and $\delta_A(K) = 0$. Thus $st(A,\mathcal{F},X)\text{-}\lim x \neq 0$ contrary to (3.3). □

## 4. The inclusion $st(\mathcal{B}, X) \subset w^p(\mathcal{B}, \mathcal{F}, X)$

First theorem gives sufficient conditions for mentioned inclusion.

**Theorem 4.1.** *Let* $\mathcal{B} \in \mathcal{R}^+$. *If* $\mathcal{F} = (f_k)$ *satisfies* (M2) *and* (M3) *then*

$$st(\mathcal{B}, X)\text{-}\lim x = l \implies w^p(\mathcal{B}, \mathcal{F}, X)\text{-}\lim x = l. \tag{4.1}$$

*Proof.* Let $st(\mathcal{B}, X)$-$\lim x = l$, $h(t) = \sup_k f_k(t)$ and choose $\varepsilon > 0$. For every $i \in \mathbb{N}$ we split the sum

$$\sigma_n(i) = \sum_k b_{nk}(i)[f_k(\|x_k - l\|)]^p$$

into two sums $\sum_1$ and $\sum_2$ over $L_\varepsilon = \{k : \|x_k - l\| \geq \varepsilon\}$ and $\{k : \|x_k - l\| < \varepsilon\}$, respectively. Then by (M3),

$$\sum\nolimits_1 \leq M^p \sup_i \sum_{k \in L_\varepsilon} b_{nk}(i) \tag{4.2}$$

and by the increase of $f_k$,

$$\sum\nolimits_2 \leq h(\varepsilon) \sup_i \sum_k b_{nk}(i).$$

Thus, using also (R2), we have

$$\lim_n \sigma_n(i) \leq M^p \delta_{\mathcal{B}}(L_\varepsilon) + h(\varepsilon)$$

which by $\delta_{\mathcal{B}}(L_\varepsilon) = 0$ and (M2) gives $\lim_n \sigma_n(i) = 0$ uniformly in $i$, i.e. $w^p(\mathcal{B}, \mathcal{F}, X)$-$\lim x = l$. □

If we examine (4.1) only for $X$-valued bounded sequences $x = (x_k)$ with $\|x\| \leq N$, then

$$f_k(\|x_k - l\|) \leq f_k(N + \|l\|) \leq h(N + \|l\|).$$

Consequently, (4.2) holds (with $h(N + \|l\|)$ instead of $M$) without the assumption (M3). Hence we have proved

**Theorem 4.2.** *Let* $\mathcal{B} \in \mathcal{R}^+$ *and let* $x = (x_k)$ *be a bounded sequence in* $X$. *The implication* (4.1) *is true if* (M2) *holds.*

Let $f$ be a modulus function and let $0 < p_k \leq \sup_k p_k = H < \infty$. Then the sequence of modulus functions $\mathcal{F}^{\mathbf{p}}$ (Example 2.7) satisfies (M2) if and only if $\inf_k p_k > 0$, and $\mathcal{F}^{\mathbf{p}}$ satisfies (M3) if and only if $f$ is bounded. Using also (2.3), from Theorems 4.1 and 4.2 we get

**Corollary 4.3.** *Let $f$ be a modulus function and let $\mathbf{p} = (p_k)$ be a positive sequence with $\sup_k p_k = H < \infty$.*

(i) *If the modulus function $f$ is bounded and $\inf_k p_k > 0$ then $st(\mathcal{B}, X)$-$\lim x = l$ implies $w^{\mathbf{p}}(\mathcal{B}, f, X)$-$\lim x = l$.*
(ii) *If $x = (x_k)$ is a bounded sequence in $X$ and $st(\mathcal{B}, X)$-$\lim x = l$ then $w^{\mathbf{p}}(\mathcal{B}, f, X)$-$\lim x = l$.*

Some special cases of Corollary 4.3 have been considered by Bilgin [3] ($\mathcal{B} = (C_1)$), Nuray and Savas [25] ($\mathcal{B} = \mathcal{B}_A^\sigma, p_k = 1$), Pehlivan [26] ($\mathcal{B} = \mathcal{B}_1, p_k = 1$) and Connor [5] ($\mathcal{B} = (A), p_k = 1$).

Let $\ell_\infty(X)$ denote the space of all $X$-valued bounded sequences. By combining Theorems 3.2, 4.1 and 4.2 we have

**Theorem 4.4.** *Let $p > 0, \mathcal{B} \in \mathcal{R}^+$ and $\mathcal{F} = (f_k)$ be a sequence of modulus functions. For any Banach space $X$*

(i) *$st(\mathcal{B}, X) = w^p(\mathcal{B}, \mathcal{F}, X)$ if (M1), (M2) and (M3) are satisfied;*
(ii) *$st(\mathcal{B}, X) \cap \ell_\infty(X) = w^p(\mathcal{B}, \mathcal{F}, X) \cap \ell_\infty(X)$ if (M1) and (M2) hold.*

In the particular case $\mathcal{F} = \mathcal{F}^{\mathbf{p}}$, using Corollaries 3.3 and 4.3, we get

**Corollary 4.5.** *Let $f$ be a modulus function, $\mathcal{B} \in \mathcal{R}^+$ and $\mathbf{p} = (p_k)$ be a positive sequence with $\sup_k p_k = H < \infty$. Then for an arbitrary Banach space $X$*

(i) *$st(\mathcal{B}, X) = w^{\mathbf{p}}(\mathcal{B}, f, X)$ if $\inf_k p_k > 0$ and $f$ is bounded;*
(ii) *$st(\mathcal{B}, X) \cap \ell_\infty(X) = w^{\mathbf{p}}(\mathcal{B}, f, X) \cap \ell_\infty(X)$.*

Suppose that $A = (a_{nk}) \in \mathcal{UT}^+$ and the sequence of moduli $\mathcal{F} = (f_k)$ satisfies (M2). By Theorem 4.1 the condition (M3) is sufficient for the implication

$$st(A, X)\text{-}\lim x = l \implies w^p(A, \mathcal{F}, X)\text{-}\lim x = l. \tag{4.3}$$

It is not difficult to see that (M3) is not necessary for (4.3) in general. Indeed, a slightly modification of the proof of Theorem 4.1 shows that (4.3) (and also (4.1)) remains true, if instead of (M3) we have

$$\sup_t \sup_{k \geq r} f_k(t) < \infty$$

for some fixed index $r$. Thus, defining, for example, $f_1(t) = t$ and $f_k(t) = t/(1+t)$ for $k \geq 2$, we get that (4.3) holds but (M3) is not satisfied.

**Remark 4.6.** Pehlivan and Fisher ([27], Theorem 3.4) assert that (M3) is necessary and sufficient for the implication (4.3) when $p = 1$, $A = A_\theta$

and $\mathcal{F}$ satisfies (M2). Our previous arguments show that (M3) may not be necessary. The proof of Pehlivan and Fisher is not correct. If $\theta = (k_r)$ is a lacunary sequence, then $\sup_t \sup_k f_k(t) = \infty$ does not follow $\sup_t \sup_r f_{k_r}(t) = \infty$ in general.

In the following we give a more precise characterization of the implication (4.3) for uniformly regular row-finite matrices $A$ (a matrix $A = (a_{nk})$ is called row-finite if for any $n$ there exists an index $k(n)$ such that $a_{nk} = 0$ if $k \geq k(n)$).

**Theorem 4.7.** *Let $A \in \mathcal{UT}^+$ be a row-finite matrix and suppose that the sequence of moduli $\mathcal{F} = (f_k)$ satisfies* (M2).

(a) *If $\mathcal{F}$ is non-decreasing then* (4.3) *is true if and only if* (M3) *holds;*
(b) *If $\mathcal{F}$ is pointwise convergent then* (M3) *implies* (4.3) *and* (4.3) *implies*

$$\sup_t \lim_k f_k(t) < \infty. \tag{4.4}$$

*Proof.* By Theorem 4.1 the implication (M3)$\Longrightarrow$(4.3) is true in both cases (a) and (b).

Suppose that (4.3) holds. For a row-finite and uniformly regular matrix $A$ we clearly have

$$\lim_n a_{n,k_n} = 0,$$

where $a_{n,k_n}$ is the last non-zero member in the $n^{\text{th}}$ row of $A$ (for sufficiently large $n$). Further, from (M2) it follows that $\sup_k f_k(t_0) < \infty$ for some $t_0 > 0$ and therefore (see [14], Lemma 2) $\sup_k f_k(t) < \infty$ for all $t > 0$. Thus, if $\mathcal{F} = (f_k)$ is non-decreasing, then $\mathcal{F}$ is pointwise convergent and (M3) reduces to (4.4). Consequently, if $\mathcal{F}$ is non-decreasing sequence failing (M3) or a pointwise convergent sequence failing (4.4), we have

$$\lim_{t\to\infty} \lim_k f_k(t) = \infty.$$

So, using also Lemma 2.4, in both cases we can find index sets $\{n(i)\}, K = \{k(i)\}$ (with $k(i) = k_{n(i)}$) and numbers $0 < t_1 < ... < t_i < t_{i+1} < ...$ such that $\delta_A(K) = 0$ and

$$f_{k(i)}(t_i) \geq (1/a_{n(i),k(i)})^{1/p} \qquad (i \in \mathbb{N}). \tag{4.5}$$

Then the sequence $x = (x_k)$, where for fixed $z \in X$ with $\|z\| = 1$,

$$x_k = \begin{cases} t_i z & \text{if } k = k(i) \\ 0 & \text{otherwise,} \end{cases}$$

converges $A$-statistically to 0 in $X$. Hence by (4.3),

$$\lim_n \sum_k a_{nk}[f_k(\|x_k\|)]^p = 0. \tag{4.6}$$

But (4.5) implies

$$a_{n(i),k(i)}[f_{k(i)}(\|x_k\|)]^p \geq 1 \qquad (i \in \mathbb{N}),$$

contrary to (4.6). Thus (4.4) must hold. □

From Theorems 3.8 and 4.7 it follows

**Corollary 4.8.** *Suppose that $A \in \mathcal{UT}^+$ is a row-finite matrix and $\mathcal{F} = (f_k)$ satisfies* (M2).

(a) *If $\mathcal{F}$ is non-decreasing then*

$$st(A, X) = w^p(A, \mathcal{F}, X) \tag{4.7}$$

*if and only if the conditions* (3.4) *and* (M3) *hold;*

(b) *If $\mathcal{F}$ is pointwise convergent then* (3.4) *and* (M3) imply (4.7), and (4.7) implies (3.4) and (4.4).

For a constant sequence $\mathcal{F} = (f)$ the conditions (3.4) and (M2) clearly hold, but the condition (M3) is equivalent to the boundedness of $f$. Hence from Corollary 4.8 we get

**Corollary 4.9.** *Let $A \in \mathcal{UT}^+$ be a row-finite matrix and let $f$ be a modulus function. For any Banach space $X$ we have*

$$st(A, X) = w^p(A, f, X) \tag{4.8}$$

*if and only if $f$ is bounded.*

In the case of $p = 1$ Maddox ([21], Theorem 2) proved Corollary 4.9 if $X$ is a locally convex space and $A = C_1$.

Since the matrix $A_\theta$ of the lacunary convergence (Example 2.1) is row-finite and $A_\theta \in \mathcal{UT}^+$, Corollaries 4.8 and 4.9 give the following corrected version of Corollary 3.5 [27].

**Corollary 4.10.** *Let $\theta$ be a lacunary sequence and let $\mathcal{F} = (f_k)$ be a non-decreasing sequence of modulus functions such that* (M2) *holds. For any Banach space $X$ and for $A = A_\theta$ the equality* (4.7) *holds if and only if*

*(3.4) and (M3) are satisfied. For $f_k = f$ and $A = A_\theta$ the equality (4.8) is true if and only if the modulus function $f$ is bounded.*

**Remark 4.11.** It should be noted that all our arguments remain valid if the norm in $X$ is replaced by a seminorm in $X$. Thus all our propositions are true also for a locally convex space $X$ if $\mathcal{B}$-statistical convergence and strong $(\mathcal{B}, p, \mathcal{F})$-summability in $X$ are defined as follows. Let $X$ be a locally convex Hausdorff topological linear space whose topology is determined by a system $G$ of continuous seminorms $g$. A sequence $x = (x_k) \in \omega(X)$ is called $\mathcal{B}$-statistically convergent to an element $l \in X$ if for each $\varepsilon > 0$ and for each $g \in G$ [22],

$$\delta_{\mathcal{B}}(\{k : g(x_k - l) \geq \varepsilon\}) = 0,$$

and strongly $(\mathcal{B}, p, \mathcal{F})$-summable to $l$ if

$$\lim_n \sum_k b_{nk}(i)[f_k(g(x_k - l))]^p = 0 \quad \text{uniformly in } i$$

for every $g \in G$.

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