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On a generalization of strong flatness

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ABSTRACT. Strong flatness or flatness of right acts over a monoid S has been the topic of many papers on homological classification of monoids during the last decades. It has turned out to be useful to consider a property called condition (P), which lies strictly between these two properties. In this paper we define a new condition (O) and show that the class of S-acts satisfying this condition lies strictly between the classes of strongly flat S-acts and S-acts satisfying condition (P).

1. Introduction

Let S (always in this paper) be a monoid with the identity element 1. A nonempty set A is called a *right S-act* (and denoted A_S) if there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions $(as)t = a(st)$ and $a1 = a$, for all $a \in A$ and all $s, t \in S$. Left S-acts ${}_SA$ are defined dually. Every right (left) ideal I of S is in a natural way a right (resp. left) S-act. If A_S and B_S are right S-acts then a mapping $f : A_S \to B_S$ is called a *homomorphism* if $f(as) = f(a)s$ for all $a \in A_S$ and $s \in S$. Analogously, homomorphisms of left S-acts are defined. If A_S and ${}_SB$ are right and left S-acts, respectively, the *tensor product* $A_S \otimes {}_SB$ can be taken as the quotient set $(A \times B)/\tau$, where τ is the smallest equivalence relation on $A \times B$ that identifies all pairs $((as, b), (a, sb))$ for $a \in A_S, b \in {}_SB, s \in S$. The τ-class of (a, b) is denoted $a \otimes b$, so $as \otimes b = a \otimes sb$ for $s \in S$. Basic properties of acts and their tensor products are discussed in [3], for example.

Pullbacks of a pair of homomorphisms of left S-acts or a pair of mappings of sets are defined as in every category. It is not difficult to see that the

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pullback of homomorphisms $f : {}_SM \to {}_SQ$ and $g : {}_SN \to {}_SQ$ of left acts is, if it exists, determined up to isomorphism and it is isomorphic to the left S-act

$$ {}_SP = \{(m,n) \in {}_SM \times {}_SN \mid f(m) = g(n)\}, $$

where $s(m,n) = (sm, sn)$ for all $s \in S, m \in {}_SM$ and $n \in {}_SN$, together with p_1 and p_2 – the restrictions of the projections on ${}_SM$ and ${}_SN$, respectively.

Consider the following, so called *pullback diagram* in S-**Act**

$$ \begin{array}{ccc} {}_SP & \xrightarrow{p_1} & {}_SM \\ {\scriptstyle p_2}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ {}_SN & \xrightarrow[g]{} & {}_SQ \end{array} $$

where ${}_SP$, p_1 and p_2 are as above. Let us denote such a diagram by $P({}_SM, {}_SN, f, g, {}_SQ)$. If A_S is a right S-act then for every homomorphism $k : {}_SB \to {}_SC$ of left S-acts let $\mathrm{id}_A \otimes k : A_S \otimes {}_SB \to A_S \otimes {}_SC$ be a mapping defined by $(\mathrm{id}_A \otimes k)(a \otimes b) = a \otimes k(b)$, $a \in A_S, b \in {}_SB$. Tensoring the pullback diagram $P({}_SM, {}_SN, f, g, {}_SQ)$ by an act A_S we get the commutative diagram

$$ \begin{array}{ccc} A_S \otimes {}_SP & \xrightarrow{\mathrm{id}_A \otimes p_1} & A_S \otimes {}_SM \\ {\scriptstyle \mathrm{id}_A \otimes p_2}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_A \otimes f} \\ A_S \otimes {}_SN & \xrightarrow[\mathrm{id}_A \otimes g]{} & A_S \otimes {}_SQ\ . \end{array} $$

For the pullback $(P', (p_1', p_2'))$ of $\mathrm{id}_A \otimes f$ and $\mathrm{id}_A \otimes g$ we may assume that

$$ P' = \{(a \otimes m, a' \otimes n) \in (A_S \otimes {}_SM) \times (A_S \otimes {}_SN) \mid a \otimes f(m) = a' \otimes g(n)\} $$

and ${p_1}'$ and ${p_2}'$ are the restrictions of the projections.

Now it follows from the definition of pullbacks that there exists a unique mapping $\varphi : A_S \otimes {}_SP \longrightarrow P'$ such that $p_1'\varphi = \mathrm{id}_A \otimes p_1$ and $p_2'\varphi = \mathrm{id}_A \otimes p_2$. It was stated in [2] that φ is given by

$$ \varphi(a \otimes (m,n)) = (a \otimes m, a \otimes n) $$

for any $a \in A_S$ and $(m,n) \in {}_SP$. In the sequel this mapping φ is called the mapping *corresponding* to the pullback diagram $P({}_SM, {}_SN, f, g, {}_SQ)$.

In [7] Stenström called a right S-act A_S strongly flat, if the functor $A_S \otimes -$ from the category of left S-acts to the category of sets preserves pullbacks and equalizers. He proved that A_S is strongly flat if and only if it satisfies the following conditions (P) and (E):

$$\text{(P)} \qquad (\forall a, a' \in A_S)(\forall s, s' \in S)(as = a's' \Rightarrow$$
$$(\exists a'' \in A_S)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs')),$$
$$\text{(E)} \qquad (\forall a \in A_S)(\forall s, s' \in S)(as = as' \Rightarrow$$
$$(\exists a' \in A_S)(\exists u \in S)(a = a'u \wedge us = us')).$$

Normak [6] was the first to consider condition (P) on its own. He showed that this condition does not imply preservation of pullbacks, although the reverse implication is true. In [2] Bulman-Fleming showed that in fact A_S is strongly flat if and only if $A_S \otimes -$ preserves pullbacks, that is, the corresponding φ is surjective and injective for every pullback diagram $P({}_SM, {}_SN, f, g, {}_SQ)$. He also proved that A_S satisfies condition (P) if and only if the corresponding φ is surjective for every pullback diagram $P({}_SM, {}_SN, f, g, {}_SQ)$. Implicitly it was even shown that condition (P) is equivalent to the surjectivity of φ for all pullback diagrams $P({}_SS, {}_SS, f, g, {}_SS)$. This motivates the question, what happens if we require surjectivity and injectivity of φ not for all pullback diagrams $P({}_SM, {}_SN, f, g, {}_SQ)$ but only for some "simpler" kind of them.

2. Condition (O)

Definition 2.1. Let us say that a right S-act A_S satisfies *condition* (O), if the corresponding φ is surjective and injective for every pullback diagram $P({}_SS, {}_SS, f, g, {}_SS)$.

It is immediate from the definition and the preceding section, that if A_S is strongly flat then it satisfies condition (O) and if it satisfies condition (O) then it satisfies condition (P).

Our first aim is to give a characterization of condition (O) which does not involve pullback diagrams but refers to A_S itself instead. For this we introduce a generalization of condition (E):

$$\text{(E}'\text{)} \qquad (\forall a \in A_S)(\forall s, s', z \in S)(as = as' \wedge sz = s'z \Rightarrow$$
$$(\exists a' \in A_S)(\exists u \in S)(a = a'u \wedge us = us')).$$

We also need the following lemmas.

Lemma 2.2. [4] *Let S be a monoid. Then $a \otimes s = a' \otimes t$ in $A_S \otimes {}_SS$ if and only if $as = a't$ in A_S.*

Lemma 2.3. [2] *If an act A_S satisfies condition* (P) *and $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_SM$ for $a, a' \in A_S$, $m, m' \in {}_SM$ then there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$, $um = vm'$.*

Proposition 2.4. *Let A_S be a right S-act. The following assertions are equivalent.*

(1) *A_S satisfies condition* (O).

(2) *A_S satisfies the following condition*

$$(\forall a, a' \in A_S)(\forall s, s', t, t', z, w \in S)$$
$$[sz = tw \wedge s'z = t'w \wedge as = a's' \wedge at = a't' \Rightarrow$$
$$(\exists a'' \in A_S)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs' \wedge ut = vt')].$$

(3) *A_S satisfies conditions* (P) *and* (E′).

Proof. (1)⇒(2). Let A_S satisfy Condition (O). Since φ is surjective for every pullback diagram $P({}_SS, {}_SS, f, g, {}_SS)$, A_S satisfies condition (P). Let $sz = tw$, $s'z = t'w$, $as = a's'$, $at = a't'$, $a, a' \in A_S$, $s, s', t, t', z, w \in S$. Let $f, g : {}_SS \to {}_SS$ be the homomorphisms of left S-acts, defined by $f(u) = uz$ and $g(u) = uw$, $u \in S$. Then $f(s) = g(t)$, $f(s') = g(t')$ and by Lemma 2.2. $a \otimes s = a' \otimes s'$ and $a \otimes t = a' \otimes t'$ in $A_S \otimes {}_SS$. This means that $\varphi(a \otimes (s, t)) = \varphi(a' \otimes (s', t'))$ for φ corresponding to the diagram $P({}_SS, {}_SS, f, g, {}_SS)$. Using injectivity of φ we get the equality $a \otimes (s, t) = a' \otimes (s', t')$ in $A_S \otimes {}_SP$. Since A_S satisfies conditon (P), by Lemma 2.3 there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $u(s, t) = v(s', t')$. But then $us = vs'$ and $ut = vt'$.

(2)⇒(3). Condition (P) follows by taking $t = s$, $t' = s'$ and $z = w = 1$ in the condition which we assume to be true. Let us show that condition (E′) holds. Suppose that $as = as'$, $sz = s'z$, $a \in A_S$, $s, s', z \in S$. The equalities $as = as'$, $a1 = a1$, $sz = 1sz$, $s'z = 1sz$ imply the existence of $a'' \in A_S$ and $u, v \in S$ such that $us = vs'$, $u1 = v1$ and $a = a''u$.

(3)⇒(1). As mentioned before, if A_S satisfies condition (P) then the corresponding φ is surjective for every pullback diagram $P({}_SS, {}_SS, f, g, {}_SS)$. Let us show that φ is also injective for every pullback diagram $P({}_SS, {}_SS, f, g, {}_SS)$. Let $f(1) = z$ and $g(1) = w$. Suppose that $\varphi(a \otimes (s, t)) = \varphi(a' \otimes (s', t'))$, $a, a' \in A_S$, $(s, t), (s', t') \in {}_SP$. By the definition of ${}_SP$ we have $f(s) = g(t)$ and $f(s') = g(t')$, or $sz = tw$ and $s'z = t'w$. Definition of φ implies that $(a \otimes s, a \otimes t) = (a' \otimes s', a' \otimes t')$, or $a \otimes s = a' \otimes s'$ and $a \otimes t = a' \otimes t'$. By Lemma 2.2. this means that $as = a's'$ and $at = a't'$. Using condition (P), from the equality $as = a's'$ we get $b \in A_S$, $u_1, v_1 \in S$ such that $a = bu_1$, $a' = bv_1$ and $u_1s = v_1s'$. We now have

$$b(u_1t) = (bu_1)t = at = a't' = (bv_1)t' = b(v_1t')$$

and

$$(u_1t)w = u_1(tw) = u_1sz = v_1s'z = v_1(t'w) = (v_1t')w.$$

Applying condition (E′) we get $a'' \in A_S$, $r \in S$ such that $b = a''r$ and $ru_1t = rv_1t'$. Denoting $u = ru_1$ and $v = rv_1$ we have $us = vs'$, $ut = vt'$, $a = bu_1 = a''ru_1 = a''u$ and, analogously, $a' = a''v$. Now

$$a \otimes (s,t) = a''u \otimes (s,t) = a'' \otimes u(s,t) = a'' \otimes (us, ut) = a'' \otimes (vs', vt')$$
$$= a'' \otimes v(s',t') = a''v \otimes (s',t') = a' \otimes (s',t')$$

in $A_S \otimes {}_SP$. Thus φ is injective. □

Lemma 2.5. *Let S be a group. Then every right S-act A_S satisfies condition* (O).

Proof. Since S is a group, by the dual of [6, Proposition 3.10] A_S satisfies condition (P). Hence it is sufficient to show that A_S satisfies condition (E′). Suppose that $as = as'$ and $sz = s'z$ for some $a \in A_S$, $s, s', z \in S$. Multiplying the equality $sz = s'z$ by z^{-1} from the right, we get $s = s'$. Thus we can take $u = 1$ and $a' = a$. □

This lemma implies that strong flatness and condition (O) are different notions. Indeed, if S is a nontrivial group then the one-element right S-act Θ_S satisfies condition (O), but it cannot be strongly flat, since it does not satisfy condition (E). Moreover, this means that condition (E′) does not imply (E), because otherwise condition (O) would imply strong flatness.

A monoid S is called *right collapsible* if for every $s, s' \in S$ there exists $z \in S$ such that $sz = s'z$. This is the dual of the definition of left collapsible monoid, which was given in [5].

Lemma 2.6. *Let S be a right collapsible monoid. Then every right S-act A_S satisfying condition* (O) *is strongly flat.*

Proof. Let A_S satisfy condition (O). It is sufficient to show that A_S satisfies condition (E). Suppose that $as = as'$, $a, a' \in A_S$, $s, s' \in S$. By the right collapsibility of S there exist $z \in S$ such that $sz = s'z$ Since A_S satisfies condition (E′) by Proposition 2.4., there exist $a' \in A_S$, $u \in S$ such that $a = a'u$, and $us = us'$. But this means that A_S satisfies condition (E) and hence it is strongly flat. □

Now we are ready to show that condition (P) does not imply condition (O).

Example 2.7. Let $T = x^*$ be a monogenic free monoid generated by an element x and let S be a monoid obtained from T by external adjoining of zero 0, i.e. $S = T \cup \{0\}$ and $s0 = 0s = 00 = 0$ for every element $s \in S$. Since S is right collapsible, a right S-act satisfies Condition (O) if and only if it is strongly flat. Let us consider the monocyclic right S-act $S/\rho(1, x)$, where

$\rho(1, x)$ is the smallest right congruence identifying the elements 1 and x. By [1, Proposition 2.10.] this act satisfies condition (P) and it is strongly flat if and only if x is an aperiodic element. Since x is not an aperiodic element, $S/\rho(1, x)$ cannot be strongly flat and hence it cannot satisfy condition (O) either.

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