

Constructive sets of real trigonometric series

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ABSTRACT. In the present paper we consider the abstract classes of trigonometric series with complementary characteristics to L^p . The concept used will generalize the method of complementary spaces introduced by Goes [2] and Tönnov [6] and the concept of T^λ -constructive spaces considered by the present author [4,5].

1. Introduction

We have introduced a concept of T^λ -constructive spaces into theory of trigonometric series [4] and [5]. Constructive spaces created a possibility to use the λ -summability method of Kangro [3] for the investigation of Fourier series. In the present paper we will generalize the concept of constructive spaces by introducing the constructive-type classes of abstract trigonometric series $\mathcal{L}_{T^\lambda}^p$. The results will link the classes of Fourier series and the classes $\mathcal{L}_{T^\lambda}^p$ of trigonometric series with Fourier coefficients of L^p .

Throughout the paper the integral is considered taken over any interval of length 2π . So, for any real number $p \geq 1$, we denote the set of equivalence classes of real-valued measurable functions f by L^p , where

$$\|f\|_p \equiv \left[\frac{1}{2\pi} \int |f(x)|^p dx \right]^{1/p}$$

is finite and the integral being extended over any interval of length 2π .

Lemma 1 (Hölder inequality (see [1], Vol. I, p. 28)). *Let $p, q \geq 1$ for $p^{-1} + q^{-1} = 1$ and $f \in L^p$, $g \in L^q$, then $f \cdot g \in L^1$ and*

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

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Lemma 2 (see [1], Vol. I, p. 190). Let $1 \leq q < \infty$ and let F be any continuous linear functional on L^q . Then there exists an essentially unique function $f \in L^p$, where $p^{-1} + q^{-1} = 1$, such that

$$F(g) = \frac{1}{2\pi} \int f(x)g(x)dx$$

for all $g \in L^q$. For any such f one has

$$\|f\|_p = \|F\| \equiv \sup\{|F(g)| : g \in L^q, \|g\|_q \leq 1\}.$$

Let $a = (a_k)$ and $b = (b_k)$ be the real sequences and $T = (\tau_{nk})$ be triangular method in the series-to-sequence form. We determine a formal trigonometric series $f^0(x) = \sum_k (a_k \cos kx + b_k \sin kx) = (a_k, b_k)$ and use notion

$$\tau_n(f(x)) := \sum_{k=1}^n \tau_{nk} (a_k \cos kx + b_k \sin kx).$$

The set of all Fourier series of functions of L^p will be denoted by \hat{L}^p . The notion $f^0(x) = (a_k, b_k)$ will be also used for the Fourier series of function $f \in L^p$.

We call a positive sequence a *rate*. Thus, $\lambda = (\lambda_n)$ is a rate, if $\lambda_n > 0$ for all $n \in \mathbb{N}$. Rates are denoted by λ and μ . The rate is called *monotonic* if $\lambda_{n+1} \geq \lambda_n$ for all n .

2. The constructive spaces $L_{T\lambda}^p$ and $\mathcal{L}_{T\lambda}^p$

Definition 1. Let $\lambda = (\lambda_n)$ be a rate and let $p \geq 1$. The set of all $f \in L^p$ for which

$$\lambda_n \|\tau_n f - f\|_p = O(1) \tag{1}$$

is called the T^λ -constructive space $L_{T\lambda}^p$.

Definition 2. Let λ be a rate and let $p \geq 1$. The set of all formal trigonometric series (a_k, b_k) for which

$$\lambda_n \|\tau_n f\|_p = O(1), \tag{2}$$

is called the T^λ -constructive space $\mathcal{L}_{T\lambda}^p$.

The space $L_{T\lambda}^p$ were introduced by the author in [4]. These spaces together with the λ -summability method of Kangro [3] were the tools applied for the investigation of the multipliers of classes $(\text{Lip}(\alpha, p), \text{Lip}(\beta, p))$, $(L^p, \text{Lip}(\alpha, p))$ etc. for $\alpha, \beta \in (0, 1)$ (see [4]).

Theorem 1. Let $p^{-1} + q^{-1} = 1$ for $p, q > 1$ and $g \in L^q$ with the Fourier series $(c_k, d_k) = \sum_{k \in \mathbb{N}} (c_k \cos kx + d_k \sin kx)$. Then $(a_k, b_k) \in \hat{\mathcal{L}}_{T\lambda}^p$ if and only if for every $g \in L^q$ the condition

$$\lambda_n \left| \sum_{k=1}^n \tau_{nk} (a_k c_k + b_k d_k) \right| = O(1) \quad (3)$$

is satisfied.

Proof. Let $(a_k, b_k) \in \hat{\mathcal{L}}_{T\lambda}^p$, then by the Parseval formula we have

$$\sum_{k=1}^n \tau_{nk} (a_k c_k + b_k d_k) = \frac{1}{2\pi} \int \tau_n f(x) g(x) dx$$

for every n . By Lemma 1 (Hölder inequality) now

$$\left| \frac{1}{2\pi} \int \tau_n f(x) g(x) dx \right| \leq \| \tau_n f \|_p \cdot \| g \|_q, \quad (4)$$

and by (2) since $f^0 \in \hat{\mathcal{L}}_{T\lambda}^p$, we have $\| \tau_n f \|_p = O(\lambda_n^{-1})$. So for any $f^0 \in \hat{\mathcal{L}}_{T\lambda}^p$ the condition (3) is fulfilled.

Let now (3) be fulfilled for every $g^0 = (c_k, d_k) \in \hat{L}^q$. Then by the Parseval formula the condition

$$\frac{\lambda_n}{2\pi} \left| \int \tau_n f(x) g(x) dx \right| = O(1) \quad (5)$$

is satisfied. By Lemma 2 we have

$$\| \tau_n(f) \|_p = \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \left| \int \tau_n f(x) g(x) dx \right|,$$

and by (5) the $f^0 \in \hat{\mathcal{L}}_{T\lambda}^p$, which completes the proof.

Theorem 2. Let $\lambda = (\lambda_n)$ be a monotonic rate and $p^{-1} + q^{-1} = 1$ for $p > 1$. Let $T = (\tau_{nk})$ be series-to-sequence matrix which for all $f \in L^p$ satisfies

$$\lim_n \| \tau_n f - f \|_p = 0. \quad (6)$$

Then $f^0 = (a_k, b_k) \in \hat{\mathcal{L}}_{T\lambda}^p$ if and only if for every $g^0 = (c_k, d_k) \in \hat{L}^q$ the

$$\lim_n \sum_{k=1}^n \tau_{nk} (a_k c_k + b_k d_k) = s \quad (7)$$

exists, and

$$\lambda_n \left| \sum_{k=1}^n \tau_{nk}(a_k c_k + b_k d_k) - s \right| = O(1). \quad (8)$$

Proof. We will show that for $f \in L^p_{T\lambda}$ and for arbitrary $g \in L^q$ the conditions (7) and (8) are satisfied. If $f \in L^p_{T\lambda}$ then also $f \in L^p$ and by Lemma 1 we have

$$\left| \frac{1}{2\pi} \int \tau_n f(x) g(x) dx - \frac{1}{2\pi} \int f(x) g(x) dx \right| \leq \|g\|_q \|\tau_n f - f\|_p. \quad (9)$$

Since $f \in L^p_{T\lambda}$ and (1) is satisfied,

$$\lim_n \sum_{k=1}^n \tau_{nk}(a_k c_k + b_k d_k) = \lim_n \frac{1}{2\pi} \int \tau_n f(x) g(x) dx = s$$

exists, and from (9) it follows that (7) and (8) are satisfied.

Let the conditions (7) and (8) be fulfilled for every $g \in L^q$. For every fixed n we form a polynomial

$$\tau_n(x) := \sum_{k=1}^n \tau_{nk}(a_k \cos kx + b_k \sin kx) \quad (10)$$

and calculate

$$\frac{1}{2\pi} \int \tau_n(x) g(x) dx = \sum_{k=1}^n \tau_{nk}(a_k c_k + b_k d_k). \quad (11)$$

By Lemma 2 we have

$$\|\tau_n(x) - \tau_{n+i}(x)\|_p = \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \left| \int (\tau_n(x) - \tau_{n+i}(x)) g(x) dx \right|.$$

By (7) and (11) from the last inequality it follows that τ_n is a Cauchy sequence in L^p . Since L^p is complete so by $\lim \tau_n(x) = f(x)$ in L^p exists. Therefore, from (11) it follows that

$$\|\tau_n f - f\|_p = \sup_{\|g\|_q \leq 1} \frac{1}{2\pi} \left| \int (\tau_n f(x) - f(x)) g(x) dx \right|.$$

Now from (7), (8) and from last equality we conclude that $f \in L^p_{T\lambda}$. This completes the proof.

The condition (6) was used only in the second part of the proof. Therefore we have the following

Corollary 1. Let $f \in L_{T\lambda}^p$, then the conditions (7) and (8) are satisfied for every $g \in L^q$.

Let E be identity method, $\lambda = (n^{\frac{1}{q}})$ and

$$\frac{1}{2} + \sum_{k=1}^n \cos kx = \frac{1}{2} D_n(x) ,$$

then $\|D_n(x)\|_p = O(\lambda_n)$ and $D_n(x) \in \mathcal{L}_{E\lambda}^p$ (see [1], Vol. I, p. 115). Let $g^0(x) = (c_k, 0) \in \hat{L}^q$. By Theorem 1 we have the following

Corollary 2. For series $\sum_{k=1}^n c_k \cos kx \in L^q$ the condition

$$\left| \sum_{k=1}^n c_k \right| = O(n^{\frac{1}{q}})$$

is satisfied.

Last corollary gives a simple asymptotic condition for Fourier coefficients, complementing so the Paley's L^p Fourier coefficients theorem ([7], Chap.12).

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