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Constructive sets of real trigonometric series

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ABSTRACT. In the present paper we consider the abstract classes of trigonometric series with complementary characteristics to L^p. The concept used will generalize the method of complementary spaces introduced by Goes [2] and Tõnnov [6] and the concept of T^λ-constructive spaces considered by the present author [4,5].

1. Introduction

We have introduced a concept of T^λ-constructive spaces into theory of trigonometric series [4] and [5]. Constructive spaces created a possibility to use the λ-summability method of Kangro [3] for the investigation of Fourier series. In the present paper we will generalize the concept of constructive spaces by introducing the conctructive-type classes of abstract trigonometric series $\mathcal{L}^p_{T\lambda}$. The results will link the classes of Fourier series and the classes $\mathcal{L}^p_{T\lambda}$ of trigonometric series with Fourier coefficients of L^p.

Throughout the paper the integral is considered taken over any interval of length 2π. So, for any real number $p \geq 1$, we denote the set of equivalence classes of real-valued measurable functions f by L^p, where

$$\| f \|_p \equiv [\frac{1}{2\pi} \int | f(x) |^p \, dx]^{1/p}$$

is finite and the integral being extended over any interval of length 2π.

Lemma 1 (Hölder inequality (see [1], Vol. I, p. 28)). *Let $p, q \geq 1$ for $p^{-1} + q^{-1} = 1$ and $f \in L^p$, $g \in L^q$, then $f \cdot g \in L^1$ and*

$$\| f \cdot g \|_1 \leqslant \| f \|_p \cdot \| g \|_q \, .$$

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Lemma 2 (see [1], Vol. I, p. 190). *Let $1 \leqslant q < \infty$ and let F be any continuous linear functional on L^q. Then there exists an essentially unique function $f \in L^p$, where $p^{-1} + q^{-1} = 1$, such that*

$$F(g) = \frac{1}{2\pi} \int f(x)g(x)dx$$

for all $g \in L^q$. For any such f one has

$$\| f \|_p = \| F \| \equiv \sup\{| F(g) | : \ g \in L^q, \ \| g \|_q \leqslant 1\}.$$

Let $a = (a_k)$ and $b = (b_k)$ be the real sequences and $T = (\tau_{nk})$ be triangular method in the series-to-sequence form. We determine a formal trigonometric series $f^0(x) = \sum_k (a_k \cos kx + b_k \sin kx) = (a_k, b_k)$ and use notion

$$\tau_n(f(x)) := \sum_{k=1}^{n} \tau_{nk}(a_k \cos kx + b_k \sin kx).$$

The set of all Fourier series of functions of L^p will be denoted by $\hat{L}^p$. The notion $f^0(x) = (a_k, b_k)$ will be also used for the Fourier series of function $f \in L^p$.

We call a positive sequence a *rate*. Thus, $\lambda = (\lambda_n)$ is a rate, if $\lambda_n > 0$ for all $n \in \mathbb{N}$. Rates are denoted by λ and μ. The rate is called *monotonic* if $\lambda_{n+1} \geq \lambda_n$ for all n.

2. The constructive spaces $L^p_{T\lambda}$ and $\mathcal{L}^p_{T\lambda}$

Definition 1. *Let $\lambda = (\lambda_n)$ be a rate and let $p \geq 1$. The set of all $f \in L^p$ for which*

$$\lambda_n \| \tau_n f - f \|_p = O(1) \tag{1}$$

is called the T^λ-constructive space $L^p_{T\lambda}$.

Definition 2. *Let λ be a rate and let $p \geq 1$. The set of all formal trigonometric series (a_k, b_k) for which*

$$\lambda_n \| \tau_n f \|_p = O(1), \tag{2}$$

is called the T^λ-constructive space $\mathcal{L}^p_{T\lambda}$.

The space $L^p_{T\lambda}$ were introduced by the author in [4]. These spaces together with the λ-summability method of Kangro [3] were the tools applied for the investigation of the multipliers of classes $(\mathrm{Lip}(\alpha, p), \mathrm{Lip}(\beta, p))$, $(L^p, \mathrm{Lip}(\alpha, p))$ etc. for $\alpha, \beta \in (0,1)$ (see [4]).

Theorem 1. *Let $p^{-1}+q^{-1}=1$ for $p,q>1$ and $g \in L^q$ with the Fourier series $(c_k,d_k)=\sum_{k\in\mathbb{N}}(c_k\cos kx+d_k\sin kx)$. Then $(a_k,b_k)\in\hat{\mathcal{L}}^p_{T\lambda}$ if and only if for every $g\in L^q$ the condition*

$$\lambda_n \mid \sum_{k=1}^{n}\tau_{nk}(a_kc_k+b_kd_k)\mid = O(1) \tag{3}$$

is satisfied.

Proof. Let $(a_k,b_k)\in\hat{\mathcal{L}}^p_{T\lambda}$, then by the Parseval formula we have

$$\sum_{k=1}^{n}\tau_{nk}(a_kc_k+b_kd_k)=\frac{1}{2\pi}\int\tau_nf(x)g(x)dx$$

for every n. By Lemma 1 (Hölder inequality) now

$$\mid\frac{1}{2\pi}\int\tau_nf(x)g(x)dx\mid\leqslant\parallel\tau_nf\parallel_p\cdot\parallel g\parallel_q, \tag{4}$$

and by (2) since $f^0\in\hat{\mathcal{L}}^p_{T\lambda}$, we have $\parallel\tau_nf\parallel_p=O(\lambda_n^{-1})$. So for any $f^0\in\hat{\mathcal{L}}^p_{T\lambda}$ the condition (3) is fulfilled.

Let now (3) be fulfilled for every $g^0=(c_k,d_k)\in\hat{L}^q$. Then by the Parseval formula the condition

$$\frac{\lambda_n}{2\pi}\mid\int\tau_nf(x)g(x)dx\mid=O(1) \tag{5}$$

is satisfied. By Lemma 2 we have

$$\parallel\tau_n(f)\parallel_p=\sup_{\|g\|_p\leqslant 1}\frac{1}{2\pi}\mid\int\tau_nf(x)g(x)dx\mid,$$

and by (5) the $f^0\in\hat{\mathcal{L}}^p_{T\lambda}$, which completes the proof.

Theorem 2. *Let $\lambda=(\lambda_n)$ be a monotonic rate and $p^{-1}+q^{-1}=1$ for $p>1$. Let $T=(\tau_{nk})$ be series-to-sequence matrix which for all $f\in L^p$ satisfies*

$$\lim_n\parallel\tau_nf-f\parallel_p=0. \tag{6}$$

Then $f^0=(a_k,b_k)\in\hat{L}^p_{T\lambda}$ if and only if for every $g^0=(c_k,d_k)\in\hat{L}^q$ the

$$\lim_n\sum_{k=1}^{n}\tau_{nk}(a_kc_k+b_kd_k)=s \tag{7}$$

exists, and

$$\lambda_n \mid \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) - s \mid = O(1). \tag{8}$$

Proof. We will show that for $f \in L^p_{T\lambda}$ and for arbitrary $g \in L^q$ the conditions (7) and (8) are satisfied. If $f \in L^p_{T\lambda}$ then also $f \in L^p$ and by Lemma 1 we have

$$\left|\frac{1}{2\pi}\int \tau_n f(x)g(x)dx - \frac{1}{2\pi}\int f(x)g(x)dx\right| \leqslant \| g \|_q \| \tau_n f - f \|_p . \tag{9}$$

Since $f \in L^p_{T\lambda}$ and (1) is satisfied,

$$\lim_n \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k) = \lim_n \frac{1}{2\pi}\int \tau_n f(x)g(x)dx = s$$

exists, and from (9) it follows that (7) and (8) are satisfied.

Let the conditions (7) and (8) be fulfilled for every $g \in L^q$. For every fixed n we form a polynomial

$$\tau_n(x) := \sum_{k=1}^{n} \tau_{nk}(a_k \cos kx + b_k \sin kx) \tag{10}$$

and calculate

$$\frac{1}{2\pi}\int \tau_n(x)g(x)dx = \sum_{k=1}^{n} \tau_{nk}(a_k c_k + b_k d_k). \tag{11}$$

By Lemma 2 we have

$$\| \tau_n(x) - \tau_{n+i}(x) \|_p = \sup_{\|g\|_p \leqslant 1} \frac{1}{2\pi}\left|\int (\tau_n(x) - \tau_{n+i}(x))g(x)dx\right|.$$

By (7) and (11) from the last inequality it follows that τ_n is a Cauchy sequence in L^p. Since L^p is complete so by $\lim \tau_n(x) = f(x)$ in L^p exists. Therefore, from (11) it follows that

$$\| \tau_n f - f \|_p = \sup_{\|g\|_q \leqslant 1} \frac{1}{2\pi}\left|\int (\tau_n f(x) - f(x))g(x)dx\right|.$$

Now from (7), (8) and from last equality we conclude that $f \in L^p_{T\lambda}$. This completes the proof.

The condition (6) was used only in the second part of the proof. Therefore we have the following

Corollary 1. *Let* $f \in L^p_{T\lambda}$, *then the conditions* (7) *and* (8) *are satisfied for every* $g \in L^q$.

Let E be identity method, $\lambda = (n^{\frac{1}{q}})$ and

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{1}{2} D_n(x) \quad ,$$

then $\| D_n(x) \|_p = O(\lambda_n)$ and $D_n(x) \in \mathcal{L}^p_{E\lambda}$ (see [1], Vol. I, p. 115). Let $g^0(x) = (c_k, 0) \in \hat{L}^q$. By Theorem 1 we have the following

Corollary 2. *For series* $\sum_{k=1}^{n} c_k \cos kx \in L^q$ *the condition*

$$|\sum_{k=1}^{n} c_k| = O(n^{\frac{1}{q}})$$

is satisfied.

Last corollary gives a simple asymptotic condition for Fourier coefficients, complementing so the Paley's L^p Fourier coefficients theorem ([7], Chap.12).

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