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Knopp's core in topological vector spaces

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ABSTRACT. The purpose of the present paper is to investigate the geometry of Knopp's core in locally convex topological vector spaces.

Let E be a Hausdorff locally convex topological vector space and let E' be its topological dual. We denote sequences in E by $x = (\xi_n)$; i.e. $\xi_n \in E$ for all $n \in \mathbb{N}$.

Let $E_n(x) = (\xi_n, \xi_{n+1}, \ldots)$ and let $R_n(x)$ be the closure of the convex hull of $E_n(x)$ in E, i.e.

$$R_n(x) = \operatorname{cl}\operatorname{conv} E_n(x).$$

Definition. *The intersection*

$$K^o(x) = \bigcap_{n=1}^{\infty} R_n(x)$$

is called Knopp's core of the sequence $x = (\xi_n)$ (see [1,2]).

It is known that sequences in $\mathbb{R}$ or $\mathbb{C}$ have following properties concerning Knopp's core (see [1], Ch. VI).

- A sequence is convergent if and only if its core is a singleton.
- A bounded sequence has nonempty core.
- For an arbitrary convex bounded and closed set K there exists a sequence $x = (\xi_n)$ that has the set K for its core, i.e.

$$K = K^o(x).$$

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The goal of the paper is to show that these properties are not valid for the sequence in an arbitrary space E.

Example 1. There exists an unbounded sequence that has a singleton for its core. Let $E = l_2$ and $x = (\xi_n)$, where

$$\xi_n = \begin{cases} ne_n, & \text{if } n = 2p+1, \ p = 0, 1, \dots \\ 0, & \text{if } n = 2p, \ p = 1, 2, \dots \end{cases}$$

Here $e_n = (\overbrace{0, \dots, 0}^{n-1}, 1, 0, \dots)$, and $0 = (0, 0, \dots)$. This sequence $x = (\xi_n)$ is not bounded in l_2. It is evident that

$$K^o(x) = \{0\},$$

therefore x is a nonconvergent sequence in l_2 where its core is a singleton.

The following obvious equation

$$K^o(x) = \{\xi \in \mathbb{R} : \ \alpha\xi \leq \limsup_n \alpha\xi_n, \ \forall \alpha \in \mathbb{R}\} \tag{1}$$

is frequently given as a definition of Knopp's core in $\mathbb{R}$. This equation can be treated as a special case ($E = \mathbb{R}$) of the following result.

Theorem 2. *If $x = (\xi_n)$ is a sequence in a real space E, then*

$$K^o(x) = \{\xi \in E : \ f(\xi) \leq \limsup_n f(\xi_n), \ \forall f \in E'\}. \tag{2}$$

Proof. We start with the observation that if B is a nonempty set in E, then

$$\operatorname{cl} \operatorname{conv} B = \{\xi \in E : \ f(\xi) \leq \sup_{\eta \in B} f(\eta) \ \forall f \in E'\}$$

(see [3], Ch. I, §6). Thus, we have

$$R_n(x) = \{\xi \in E : \ f(\xi) \leq \sup_{k \geq n} f(\xi_k) \ \forall f \in E'\}. \tag{3}$$

If $\xi \in K^o(x)$ then $\xi \in R_n(x)$ for every n and consequently by (3)

$$f(\xi) \leq \limsup_n f(\xi_n) \ \forall f \in E'$$

and therefore (2) is valid. □

Corollary 3. *If $x = (\xi_n)$ is a sequence in a real space E, then*

$$K^o(x) = \bigcap_{f \in E'} \{\xi \in E : \ f(\xi) \in K^o((f(\xi_n)))\}.$$

Corollary 4. *If a sequence $x = (\xi_n) \subset E$ is weakly convergent, then its Knopp's core is a singleton.*

We shall now take E to be a normed space. Let $m^\sharp(E)$ be the set of all such sequences in E which cores are bounded and nonempty. Let $m(E)$ denote the space of all bounded sequences in E, i.e.

$$m(E) = \{x = (\xi_n) \subset E : \ \sup_n \| \xi_n \| < \infty\}.$$

For $\mathbb{R}$ we have that $m^\sharp(\mathbb{R}) = m(\mathbb{R})$ and by using Corollary 2, we get that if E is finite-dimensional, then

$$m^\sharp(E) = m(E).$$

In general case of E the last equality is not true (see Example 1).

Example 5. There exists a bounded sequence that has the empty core. Let $E = c$, and let $\xi_n = \sum_{i=1}^n e_i$. The sequence $x = (\xi_n)$ is bounded, i.e. $x = (\xi_n) \in m(E)$. The core $K^o(x)$ is empty.

Proposition 6. *Let E be a Banach space. If E is reflexive, then*

$$m(E) \subset m^\sharp(E).$$

Proof. According to the definition of Knopp's core a bounded sequence $x = (\xi_n)$ has a bounded core. If E is reflexive, then every $R_n(x)$ is weakly compact and therefore this core is not empty. This proves the proposition. □

Proposition 7. *For every convex and compact set K in a Banach space E there exists a sequence $x = (\xi_n) \subset E$ such that*

$$K^o(x) = K.$$

Proof. Due to the compactness of K, for every $n \in \mathbb{N}$ there exists a finite set $\{\xi_{n1}, \xi_{n2}, ..., \xi_{nk_n}\}$ such that

$$K \subset \bigcup_{i=1}^{k_n} B(\xi_{ni}, \frac{1}{n}),$$

where $B(\xi, r) = \{\eta \in E : \| \xi - \eta \| < r\}$.

We will choose these finite sets so that for every ξ_{ni} there exists $\eta \in K$ such that

$$\| \xi_{ni} - \eta \| < \frac{1}{n}. \tag{4}$$

Let x be the sequence

$$x = (\xi_{11}, ..., \xi_{1k_1}, \xi_{21}, ..., \xi_{2k_2}, ...).$$

It follows directly from the construction of x that

$$K \subset K^o(x).$$

We show next that

$$K^o(x) \subset K.$$

Let

$$K_n = \text{cl} \bigcup_{\eta \in K} B(\eta, \frac{1}{n}).$$

Since K is convex, K_n is convex. A straightforward verification shows that

$$K = \bigcap_{n=1}^{\infty} K_n.$$

Let

$$E_{ni}(x) = (\xi_{ni}, \xi_{ni+1}, ..., \xi_{nk_n}, \xi_{n+1,1}, ..., \xi_{n+1,k_{n+1}}, ...),$$

and

$$R_{ni}(x) = \text{cl conv}\, E_{ni}(x),$$

then

$$K^o(x) = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{k_n} R_{ni}(x).$$

On account of (4)

$$E_{ni}(x) \subset K_n \quad \forall i = 1, ..., k_n.$$

Since K_n is closed and convex,

$$R_{ni}(x) \subset K_n \quad \forall i = 1, ..., k_n$$

and

$$\bigcap_{i=1}^{k_n} R_{ni}(x) \subset K_n.$$

This gives

$$K^o(x) \subset \bigcap_{n=1}^{\infty} K_n = K.$$

□

Theorem 8. *A normed space E admits a sequence with ball-shape core if and only if this space is separable.*

Proof. Necessity. There is no loss of generality in assuming that there exists a sequence $x = (\xi_n)$ such that

$$K^o(x) = \mathrm{cl}\, B(0,1).$$

If z is an arbitrary nonzero element in E, then

$$\frac{z}{\| z \|} \in \mathrm{cl}\, B(0;1) \subset R_n(x) \quad \forall n \in \mathbb{N}. \tag{5}$$

Furthermore,

$$R_n(x) = \mathrm{cl\ conv}\, (\xi_n, \xi_{n+1}, ...) \subset \mathrm{cl\ span}\, (\xi_n, \xi_{n+1}, ...).$$

The set

$$L = \mathrm{cl\ span}\, (\xi_n, \xi_{n+1}, ...)$$

is separable. It follows from (5) that $z \in L$, i.e. $E = L$, and consequently E is separable.

Sufficiency. Let $y = (\eta_n)$ be a sequence of elements from E that is dense in $B(0,1)$. The sequence

$$x = (\eta_1, \eta_1, \eta_2, \eta_1, \eta_2, \eta_3, \eta_1, ..., \eta_{k-1}, \eta_1, \eta_2, ..., \eta_k, \eta_1, \eta_2, ..., \eta_{k+1}, \eta_1, ...)$$

has *cl* $B(0,1)$ for its core. □

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