ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 3, 1999

# On the generalized Cesàro summability factors

Hikmet Seyhan

Abstract. In this paper, an extension of a result of Bor on $| C, 1 |_k$ summability is proved.

## 1. Definitions

Let $\sum a_n = \sum_{n=0}^{\infty} a_n$ be an infinite series with partial sums $(s_n)$. We denote by $u_n^\alpha$ and $t_n^\alpha$ the Cesàro means of order $\alpha$, with $\alpha > -1$, of the sequences $(s_n)$ and $(na_n)$, respectively, i.e.

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v.$$

Let $(\psi_n)$ be a sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\psi$-$| C, \alpha; \delta |_k$, $k \geq 1$, $\alpha > -1$ and $\delta \geq 0$, if

$$\sum_{n=1}^{\infty} \psi_n^{\delta k + k - 1} | u_n^\alpha - u_{n-1}^\alpha |^k < \infty. \tag{1}$$

But since $t_n^\alpha = n(u_n^\alpha - u_{n-1}^\alpha)$ (see [4]), condition (1) can also be written as

$$\sum_{n=1}^{\infty} \psi_n^{\delta k + k - 1} n^{-k} | t_n^\alpha |^k < \infty.$$

If we take $\delta = 0$ and $\psi_n = n$ (resp. $\delta = 0$, $\alpha = 1$ and $\psi_n = n$), then $\psi$-$| C, \alpha; \delta |_k$ summability is the same as $| C, \alpha |_k$ (resp. $| C, 1 |_k$) summability (see [3]).

Received June 21, 1996; revised December 2, 1997 and March 25, 1999.
1991 *Mathematics Subject Classification.* 40D15, 40F05, 40G05.
*Key words and phrases.* Absolute summability factors, Cesàro methods of summability.

## 2. Main result

The following theorem is known.

**Theorem A** ([1]). *Let $(X_n)$ be a positive non-decreasing sequence and let there be sequences $(\beta_n)$ and $(\lambda_n)$ such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{2}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{3}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{4}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{5}$$

*If*

$$\sum_{n=1}^{m} \frac{1}{n} \mid t_n \mid^k = O(X_m) \quad as \quad m \to \infty,$$

*then the series $\sum a_n \lambda_n$ is summable $\mid C, 1 \mid_k$, $k \geq 1$.*

The aim of this paper is to prove the following extension of Theorem A.

**Theorem B.** *Let $k \geq 1$, $\delta \geq 0$, $0 < \alpha \leq 1$ and $\alpha k + \epsilon > 1$. Let $(X_n)$ be a positive non-decreasing sequence and the sequences $(\beta_n)$ and $(\lambda_n)$ such that conditions (2)–(5) of Theorem A are satisfied. If there exists an $\varepsilon > 0$ such that the sequence $(n^{\varepsilon-k}\psi_n^{\delta k+k-1})$ is non-increasing and the sequence $(w_n^\alpha)$, defined by*

$$w_n^\alpha = \begin{cases} \mid t_n^\alpha \mid, & \alpha = 1, \\ \max\limits_{1 \leq v \leq n} \mid t_v^\alpha \mid, & 0 < \alpha < 1, \end{cases}$$

*satisfies the condition*

$$\sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} (w_n^\alpha)^k = O(X_m) \quad as \quad m \to \infty,$$

*then the series $\sum a_n \lambda_n$ is summable $\psi$-$\mid C, \alpha; \delta \mid_k$.*

If we take $\delta = 0$, $\varepsilon = 1$, $\alpha = 1$ and $\psi_n = n$ in Theorem B, then we get Theorem A.

## 3. Proof of the main result

We need the following lemmas for the proof of our theorem.

**Lemma 1** ([2]). *If* $0 < \alpha \leq 1$ *and* $1 \leq v \leq n$, *then*

$$\left| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p \right| .$$

**Lemma 2** ([5]). *If conditions* (2)–(5) *on* $(X_n)$, $(\beta_n)$ *and* $(\lambda_n)$ *are satisfied, then* $\sum_{n=1}^{\infty} \beta_n X_n < \infty$ *and* $n\beta_n X_n = O(1)$ *as* $n \to \infty$.

*Proof of Theorem B.* Let $0 < \alpha \leq 1$ and let $(T_n^\alpha)$ be $(C, \alpha)$ means of the sequence $(na_n\lambda_n)$. Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v.$$

So by Lemma 1, we have

$$\begin{aligned} | T_n^\alpha | &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} | \Delta\lambda_v || \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p | + \frac{| \lambda_n |}{A_n^\alpha} | \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v | \\ &\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha | \Delta\lambda_v | + | \lambda_n | w_n^\alpha = T_{n,1}^\alpha + T_{n,2}^\alpha \end{aligned}$$

and we will proceed further by using the inequality

$$| T_{n,1}^\alpha + T_{n,2}^\alpha |^k \leq 2^k (| T_{n,1}^\alpha |^k + | T_{n,2}^\alpha |^k).$$

Now, when $k > 1$, applying Hölder's inequality, we get

$$\begin{aligned} \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} | T_{n,1}^\alpha |^k &= \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} \left| \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha | \Delta\lambda_v | \right|^k \\ &\leq \sum_{n=2}^{m+1} \psi_n^{\delta k+k-1} n^{-k} (A_n^\alpha)^{-k} \sum_{v=1}^{n-1} (A_v^\alpha)^k (w_v^\alpha)^k \beta_v \times \{\sum_{v=1}^{n-1} \beta_v\}^{k-1} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \beta_v \sum_{n=v+1}^{m+1} \frac{\psi_n^{\delta k+k-1} n^{\varepsilon-k}}{n^{\alpha k+\varepsilon}} \\ &= O(1) \sum_{v=1}^{m} v^{\alpha k} (w_v^\alpha)^k \beta_v v^{\varepsilon-k} \psi_v^{\delta k+k-1} \int_v^\infty \frac{dx}{x^{\alpha k+\varepsilon}} \\ &= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} r^{-k} (w_r^\alpha)^k \psi_r^{\delta k+k-1} \end{aligned}$$

$$+ O(1) m\beta_m \sum_{v=1}^{m} v^{-k} (w_v^{\alpha})^k \psi_v^{\delta k+k-1}$$

$$= O(1) \sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1) m\beta_m X_m$$

$$= O(1) \sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1) \sum_{v=1}^{m-1} \mid \beta_{v+1} \mid X_v + O(1) m\beta_m X_m$$

$$= O(1),$$

by virtue of the hypotheses of Theorem B and Lemma 2.

Again, since $\mid \lambda_n \mid = O(1/X_n) = O(1)$, by (5), we have

$$\sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid T_{n,2}^{\alpha} \mid^k = \sum_{n=1}^{m} \psi_n^{\delta k+k-1} n^{-k} \mid\mid \lambda_n \mid w_n^{\alpha} \mid^k$$

$$= O(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} \psi_v^{\delta k+k-1} v^{-k} (w_v^{\alpha})^k$$

$$+ O(1) \mid \lambda_m \mid \sum_{v=1}^{m} \psi_v^{\delta k+k-1} v^{-k} (w_v^{\alpha})^k$$

$$= O(1) \sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m$$

$$= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) \mid \lambda_m \mid X_m = O(1),$$

by virtue of the hypotheses of Theorem B and Lemma 2.

We note that for $k = 1$ the proof of Theorem B is trivial.

## References


1. H. Bor, *On the absolute Cesàro summability factors*, Anal. Numer. Theor. Approx., **20** (1991), 11–14.
2. S. Bosanquet, *A mean value theorem*, J. London Math. Soc. **16** (1941), 146–148.
3. T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7** (1957), 113–141.
4. E. Kogbetliantz, *Sur les séries absolument sommables par la méthode des moyannes arithmétiques*, Bull. Sci. Math. **49** (1925), 234–256.
5. K. N. Mishra and R. S. L. Srivastava, *On absolute Cesàro summability factors of infinite series*, Portugal. Math. **42** (1983/1984), 53–61.



Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey

*E-mail address*: seyhan@zirve.ericiyes.edu.tr