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Complete proofs of the main theorems on summability factors

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ABSTRACT. In this article the shortest proofs up to the present of the main theorems on summability factors for Cesàro methods are given. The necessity part of these theorems is proved for more general methods. Especially comparatively short is the proof of the sufficiency part of the theorem for summability factors of types $(C_O^\alpha, |C^\beta|)$ and $(C^\alpha, |C^\beta|)$.

1. Introduction

Let $\varepsilon = (\varepsilon_n)$ be a sequence of complex numbers. The numbers ε_n are called summability factors of type (A, B) (respectively (A_O, B) or $(|A|, B)$), if for each A-summable (respectively A-bounded or absolutely A-summable) series

$$\sum u_n \tag{1.1}$$

the series

$$\sum \varepsilon_n u_n$$

is B-summable. Briefly, we write $\varepsilon \in (A, B)$ (respectively $\varepsilon \in (A_O, B)$ or $\varepsilon \in (|A|, B)$). Summability factors of the types $(A_O, |B|)$, $(A, |B|)$ and $(|A|, |B|)$ are defined analogously.

Let C^α be the Cesàro method of order $\alpha \geq 0$. The series (1.1) is called C^α-summable if

$$\lim_n \left(\sum_{k=0}^{n} A_{n-k}^\alpha u_k / A_n^\alpha \right)$$

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exists and absolutely C^α-summable, or $|C^\alpha|$-summable, if

$$\sum_{n=1}^{\infty}(nA_n^\alpha)^{-1}\left|\sum_{k=1}^{n}A_{n-k}^{\alpha-1}ku_k\right|<\infty.$$

Denote

$$A_n^\lambda := \binom{n+\lambda}{n},$$

$$\Delta^\lambda\varepsilon_n := \Delta_n^\lambda\varepsilon_n := \sum_{\nu=n}^{\infty}A_{\nu-n}^{-\lambda-1}\varepsilon_\nu, \qquad \Delta\varepsilon_n := \Delta^1\varepsilon_n = \varepsilon_n - \varepsilon_{n+1},$$

if the last series converges. Unless otherwise indicated, the summation at $\sum$ is likewise over $0, 1, 2, \dots$ and for the symbols O and o over $1, 2, \dots$.

In the present paper complete proofs of the following theorems are given, where $\alpha, \beta \geq 0$.

Theorem 1.1. *In order that $\varepsilon \in (|C^\alpha|, C^\beta)$ or $\varepsilon \in (|C^\alpha|, |C^\beta|)$, it is necessary and sufficient that*

$$\varepsilon_n = O(1), \tag{1.2}$$

$$\varepsilon_n = O(n^{\beta-\alpha}), \tag{1.3}$$

$$\Delta^\alpha\varepsilon_n = O(n^{-\alpha}) \tag{1.4}$$

be satisfied.

Theorem 1.2. *In order that $\varepsilon \in (C^\alpha, C^\beta)$, it is necessary and sufficient that* (1.2), (1.3) *and*

$$\sum(n+1)^\alpha|\Delta^{\alpha+1}\varepsilon_n| < \infty \tag{1.5}$$

be satisfied.

Theorem 1.3. *In order that $\varepsilon \in (C_O^\alpha, C^\beta)$, it is necessary and sufficient that* (1.5) *and*

$$\varepsilon_n = o(1), \tag{1.6}$$

$$\varepsilon_n = o(n^{\beta-\alpha}) \tag{1.7}$$

be satisfied.

Theorem 1.4. *In order that $\varepsilon \in (C^\alpha_O, |C^\beta|)$ or $\varepsilon \in (C^\alpha, |C^\beta|)$, it is necessary and sufficient that* (1.5) *and*

$$\sum \frac{|\varepsilon_n|}{n+1} < \infty, \tag{1.8}$$

$$\sum (n+1)^{\alpha-\beta} |\varepsilon_n| < \infty \tag{1.9}$$

be satisfied.

It should be noted that these theorems are given in the above order since the conditions of the previous theorem are also necessary for the summability factors in the following theorem.

The theory of summability factors originated in the beginning of the 20th century when the Theorem of Dedekind and Hadamard [20] (that is the case $\beta = \alpha = 0$ of Theorems 1.2 and 1.3) as well as first generalizations were published. After the Dedekind–Hadamard Theorem, various authors dealt with the case of series, summable by the Cesàro method of integer order.

The first important generalizations of the Dedekind–Hadamard Theorem were obtained in 1907–1909 by Bohr [10, 11], Hardy [22, 23] (the case $\beta = \alpha$ in Theorems 1.2 and 1.3) and Bromwich [16] (the case $\beta = 0$ of Theorem 1.2). Note that they as well as Chapman [17] and Andersen [2, 3] found only sufficient conditions, replacing condition (1.3) by (1.7). A proof of the Bohr–Hardy Theorem is also brought in the famous book of Hardy ([24], Theorem 71). The necessity of the conditions of the Bohr–Hardy Theorem were proved in 1916 by Fekete [19], and of the Bromwich Theorem in 1917 by Kojima [29]. For arbitrary real $\alpha \geq 0$ the Bromwich Theorem was generalized by Chapman [17] in 1910 and the Bohr–Hardy Theorem was generalized by Andersen ([2], pp. 45–53). Another proof of the Bohr–Hardy Theorem for any $\alpha \geq 0$ was given by Andersen [3] in 1926 and Bosanquet [12] in 1942, which also proves the necessity of the conditions. A recent, beautiful proof of the Bohr–Hardy Theorem was given by Lorentz and Zeller [31].

For arbitrary real $\alpha, \beta \geq 0$, Theorems 1.2 and 1.3 were formulated in 1918 by Schur [41]. Omitting their proof, Schur stated that it "requires enough difficult calculations", although the conditions are simple. The complete proof of Theorems 1.2 and 1.3 for arbitrary $\alpha, \beta \geq 0$ was given by Bosanquet [14] in 1946. Independently, Knopp [27] proved this theorem, however, he assumed $\alpha \geq 0$ to be integer.

Theorem 1.1 for $\varepsilon \in (|C^\alpha|, |C^\beta|)$, was proved by Fekete [19] in the case of integer $\beta = \alpha$, by Bosanquet [13] in 1944 for integer $\alpha, \beta \geq 0$, by Andersen [4] (only the sufficiency), and by Peyerimhoff [36] in 1953 for arbitrary $\alpha, \beta \geq 0$.

Theorem 1.4 for the cases $\beta = \alpha$ and $\beta = \alpha + 1$ with integer $\alpha \geq 0$ was given by Fekete [19] in 1916 and for any integer $\alpha, \beta \geq 0$ by Bosanquet [13] in

1944 for $\beta \leq \alpha+1$ and by Tatchell [42] in 1953 for $\beta > \alpha+1$. The complete proof of Theorem 1.4 for any real $\alpha, \beta \geq 0$ was given by Peyerimhoff [38] in 1956.

Chow [18] in 1952 also proved two theorems similar to Theorem 1.1 and 1.4 with other conditions. Bosanquet and Chow [15] in 1957 showed that the Chow Theorems are equivalent to Theorems 1.1 and 1.4, although Chow considers a type of summability factors different than those in Theorem 1.4. We note that these Theorems are proved [8, 9] also for negative $\beta > -1$. In case of integer $\alpha \geq 0$ Volkov [43, 44] extends Theorem 1.2 for complex β with Re $\beta = \alpha$. Moore (see [32], p. 46, [6], p. 192) generalizes the Bromwich Theorem for complex α with Re $\alpha > 0$, and Volkov (see [43], p. 162) considers also the case of imaginary α with Re $\alpha = 0$. Abel [1] generalizes Theorems 1.1 – 1.4 for complex β and Theorems 1.1 and 1.4 even for complex α and β with Re $\alpha > 0$ and $-1 <$ Re $\beta < 0$. However in the present paper the cases of negative or complex α and β are not dealt with.

We mention the methods employed by various authors for the proofs of Theorems 1.1 – 1.4. We stress three methods: a) the direct method; b) the method of inverse transformation; c) the method of functional analysis.

The direct method is the same as that used by Hardy [22, 23], Bohr [11], Bromwich [16], Chapman [17], Fekete [19], Andersen [2, 3, 4], Bosanquet [12, 13, 14].

The method of inverse transformation for the Cesàro method of summability was used by Knopp [27], Chow [18] and Peyerimhoff [36]. This method was already indicated by Schur ([41], p. 106). The basic idea of it is that by means of the inverse matrix, the problem of finding summability factors reduces to the investigation of certain matrix transformation. By applying to this transformation the corresponding theorem for $\ell \to c$, $\ell \to \ell$, $c \to c$, $m \to c$, $c \to \ell$ or $m \to \ell$, we find necessary and sufficient conditions for the corresponding summability factors ε_n. However, the derived conditions are difficult to verify in practice, they are not effective. But effective necessary conditions are deduced from them and from the latter, in turn, the mentioned non-effective conditions are deduced.

The method of functional analysis in order to prove theorems on summability factors was developed by Peyerimhoff [34, 35]. In 1951 (see [34], p. 29, cf. [6], p. 213), he proved that for $\varepsilon \in (A, B)$ is necessary the existence of a continuous linear functional f in the summability field of the method A, satisfying the functional condition

$$\Delta\varepsilon_n = fe_n, \tag{1.10}$$

where $e_n = (\delta_{nk})$, provided A and B are regular methods. As Kangro mentions ([26], p. 12), the problem arises which additional conditions should be added to the necessity condition (1.10) in order to obtain sufficiency conditions for $\varepsilon \in (A, B)$. If A is a reversible method with the sequence-to-sequence transformation matrix (a_{nk}), then the functional condition (1.10) is equivalent to (cf. [34], p. 31, [40], p. 39, [6], p. 215)

$$\Delta\varepsilon_k = \sum_n c_n a_{nk}, \quad \sum |c_n| < \infty. \tag{1.11}$$

Similar conditions for absolute summability were also found by Peyerimhoff ([35], p. 282). The method of functional analysis is especially suitable for $B = A$ and $B = E$, where $E = C^0$ is the convergence method, as is obvious from the results of Jurkat and Peyerimhoff [34, 35, 25] (cf. also [6], p. 215–217).

We now return to the method of inverse transformation. It should be pointed out that each author utilizes this method differently and at the same time creates his own method of proof for summability factors theorems. The simplest of these methods is that of Knopp [27], whereby he proves Theorems 1.2 and 1.3. However, Knopp has to restrict himself to integer $\alpha \geq 0$, since the formula for difference of product of sequences

$$\Delta^\alpha (x_n \varepsilon_n) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \Delta^{\alpha-i} x_{n+i} \cdot \Delta^i \varepsilon_n \tag{1.12}$$

applied by him is not valid for noninteger α.

In 1955 Peyerimhoff [37, 38] found a new method of proof for Theorems 1.1, 1.2 and 1.4. In [38] Peyerimhoff gave generalizations of the formula (1.12) for nonintegers $\alpha \geq 0$ by adding remainder terms to (1.12). The formulas of Peyerimhoff are very important also for finding absolute summability factors in a sequence [7], in particular, not only to prove the sufficiency but the necessity as well. Peyerimhoff [37], applying the transpose of the matrix (a_{nk}), obtained theorems, equivalent to Theorems 1.1, 1.2 and 1.4, by means of which he proves these Theorems.

Chow [18] in 1954 and Baron [5] in 1960 extended (1.12) by "cutting off" the noninteger part of α. However, Baron [5] mainly proves the sufficiency in Theorems 1.1 – 1.4 for $0 \leq \beta \leq \alpha$ only.

In the present paper, complete proofs of Theorems 1.1 – 1.4 are given, applying for the necessity part the arguments of both [37] and [34] for arbitrary normal matrix methods A and B, while the arguments of [5] for the sufficiency part. The main aim of this work is to represent in one paper the today shortest proofs of the necessity and sufficient parts of Theorems 1.1 – 1.4. The sufficiency in Theorem 1.4 is of special difficulty; its proof (cf. [9]) is considerably shorter than that in [5]. Also in the proof of the sufficiency in Theorem 1.1 we supply in [5] some absent parts. Improved are also the proofs of Theorems 1.2 and 1.3.

2. Necessity conditions

We will consider matrix tranformations

$$y_n = \sum_{k=0}^{n} a_{nk} x_k, \tag{2.1}$$

where x_n, y_n, a_{nk} are complex numbers, and $A = (a_{nk})$ is a normal matrix, i.e., triangular with $a_{nn} \neq 0$. Then the transformation (2.1) possesses an inverse transformation, denote it by

$$x_n = \sum_{\nu=0}^{n} a'_{n\nu} y_\nu. \tag{2.2}$$

Let $B = (b_{nk})$ be a triangular matrix with complex entries and let

$$z_n = \sum_{k=0}^{n} b_{nk} \varepsilon_k x_k. \tag{2.3}$$

By substituting (2.2) in (2.3) we obtain

$$z_n = \sum_{\nu=0}^{n} g_{n\nu} y_\nu, \tag{2.4}$$

where

$$g_{n\nu} = \sum_{k=\nu}^{n} b_{nk} a'_{k\nu} \varepsilon_k. \tag{2.5}$$

Let, further, A and B be normal matrix methods, $G := (g_{nk})$.

Lemma 2.1. *Let A preserve the absolute convergence and B be regular. Then for $\varepsilon \in (|A|, B_O)$ the conditions* (1.2) *and*

$$\varepsilon_n = O(a_{nn}/b_{nn}) \tag{2.6}$$

are necessary.

Proof. Let A be in series-to-series form and B in series-to-sequence form. By applying Theorem of Hahn ([21], p. 29, see also [35], p. 269, [6], p. 30) for $G : \ell \to m$ to the transformation (2.4) we obtain

$$g_{n\nu} = O(1). \tag{2.7}$$

Since $g_{nn} = b_{nn} a'_{nn} \varepsilon_n = b_{nn} \varepsilon_n / a_{nn}$, condition (2.6) follows from (2.7). Observe that $\sum_k \delta_{k\mu}$ is absolutely convergent for each μ, and therefore, it is

$|A|$-summable. Setting $x_k = \delta_{k\mu}$ in (2.1) and (2.3) we obtain $y_\nu = a_{\nu\mu}$ and $z_n = b_{n\mu}\varepsilon_\mu$, respectively. Putting these in (2.4) we obtain

$$b_{n\mu}\varepsilon_\mu = \sum_{\nu=0}^{n} g_{n\nu}a_{\nu\mu} = \sum_{\nu=\mu}^{n} g_{n\nu}a_{\nu\mu}. \tag{2.8}$$

Since A preserves absolute convergence, by Theorem of Knopp–Lorentz ([28], p. 11, [6], p. 34) for $A : \ell \to \ell$ we deduce from (2.8) by (2.7)

$$|b_{n\mu}\varepsilon_\mu| = O(1)\sum_{\nu=\mu}^{n} |a_{\nu\mu}| = O(1), \tag{2.9}$$

and since B is regular and consequently

$$b_{nk} = O(1), \quad \lim_n b_{nk} = 1 \tag{2.10}$$

(see [21], p. 33, [6], p. 20), then (2.9) yields (1.2).

Otherwise, it is possible to prove the necessity of (1.2) in the following way. Since A preserves absolute convergence, then the conditions for $\varepsilon \in (|E|, B_O)$ are necessary also for $\varepsilon \in (|A|, B_O)$. For E we have $a'_{k\nu} = \delta_{k\nu}$ in (2.2) and therefore, $y_\nu = x_\nu$ and $g_{n\nu} = b_{n\nu}\varepsilon_\nu$ in (2.4). Applying right now above-mentioned Theorem of Hahn we obtain (2.7), from which (1.2) immediately follows by (2.10). □

Lemma 2.2. *If A is in series-to-series form and preserves absolute convergence and B is regular, then for $\varepsilon \in (|A|, B)$ the condition*

$$\sum_{k=\nu}^{\infty} a'_{k\nu}\varepsilon_k = O(1) \tag{2.11}$$

is necessary provided the series converges absolutely for every ν.

Proof. Let B be in series-to-sequence form. Applying Theorem of Hahn ([21], p. 29, [6], p. 25) for $G : \ell \to c$ to the transformation (2.4), we obtain that (2.7) and the existence of

$$g_\nu := \lim_n g_{n\nu} \tag{2.12}$$

are necessary and sufficient for $\varepsilon \in (|A|, B)$. ¿From (2.7), (2.5) and (2.12), by (2.10) we obtain (2.11), since B is regular and g_ν is the absolutely convergent series in (2.11). □

Lemma 2.3. *If A is in series-to-sequence form, A preserves the convergence and B is regular, then for $\varepsilon \in (A,B)$ the conditions* (1.2), (2.6) *and*

$$\sum_{\nu} \Big| \sum_{k=\nu}^{\infty} a'_{k\nu}\varepsilon_k \Big| < \infty, \tag{2.13}$$

are necessary provided the series in (2.11) *converges absolutely for every ν.*

Proof. Let B be in series-to-sequence form. Applying Theorem of Kojima–Schur ([29], p. 297, [41], p. 82, [40], p. 12, [6], p. 13) for $G : c \to c$ to the transformation (2.4), we obtain that

$$\sum_{\nu} |g_{n\nu}| = O(1) \tag{2.14}$$

and (2.12) are necessary for $\varepsilon \in (A,B)$. From (2.14) for any $k = 0, 1, \ldots$ it follows that

$$\sum_{\nu=0}^{k} |g_{n\nu}| = O(1)$$

which together with (2.12) and (2.10) yields

$$\sum |g_{\nu}| < \infty, \tag{2.15}$$

and this is (2.13), since B is regular and (2.11) converges absolutely. From (2.14) condition $g_{nn} = O(1)$ follows and hence (2.6) is necessary. Since A preserves convergence each condition for $\varepsilon \in (E,B)$ is necessary also for $\varepsilon \in (A,B)$. Therefore, from (2.13) with $A = E$ the necessity of

$$\sum |\Delta\varepsilon_{\nu}| < \infty \tag{2.16}$$

follows, because of $a'_{k\nu} = A^{-2}_{k-\nu}$ (see [6], p. 86), and (2.16) implies (1.2). □

Lemma 2.4. *If A preserves the boundedness and B is regular, then for $\varepsilon \in (A_O, B)$ the conditions* (1.6) *and*

$$\varepsilon_n = o(a_{nn}/b_{nn}) \tag{2.17}$$

are necessary.

Proof. Let B be in series-to-sequence form. Since A preserves boundedness, then the conditions for $\varepsilon \in (E_O, B)$ are necessary also for $\varepsilon \in (A_O, B)$. Therefore, we can assume $y_{\nu} = x_{\nu}$ in (2.4), and with this $g_{n\nu} = b_{n\nu}\varepsilon_{\nu}$.

Hence by Theorem of Nigam ([33], p. 123) we obtain that for $\varepsilon \in (A_O, B)$ the conditions

$$\lim_{\nu} g_{n\nu} = 0, \quad \lim_{n} \sum_{\nu} |\Delta g_{n\nu}| = 0 \tag{2.18}$$

are necessary. From (2.18) we now obtain by the second condition of (2.10)

$$\lim_{k} \varepsilon_k = \lim_{k} \lim_{n} g_{nk} = \lim_{k} \lim_{n} \lim_{m} (g_{nk} - g_{nm}) =$$

$$= \lim_{k} \lim_{n} \lim_{m} \sum_{\nu=k}^{m-1} \Delta g_{n\nu} = 0,$$

and therefore (1.6) is necessary.

Let now also A be in series-to-sequence form. Then by applying Theorem of Schur ([41], p. 82, [40], p.13, [46], p. 58, [6], p. 22) for $G : m \to c$ to the transformation (2.4), we obtain that (2.12), (2.14) and

$$\lim_{n} \sum_{\nu} |g_{n\nu} - g_{\nu}| = 0 \tag{2.19}$$

are necessary and sufficient for $\varepsilon \in (A_O, B)$. From (2.19) we get

$$\lim (g_{nn} - g_n) = 0. \tag{2.20}$$

Since the necessity of $g_n = o(1)$ follows from (2.15), applying (2.20) gives $g_{nn} = o(1)$, and this by (2.5) is (2.17). $\square$

We go on to find necessary conditions for $\varepsilon \in (A_O, |B|)$ and $\varepsilon \in (A, |B|)$. For this we represent A in the series-to-sequence form and B in the series-to-series form and associate with the transformation (2.4) the transposed transformation

$$z_n = \sum_{\nu=0}^{n} g^*_{n\nu} y_\nu = \sum_{\nu=0}^{n} g_{\nu n} y_\nu, \tag{2.21}$$

assuming $g^*_{n\nu} := g_{\nu n}$. Since for any finite subset $\mathfrak{N} \subseteq \mathrm{N}$ of nonnegative integers we have

$$\sum_{n} |\sum_{\nu \in \mathfrak{N}} g^*_{n\nu}| = \sum_{\nu} |\sum_{n \in \mathfrak{N}} g_{n\nu}|,$$

it follows from Theorems of Zeller–Lorentz ([45], p. 344, [30], p. 244) and Peyerimhoff ([37], p. 142, [6], p. 38) that the transformations (2.4) and (2.21) simultaneously transfer all convergent or all bounded sequences into absolute convergent series. As is obvious from (2.21) for $G : m \to \ell$, the condition

$$\sum_{n} |\sum_{\nu=n}^{\infty} g_{\nu n} y_\nu| < \infty \qquad \forall (y_\nu) \in m,$$

equivalent, in turn, to the condition

$$\sum_{\nu} \Big| \sum_{n=\nu}^{\infty} g_{n\nu} y_n \Big| < \infty \qquad \forall (y_n) \in m, \tag{2.22}$$

is necessary and sufficient. Since each necessary condition for the transformation $G : m \to \ell$ is also necessary for $G : c \to \ell$ and vice versa, condition (2.22) is necessary and sufficient for $\varepsilon \in (A_O, |B|)$ and $\varepsilon \in (A, |B|)$. If B is absolutely regular, then by Theorem of Knopp–Lorentz for $B : \ell \to \ell$

$$\sum_{n=k}^{\infty} |b_{nk}| = O(1), \quad \sum_{n=k}^{\infty} b_{nk} = 1,$$

and if the series in (2.11) converges absolutely, then from (2.5)

$$\sum_{n=\nu}^{\infty} g_{n\nu} = \sum_{n=\nu}^{\infty} \sum_{k=\nu}^{n} b_{nk} a'_{k\nu} \varepsilon_k = \sum_{k=\nu}^{\infty} a'_{k\nu} \varepsilon_k \sum_{n=k}^{\infty} b_{nk} = \sum_{k=\nu}^{\infty} a'_{k\nu} \varepsilon_k.$$

Now from (2.22) with $y_n = 1$ we again obtain condition (2.13). If A preserves boundedness, then the conditions for $\varepsilon \in (E_O, |B|)$ are necessary also for $\varepsilon \in (A_O, |B|)$. Therefore, from (2.13) and (2.22) with $A = E$ we deduce the necessity of (2.16), since $a'_{k\nu} = A^{-2}_{k-\nu}$ and just as above

$$\sum_{n=\nu}^{\infty} g_{n\nu} y_n = \sum_{k=\nu}^{\infty} A^{-2}_{k-\nu} \varepsilon_k \sum_{n=k}^{\infty} b_{nk} y_n = \Delta_\nu \Big(\varepsilon_\nu \sum_{n=\nu}^{\infty} b_{n\nu} y_n\Big).$$

Putting here $B = C^\beta$ in the series-to-series form (see [36], p. 417, [6], p. 84) with $\beta \geq 0$ and $y_n = A^\beta_n / A^{\beta+i\rho}_n$ for real $\rho \neq 0$ by Chow's formula

$$\sum_{n=\nu}^{\infty} \frac{1}{n A^\sigma_n} A^\delta_{n-\nu} = \frac{1}{\nu A^{\sigma-\delta-1}_\nu}, \tag{2.23}$$

valid for $\operatorname{Re} \sigma > -1$, $\operatorname{Re}(\sigma - \delta) > 0$, $\nu = 1, 2 \ldots$ (see [18], p. 461, [36], p. 418, [6], p. 80), we obtain that

$$\begin{aligned}
\sum_{n=\nu}^{\infty} g_{n\nu} y_n &= \Delta \left(\varepsilon_\nu \sum_{n=\nu}^{\infty} \nu A^{\beta-1}_{n-\nu} / (n A^\beta_n) \cdot y_n \right) \\
&= \Delta \left(\varepsilon_\nu \nu \sum_{n=\nu}^{\infty} A^{\beta-1}_{n-\nu} / (n A^{\beta+i\rho}_n) \right) \\
&= \Delta(\varepsilon_\nu / A^{i\rho}_\nu),
\end{aligned}$$

and by (2.22) we get the necessary condition

$$\sum |\Delta(\varepsilon_\nu / A_\nu^{i\rho})| < \infty. \tag{2.24}$$

Since

$$\begin{aligned}\Delta(\varepsilon_\nu / A_\nu^{i\rho}) &= \varepsilon_\nu \Delta(1/A_\nu^{i\rho}) + (\Delta\varepsilon_\nu)/A_{\nu+1}^{i\rho} \\ &= \frac{i\rho}{1+i\rho}\varepsilon_\nu / A_\nu^{1+i\rho} + (\Delta\varepsilon_\nu)/A_{\nu+1}^{i\rho},\end{aligned}$$

in view of $1/A_{\nu+1}^{i\rho} = O(1)$, from (2.16) and (2.24) we derive

$$\sum |\varepsilon_\nu / A_\nu^{1+i\rho}| < \infty$$

which is equivalent to (1.8), because of $|A_\nu^{1+i\rho}| \sim (\nu+1)/|\Gamma(1+i\rho)|$.

Finally, from (2.5) by Lemma of Chow ([18], p. 462, Lemma 6, [39], p. 34, [6], p. 42) we obtain the necessity of $\sum |g_{nn}| < \infty$, which is the condition

$$\sum |b_{nn}\varepsilon_n / a_{nn}| < \infty. \tag{2.25}$$

Thus we proved

Lemma 2.5. *Let A be in the series-to-sequence form and preserve the boundedness, and let B be absolutely regular. Then for $\varepsilon \in (A_O, |B|)$ and $\varepsilon \in (A, |B|)$ the conditions* (2.25) *and* (2.13) *are necessary if the series in* (2.11) *converges absolutely. For $\varepsilon \in (A_O, |C^\beta|)$ and $\varepsilon \in (A, |C^\beta|)$ with $\beta \geq 0$ condition* (1.8) *is necessary.*

The Cesàro method C^α with $\alpha \geq 0$ is regular and preserves the absolute convergence (see [6], pp. 61, 82 and 85), and hence satisfies the above requirements on A and B.

3. Proof of Theorem 1.1

For $A = C^\alpha$ and $B = C^\beta$ as $\alpha, \beta \geq 0$ the condition (2.6) of Lemma 2.1 is the condition (1.3). We have

$$a'_{k\nu} = \frac{\nu}{k} A_\nu^\alpha A_{k-\nu}^{-\alpha-1}$$

(see [6], p. 86), putting A is in series-to-series form, and (2.11) reduces to

$$\nu A_\nu^\alpha \Delta^\alpha(\varepsilon_\nu/\nu) = O(1), \tag{3.1}$$

since the series (2.11) in Lemma 2.2 converges absolutely by condition (1.2). Peyerimhoff ([36], Lemmas 3 and 4) and also Bosanquet and Chow ([15], Theorem X) proved that (1.4) follows from (3.1) and (1.2). According to Lemmas 2.1 and 2.2, the conditions (1.2), (1.3) and (1.4) are necessary for $\varepsilon \in (|C^\alpha|, C^\beta)$ and hence also for $\varepsilon \in (|C^\alpha|, |C^\beta|)$.

It suffices to prove that these conditions are sufficient for $\varepsilon \in (|C^\alpha|, |C^\beta|)$. Let A and B be in the series-to-series form, by Theorem of Knopp–Lorentz for $G : \ell \to \ell$ it remains to prove that (1.2) – (1.4) imply $\sum_n |g_{n\nu}| = O(1)$, that is

$$\sum_{n=\nu}^{\infty} \frac{1}{nA_n^\beta} \left| \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-1} A_{n-k}^{\beta-1} \varepsilon_k \right| = O(\nu^{-\alpha-1}). \tag{3.2}$$

In the proof we employ the following lemmas. Here and in what follows we take into account that for all $\lambda > -1$

$$A_n^\lambda \sim n^\lambda / \Gamma(\lambda + 1).$$

Lemma 3.1. *If* $\sigma > -1$, $\alpha \geq 0$ *and* $\alpha + \sigma > 0$ (*respectively* $\sigma \geq -1$, $\alpha \geq 0$ *and* $\alpha + \sigma \geq 0$), *then condition* (1.2) (*condition* (1.6), *respectively*) *implies the equation*

$$\Delta^\sigma (\Delta^\alpha \varepsilon_n) = \Delta^{\alpha+\sigma} \varepsilon_n.$$

Lemma 3.1 is due to Andersen ([2], p. 20, see also [6], p. 177). Another proof was given by Bosanquet [12].

Lemma 3.2. *For any* $0 \leq \sigma \leq \alpha$ *the conditions* (1.2) *and* (1.4) *yield*

$$\Delta^\sigma \varepsilon_n = O(n^{-\sigma}).$$

Lemma 3.2 was proved by Chow ([18], Lemma 13) and for integers α and σ it was proved already by Bosanquet [13].

Let $0 \leq \beta \leq \alpha$. In what follows we employ the notations

$$a = [\alpha], \quad b = [\beta].$$

To prove Theorem 1.1, is of importance the following formula

$$\sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-1} A_{n-k}^{\beta-1} \varepsilon_k = \sum_{i=0}^{a} \binom{a}{i} \sum_{k=\nu}^{n-i} A_{k-\nu}^{a-\alpha-1} A_{n-i-k}^{i+\beta-a-1} \Delta^i \varepsilon_k. \tag{3.3}$$

In order to prove (3.3) we apply successively the partial summation

$$\sum_{k=0}^{m} u_k v_k = \sum_{k=0}^{m} (\Delta u_k) V_k + u_{m+1} V_m, \quad V_n = \sum_{k=0}^{n} v_k, \tag{3.4}$$

and (1.12) to the left side of (3.3). Then by formula (3.5) stated below we obtain

$$\sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-1} A_{n-k}^{\beta-1}\varepsilon_k = \sum_{k=\nu}^{n} A_{k-\nu}^{a-\alpha-1}\Delta_k^a (A_{n-k}^{\beta-1}\varepsilon_k)$$
$$= \sum_{k=\nu}^{n} A_{k-\nu}^{a-\alpha-1}\sum_{i=0}^{a}\binom{a}{i}(\Delta_k^{a-i}A_{n-k-i}^{\beta-1})\Delta^i\varepsilon_k,$$

from which (3.3) follows immediately, since for any $\varkappa$ and λ we have

$$\Delta_k^{\lambda} A_{n-k}^{\varkappa} = \sum_{\nu=k}^{n} A_{\nu-k}^{-\lambda-1} A_{n-\nu}^{\varkappa} = A_{n-k}^{\varkappa-\lambda}$$

by the formula (cf. [24], p. 97, [6], p. 77)

$$\sum_{\nu=k}^{n} A_{\nu-k}^{\varkappa} A_{n-\nu}^{\lambda} = A_{n-k}^{\varkappa+\lambda+1}. \tag{3.5}$$

By condition (1.3) we obtain

$$\sum_{n=\nu}^{\nu+a}\frac{1}{nA_n^{\beta}}\Big|\sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-1}A_{n-k}^{\beta-1}\varepsilon_k\Big| = O(\nu^{-\alpha-1}).$$

Therefore, we can apply (3.3) with n in place of $n-i$. Applying (3.3) and

$$nA_n^{\chi-1} = \chi A_{n-1}^{\chi} = \chi(A_n^{\chi} - A_n^{\chi-1}), \tag{3.6}$$

to (3.2), we conclude that in order to prove Theorem 3.1, it suffices, instead of (3.2), to show that (1.2) – (1.4) yield the following relations

$$\sum_{n=\nu}^{\infty}\frac{1}{A_n^{\beta+1}}|B_i| = O(\nu^{-\alpha-1}), \tag{3.7}$$

where $i = 0, \ldots, a$ and

$$B_i = \sum_{k=\nu}^{n} A_{k-\nu}^{a-\alpha-1}A_{n-k}^{i+\beta-a-1}\Delta^i\varepsilon_k.$$

To prove this statement, we have to consider separately three cases, depending on the behavior of the numbers $A_n^{i+\beta-a-1}$.

1. The easiest thing is to estimate these terms in (3.7) with i such that $\sum |A_n^{i+\beta-a-1}| < \infty$. Therefore, for all $i = 0, \ldots, a-b-1$, if $\beta > b$ (for $a = b$ no such i exists), and for all $i = 0, \ldots a-b$, if $\beta = b$, condition (1.3) implies

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |B_i| = \sum_{k=\nu}^{\infty} |A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k| O(k^{-\beta-1}) \sum_{n=k}^{\infty} |A_{n-k}^{i+\beta-a-1}| = O(\nu^{-\alpha-1}).$$

2. For the terms in (3.7), for which $\sum |A_n^{i+\beta-a-1}| = \infty$, we are aided by the following formula

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\tau}} A_{n-\nu}^{\delta} = \frac{\tau}{\tau-\delta-1} \, \frac{1}{A_\nu^{\tau-\delta-1}}, \tag{3.8}$$

valid for $\tau > 0$, $\tau - \delta > 1$, $\nu = 0, 1, \ldots$, which follows from (2.23) and (3.6) (see [35], p. 288, [6], p. 81).

By (3.5)

$$B_i = \sum_{k=\nu}^{n} A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k \cdot \sum_{\varkappa=0}^{n-k} A_\varkappa^{i+\beta-a-2} = \sum_{\varkappa=0}^{n-\nu} A_\varkappa^{i+\beta-a-2} \sum_{k=\nu}^{n-\varkappa} A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k .$$

Therefore, by (3.5) and Lemma 3.1 condition (1.2) for all $i \geq 0$ yields

$$B_i = C_i - D_i, \tag{3.9}$$

where

$$C_i = A_{n-\nu}^{i+\beta-a-1} \Delta^{i+\alpha-a} \varepsilon_\nu,$$

$$D_i = \begin{cases} \sum\limits_{\varkappa=0}^{n-\nu} A_\varkappa^{i+\beta-a-2} \sum\limits_{k=n-\varkappa+1}^{\infty} A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k & \text{as } \alpha > a, \\ 0 & \text{as } \alpha = a. \end{cases}$$

By (3.5) and (3.8) and Lemma 3.2 we obtain from (1.2) and (1.4) for all $i = a-b, \ldots, a$ (i.e., for i satisfying $A_n^{i+\beta-a-1} \geq 0$)

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |C_i| \leq |\Delta^{i+\alpha-a} \varepsilon_\nu| \frac{\beta+1}{A_\nu^{a+1-i}} = O(\nu^{-\alpha-1}).$$

If $\alpha > a$, then for all $i = a-b+1, \ldots, a$ (i.e., for i satisfying $A_\varkappa^{i+\beta-a-2} \geq 0$; such i do not exist in the case $a = 0$) by Lemma 3.2 and formulas (3.5) and

(3.8), conditions (1.2) and (1.4) imply

$$\begin{aligned}\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |D_i| &\le \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} \sum_{\varkappa=0}^{n-\nu} A_{\varkappa}^{i+\beta-a-2} \sum_{k=n-\nu-\varkappa+1}^{\infty} |A_k^{a-\alpha-1} \Delta^i \varepsilon_{k+\nu}| \\ &= O(\nu^{-i}) \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} \sum_{\varkappa=0}^{n-\nu} A_{\varkappa}^{i+\beta-a-2} A_{n-\nu-\varkappa}^{a-\alpha} \\ &= O(\nu^{-i}) \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} A_{n-\nu}^{i+\beta-\alpha-1} = O(\nu^{-\alpha-1}).\end{aligned}$$

3. It remains to prove the corresponding estimate for D_{a-b} with $\alpha > a$ and $\beta > b$. These terms have to be estimated separately, because at $i = a-b$ the numbers $A_n^{i+\beta-a-1} = A_n^{\beta-b-1}$ are terms of a divergent series and the numbers $A_{\varkappa}^{\beta-b-2}$ are not of constant sign. It is more suitable to consider B_{a-b} instead of D_{a-b}. By (3.4) and (3.5) we obtain

$$B_{a-b} = E + F, \tag{3.10}$$

where

$$\begin{aligned} E &= A_{n-\nu}^{\beta-b+a-\alpha-1} \Delta^{a-b} \varepsilon_{n+1}, \\ F &= \sum_{k=\nu}^{n} \Delta^{a+1-b} \varepsilon_k \cdot \sum_{\varkappa=\nu}^{k} A_{\varkappa-\nu}^{a-\alpha-1} A_{n-\varkappa}^{\beta-b-1}. \end{aligned}$$

Estimates for E are simple. Indeed, if $\beta - b + a - \alpha \le 0$, then (1.3) implies

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |E| = \frac{1}{A_\nu^{\beta+1}} O(\nu^{\beta-\alpha}) \sum_{n=\nu}^{\infty} |A_{n-\nu}^{\beta-b+a-\alpha-1}| = O(\nu^{-\alpha-1}).$$

If $\beta - b + a - \alpha > 0$, then by Lemma 3.2 and (3.8), conditions (1.2) and (1.4) yield

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |E| = O(\nu^{b-a}) \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} A_{n-\nu}^{\beta-b+a-\alpha-1} = O(\nu^{-\alpha-1}).$$

In order to complete the proof of Theorem 1.1 it remains to estimate F. This needs much technicalies. By Lemma 3.1 from (1.2) it follows that

$$\Delta^{a+1-b} \varepsilon_k = \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-2} \Delta^{\alpha-b} \varepsilon_s.$$

Inserting this last equation in F, we obtain

$$F = G + H,$$

where

$$G = \sum_{s=\nu}^{n} \Delta^{\alpha-b}\varepsilon_s \cdot \sum_{k=\nu}^{s} A_{s-k}^{\alpha-a-2} \sum_{\varkappa=\nu}^{k} A_{\varkappa-\nu}^{a-\alpha-1} A_{n-\varkappa}^{\beta-b-1},$$

$$H = \sum_{s=n+1}^{\infty} \Delta^{\alpha-b}\varepsilon_s \cdot \sum_{k=\nu}^{n} A_{s-k}^{\alpha-a-2} \sum_{\varkappa=\nu}^{k} A_{\varkappa-\nu}^{a-\alpha-1} A_{n-\varkappa}^{\beta-b-1}.$$

By (3.5) we obtain

$$G = \sum_{s=\nu}^{n} \Delta^{\alpha-b}\varepsilon_s \cdot \sum_{\varkappa=\nu}^{s} A_{\varkappa-\nu}^{a-\alpha-1} A_{s-\varkappa}^{\alpha-a-1} A_{n-\varkappa}^{\beta-b-1}.$$

Applying partial summation and (3.5) in the inner sum and changing the order of summation afterwards, we discover that

$$G = C_{a-b} + J,$$

and therefore

$$F = C_{a-b} + J + H, \tag{3.11}$$

where

$$J = \sum_{\varkappa=\nu}^{n-1} A_{n-\varkappa}^{\beta-b-2} \sum_{s=\varkappa+1}^{n} \Delta^{\alpha-b}\varepsilon_s \cdot \sum_{p=\nu}^{\varkappa} A_{s-p}^{\alpha-a-1} A_{p-\nu}^{a-\alpha-1}.$$

The expression C_{a-b} is estimated in part 2. Now we use the following formula of Bosanquet ([14], p. 487, [6], p. 82) to estimate sums of binomial coefficients:

$$\sum_{\rho=\mu}^{\nu} A_{\rho-\mu}^{-\varkappa-1} A_{n-\rho}^{\lambda-1} = O(1) A_{n-\mu}^{\lambda-1} A_{\nu-\mu}^{-\varkappa} \tag{3.12}$$

for $0 \le \mu \le \nu \le n$, $0 \le \varkappa < 1$, $0 < \lambda < 1$.

Applying it to the inner sum in J, we obtain

$$\begin{aligned}
\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |J| &= O(1) \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} \sum_{\varkappa=\nu}^{n} \left| A_{n-\varkappa}^{\beta-b-2} \right| \sum_{s=\varkappa}^{n} |\Delta^{\alpha-b}\varepsilon_s| A_{s-\nu}^{\alpha-a-1} A_{\varkappa-\nu}^{a-\alpha} \\
&= O(1) \sum_{s=\nu}^{\infty} A_{s-\nu}^{\alpha-a-1} |\Delta^{\alpha-b}\varepsilon_s| \sum_{\varkappa=\nu}^{s} A_{\varkappa-\nu}^{a-\alpha} \sum_{n=s}^{\infty} \frac{1}{A_n^{\beta+1}} |A_{n-\varkappa}^{\beta-b-2}| \\
&= O(1) \sum_{s=\nu}^{\infty} \frac{1}{A_s^{\beta+1}} A_{s-\nu}^{\alpha-a-1} |\Delta^{\alpha-b}\varepsilon_s| \sum_{\varkappa=\nu}^{s} A_{\varkappa-\nu}^{a-\alpha} A_{s-\varkappa}^{\beta-b-1}.
\end{aligned}$$

Use (3.5), taking into account $\beta - b + a - \alpha > -1$, and (3.8) with the help of Lemma 3.2. By this (1.2) and (1.4) yield

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |J| = O(1) \sum_{s=\nu}^{\infty} \frac{1}{A_s^{\beta+1}} A_{s-\nu}^{\beta-b-1} |\Delta^{\alpha-b} \varepsilon_s| \tag{3.13}$$
$$= O(\nu^{-\alpha-1}).$$

It is easy to estimate H. Indeed, by Lemma 3.2 and formulas (3.12), (3.5) and (3.8), it follows from conditions (1.2) and (1.4) that

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |H| = O(\nu^{b-\alpha}) \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} A_{n-\nu}^{\beta-b-1} \sum_{k=\nu}^{n} A_{k-\nu}^{a-\alpha} \sum_{s=n+1}^{\infty} |A_{s-k}^{\alpha-a-2}|$$
$$= O(\nu^{-\alpha-1}).$$

Thus we have shown that (3.7) for all $i \leq a$ follow from (1.2), (1.3) and (1.4), and consequently (3.2) is established for $0 \leq \beta \leq \alpha$.

Let $\beta > \alpha \geq 0$. As above, (1.2) and (1.4) are sufficient for $\varepsilon \in (|C^\alpha|, |C^\alpha|)$, and by the inclusion $|C^\alpha| \subset |C^\beta|$ (see [6], p. 89) these conditions are also sufficient for $\varepsilon \in (|C^\alpha|, |C^\beta|)$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let $A = C^\alpha$ and $B = C^\beta$ with $\alpha, \beta \geq 0$ be in the series-to-sequence form. The necessity of (1.2) and (1.3) for $\varepsilon \in (C^\alpha, C^\beta)$ follows from Lemma 2.3 and also from Lemma 2.1, since C^α preserves the absolute convergence. We have

$$a'_{k\nu} = A_\nu^\alpha A_{k-\nu}^{-\alpha-2}$$

(see [6], p. 86), and therefore (1.2) implies the existence of (2.12), where

$$g_\nu = A_\nu^\alpha \Delta^{\alpha+1} \varepsilon_\nu,$$

and condition (2.13) reduces to

$$\sum A_\nu^\alpha |\Delta^{\alpha+1} \varepsilon_\nu| < \infty, \tag{4.1}$$

which is equivalent to (1.5) and is necessary by Lemma 2.3. The necessity of (4.1) also follows immediately from the functional condition (1.11), that is from

$$\Delta\varepsilon_k = \Delta^{-\alpha}(c_k / A_k^\alpha), \quad \sum |c_k| < \infty,$$

whence using (1.2) by Lemma 3.1 and, since $A^{\alpha-1}_{\nu-k}/A^{\alpha}_{\nu} = O(1)$ for $\nu \geq k$, by (3.5) we obtain

$$\Delta^{\alpha+1}\varepsilon_k = \Delta^{\alpha}\Delta^{-\alpha}(c_k/A^{\alpha}_k) = c_k/A^{\alpha}_k.$$

Let us prove the sufficiency of these conditions for $\varepsilon \in (C^{\alpha}, C^{\beta})$. Applying Theorem of Kojima–Schur for $G : c \to c$ to the transformation (2.4), we see that it remains to prove that (1.2), (1.3) and (1.5) imply (2.14), that is

$$\sum_{\nu=0}^{n} A^{\alpha}_{\nu} \,|\sum_{k=\nu}^{n} A^{-\alpha-2}_{k-\nu} A^{\beta}_{n-k}\varepsilon_k| = O(n^{\beta}), \tag{4.2}$$

since for Cesàro methods $\sum_{\nu=0}^{n} g_{n\nu} = \varepsilon_0$.

In the proof we employ the following lemmas.

Lemma 4.1. *For any* $0 < \sigma \leq \alpha + 1$, *conditions* (1.2) *and* (1.5) *imply*

$$\sum (n+1)^{\sigma-1}|\Delta^{\sigma}\varepsilon_n| < \infty.$$

Lemma 4.2. *For any* $0 < \sigma \leq \alpha$, *conditions* (1.2) *and* (1.5) *imply*

$$\Delta^{\sigma}\varepsilon_n = o(n^{-\sigma}).$$

Lemmas 4.1 and 4.2 are due to Andersen ([2], p. 31, see also [6], p. 179 and 180). Another proof is given by Bosanquet [12]. Lemma 4.1 for $\varepsilon_n = o(1)$ and integer $\sigma \geq 1$ was already proved by Bromwich ([16], p. 361).

Lemma 4.3. *Conditions* (1.2) *and* (1.5) *imply the uniform convergence in n of the series*

$$\sum_{\nu} (A^{\beta}_n)^{-1} A^{\alpha}_{\nu}\, |C_i|$$

for all $i = a - b + 1, \ldots, a + 1$.

Proof. Since $i + \beta - a - 1 \geq 0$ and $a + 1 - i \geq 0$, it follows that

$$(A^{\beta}_n)^{-1} A^{\alpha}_{\nu}\, |C_i| =$$

$$= O(1)\left(\frac{n+1-\nu}{n+1}\right)^{i+\beta-a-1}\left(\frac{\nu+1}{n+1}\right)^{a+1-i}(\nu+1)^{i+\alpha-a-1}|\Delta^{i+\alpha-a}\varepsilon_{\nu}|$$
$$= O(1)(\nu+1)^{i+\alpha-a-1}|\Delta^{i+\alpha-a}\varepsilon_{\nu}|,$$

whence by Lemma 4.1, the claim of the lemma follows from (1.2) and (1.5).

Now we can go back to the proof of Theorem 1.2.

Let $0 \leq \beta \leq \alpha$. Similarly to (3.3), we can prove the following important formula

$$\sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-2} A_{n-k}^{\beta} \varepsilon_k = \sum_{i=0}^{a+1} \binom{a+1}{i} \sum_{k=\nu}^{n-i} A_{k-\nu}^{a-\alpha-1} A_{n-i-k}^{i+\beta-a-1} \Delta^i \varepsilon_k. \tag{4.3}$$

By means of (4.3) we can instead of (4.2) prove that (1.3) and (1.5) yield

$$\sum_{\nu=0}^{n} A_\nu^\alpha |B_i| = O(n^\beta), \tag{4.4}$$

where $i = 0, \ldots, a+1$. In order to obtain (4.4) from (1.3) and (1.5), for the same reason as in the proof of Theorem 1.1, we will separately consider three cases. To this end we need to consider only the case $\beta > 0$, since for $\beta = 0$ conditions (1.3) and (1.5) yield (4.4) immediately. Indeed, even if $\alpha > 0$ we have

$$\begin{aligned}\sum_{\nu=0}^{n} A_\nu^\alpha \left| \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-2} \varepsilon_k \right| &\leq \sum_{\nu=0}^{n} A_\nu^\alpha \left|\Delta^{\alpha+1}\varepsilon_\nu\right| + \sum_{\nu=0}^{n} A_\nu^\alpha \left| \sum_{k=n+1}^{\infty} A_{k-\nu}^{-\alpha-2} \varepsilon_k \right| \\ &= O(1) + \sum_{\nu=0}^{n} A_\nu^\alpha \sum_{k=n+1}^{\infty} (k-\nu)^{-\alpha-2}\, O(k^{-\alpha}) \\ &= O(1) + O(1) \sum_{\nu=0}^{n} (n+1-\nu)^{-\alpha-1} = O(1).\end{aligned}$$

Therefore assume that $\alpha > 0$ and $\beta > 0$.

1. For all $i = 0, \ldots, a-b-1$ with $\beta > b$ (if $a = b$ such i do not exist) and for all $i = 0, \ldots, a-b$ with $\beta = b$, condition (1.3) immediately yields

$$\sum_{\nu=0}^{n} A_\nu^\alpha |B_i| = \sum_{k=0}^{n} \left|A_{n-k}^{i+\beta-a-1}\right| \sum_{\nu=0}^{k} O(A_\nu^\beta)\, \left|A_{k-\nu}^{a-\alpha-1}\right| = O(n^\beta).$$

2. By condition (1.2), equation (3.9) also holds here. By Lemma 4.3 for all $i = a-b+1, \ldots, a+1$ conditions (1.2) and (1.5) immediately yield

$$\sum_{\nu=0}^{n} A_\nu^\alpha |C_i| = O(n^\beta).$$

Before estimating D_i for $\alpha > a$, observe that by Lemma 3.1 condition (1.2) yields

$$\Delta^i \varepsilon_k = \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-1} \Delta^{i+\alpha-a} \varepsilon_s \tag{4.5}$$

for all $i \geq 0$. Hence for all $i = a - b + 1, \ldots, a + 1$ with the aid of Lemma 4.1 (used for justifying the rearranging the order of summation), and by (3.5), we deduce from (1.2) and (1.5)

$$\sum_{k=n-\varkappa+1}^{\infty} A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k = \sum_{k=n-\varkappa+1}^{\infty} A_{k-\nu}^{a-\alpha-1} \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-1} \Delta^{i+\alpha-a} \varepsilon_s$$
$$= - \sum_{s=n-\varkappa+1}^{\infty} \Delta^{i+\alpha-a} \varepsilon_s \cdot \sum_{k=\nu}^{n-\varkappa} A_{k-\nu}^{a-\alpha-1} A_{s-k}^{\alpha-a-1},$$

whence by Bosanquet's formula (3.12) we have

$$\sum_{k=n-\varkappa+1}^{\infty} A_{k-\nu}^{a-\alpha-1} \Delta^i \varepsilon_k = O(1) A_{n-\nu-\varkappa}^{a-\alpha} \sum_{s=\nu}^{\infty} A_{s-\nu}^{\alpha-a-1} |\Delta^{i+\alpha-a} \varepsilon_s|. \tag{4.6}$$

Putting this in D_i we obtain by means of (3.5)

$$\sum_{\nu=0}^{n} A_\nu^\alpha |D_i| = O(1) \cdot (K_i + L_i)$$

for all $i = a - b + 1, \ldots, a + 1$, where

$$K_i = \sum_{s=0}^{n} |\Delta^{i+\alpha-a} \varepsilon_s| \sum_{\nu=0}^{s} A_\nu^\alpha A_{n-\nu}^{i+\beta-\alpha-1} A_{s-\nu}^{\alpha-a-1},$$

$$L_i = \sum_{s=n+1}^{\infty} |\Delta^{i+\alpha-a} \varepsilon_s| \sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^{i+\beta-\alpha-1} A_{s-\nu}^{\alpha-a-1}.$$

First we estimate K_i and L_i for $i > a - \beta + 1$. In fact, in this case we can choose a number $\xi > 0$ such that $\xi \leq \beta$ and

$$a + 1 - i < \xi < \alpha + 1 - i.$$

Then $-1 < i + \xi - \alpha - 1 < 0$ and $i + \xi - a - 2 > -1$. Therefore,

$$(n+1)^{\xi-\beta} A_{n-\nu}^{i+\beta-\alpha-1} A_{s-\nu}^{\alpha-a-1} = O(1) \left(\frac{n+1-\nu}{n+1} \right)^{\beta-\xi} A_{n-\nu}^{i+\xi-\alpha-1} A_{s-\nu}^{\alpha-a-1}$$
$$= O(1) A_{s-\nu}^{i+\xi-a-2}$$

if $s \leq n$, and
$$A_{n-\nu}^{i+\beta-\alpha-1} A_{s-\nu}^{\alpha-a-1} = O(1) A_{n-\nu}^{i+\beta-a-2}$$

if $s > n$. Consequently, by (3.5) and Lemma 4.1, conditions (1.2) and (1.5) imply

$$\begin{aligned}\lim_n (n+1)^{-\beta} K_i &= \lim_n (n+1)^{-\xi} \sum_{s=0}^{n} |\Delta^{i+\alpha-a}\varepsilon_s| \sum_{\nu=0}^{s} A_\nu^\alpha A_{s-\nu}^{i+\xi-a-2} \\ &= \lim_n O(1) \sum_s \left(\frac{s+1}{n+1}\right)^{\xi} (s+1)^{i+\alpha-a-1} |\Delta^{i+\alpha-a}\varepsilon_s| = 0,\end{aligned}$$

and also

$$\begin{aligned}\lim_n (n+1)^{-\beta} L_i &= \lim_n O(1)(n+1)^{-\beta} \sum_{s=n+1}^{\infty} |\Delta^{i+\alpha-a}\varepsilon_s| \sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^{i+\beta-a-2} \\ &= \lim O(1) \sum_{s=n}^{\infty} (s+1)^{i+\alpha-a-1} |\Delta^{i+\alpha-a}\varepsilon_s| = 0.\end{aligned}$$

If $i = a - \beta + 1$, we have $0 < i \leq \alpha$ since $a + 1 > \beta = b \geq 1$. Hence by Lemma 4.2, conditions (1.2) and (1.5) together with (3.5) imply

$$\begin{aligned}\sum_{\nu=0}^{n} A_\nu^\alpha |D_{a-\beta+1}| &= \sum_{\nu=0}^{n} A_\nu^\alpha \,|\sum_{k=n+1}^{\infty} A_{k-\nu}^{a-\alpha-1} \Delta^{a-\beta+1}\varepsilon_k| \\ &= o(n^{\beta-a-1}) \sum_{\nu=0}^{n} A_\nu^\alpha A_{n-\nu}^{a-\alpha} = o(n^\beta).\end{aligned}$$

Thus we proved that for all $i = a - b + 1, \ldots, a + 1$ conditions (1.2) and (1.5) imply

$$\lim_n (A_n^\beta)^{-1} \sum_{\nu=0}^{n} A_\nu^\alpha |D_i| = 0. \tag{4.7}$$

3. It remains to prove the corresponding estimate for B_{a-b}, with $\alpha > a$ and $\beta > b$. In view of (3.10) we can separately estimate E and F. In fact, if $\beta - b + a - \alpha \leq 0$ then (1.3) yields

$$\begin{aligned}\sum_{\nu=0}^{n} A_\nu^\alpha |E| &\leq A_n^\alpha |\Delta^{a-b}\varepsilon_{n+1}| \sum_{\nu=0}^{n} |A_{n-\nu}^{\beta-b+a-\alpha-1}| \\ &= O(n^\beta).\end{aligned} \tag{4.8}$$

If, however $\beta - b + a - \alpha > 0$, then also $a - b > 0$. Therefore, by (3.5) and Lemma 4.2, conditions (1.2) and (1.5) imply

$$\sum_{\nu=0}^{n} A_\nu^\alpha |E| = o(n^\beta). \tag{4.9}$$

Now applying Bosanquet's formula (3.12) and the relation

$$\Delta^{a+1-b}\varepsilon_k = \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-1}\Delta^{\alpha+1-b}\varepsilon_s,$$

which arises from (1.2) by Lemma 3.1, as well as (3.5) to F, we obtain

$$\begin{aligned} F &= O(1)A_{n-\nu}^{\beta-b-1}\sum_{s=\nu}^{\infty}|\Delta^{\alpha+1-b}\varepsilon_s| \\ &= O(1)(\nu+1)^{b-\alpha}A_{n-\nu}^{\beta-b-1}\sum_{s=\nu}^{\infty}(s+1)^{\alpha-b}|\Delta^{\alpha+1-b}\varepsilon_s| \\ &= o(1)(\nu+1)^{b-\alpha}A_{n-\nu}^{\beta-b-1}. \end{aligned}$$

Hence, by Lemma 4.1 and formula (3.5), conditions (1.2) and (1.5) imply

$$\sum_{\nu=0}^{n} A_\nu^\alpha |F| = o(n^\beta). \tag{4.10}$$

Thus we proved that (4.4) follows from (1.2), (1.3) and (1.5) for all $i \leq a+1$ and hence (4.2) is established for $0 \leq \beta \leq \alpha$.

Let $\beta > \alpha \geq 0$. As above, (1.2) and (1.5) are sufficient for $\varepsilon \in (C^\alpha, C^\alpha)$, but by the inclusion $C^\alpha \subset C^\beta$ (see [6], p. 87) these conditions are also sufficient for $\varepsilon \in (C^\alpha, C^\beta)$. Therefore Theorem 1.2 is completely proved.

5. Proof of Theorem 1.3

The necessity of (1.5) – (1.7) for $\varepsilon \in (C_O^\alpha, C^\beta)$ follows from Theorem 1.2 and Lemma 2.4, since for the methods of Cesàro, condition (2.17) is just (1.7).

let us prove the sufficiency of these conditions for $\varepsilon \in (C_O^\alpha, C^\beta)$. Applying Schur's Theorem ([40], p. 13, [46], p. 58, [6], p. 22) for $G : m \to c$ to the transformation (2.4) with $A = C^\alpha$ and $B = C^\beta$ in series-to-sequence form, and taking into account the proof of Theorem 1.2, it remains to prove that (1.5) – (1.7) imply (2.19), that is

$$\lim_n \sum_{\nu=0}^{n} A_\nu^\alpha \left| \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-2} A_{n-k}^\beta \varepsilon_k / A_n^\beta - \Delta^{\alpha+1}\varepsilon_\nu \right| = 0. \tag{5.1}$$

Let $0 \leq \beta \leq \alpha$. By virtue of (1.2), we obtain by (1.12) that

$$\Delta^{\alpha+1}\varepsilon_\nu = \sum_{i=0}^{a+1} \binom{a+1}{i} \Delta^{a+1-i}1 \cdot \Delta^{i+\alpha-a}\varepsilon_\nu.$$

From this and (4.3) we see that instead of proving (5.1) it suffices to show that (1.5) and (1.7) yield

$$\lim_n \sum_{\nu=0}^{n} A_\nu^\alpha |M_i| = 0, \tag{5.2}$$

where $i = 0, \ldots, a+1$ and

$$M_i = (A_{n+i}^\beta)^{-1} B_i - \Delta^{a+1-i} 1 \cdot \Delta^{i+\alpha-a} \varepsilon_\nu.$$

In order to deduce (5.2) from (1.5) and (1.7), we again will consider separately three cases. As above, we need to consider only the case $\beta > 0$, since if $\beta = 0$, condition (5.1) immediately follows from (1.7). Indeed, even if $\alpha > 0$ we have

$$\begin{aligned}\sum_{\nu=0}^{n} A_\nu^\alpha \left| \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-2} \varepsilon_k - \Delta^{\alpha+1} \varepsilon_\nu \right| &= \sum_{\nu=0}^{n} A_\nu^\alpha \left| \sum_{k=n+1}^{\infty} A_{k-\nu}^{-\alpha-2} \varepsilon_k \right| \\ &= o(n^{-\alpha}) \sum_{\nu=0}^{n} A_\nu^\alpha (n+1-\nu)^{-\alpha-1} = o(1).\end{aligned}$$

Therefore assume that $\alpha > 0$ and $\beta > 0$.

1. For all $i = 0, \ldots, a-b-1$ when $\beta > b$ (if $a = b$ such i do not exist) and for all $i = 0, \ldots, a-b$ when $\beta = b$, condition (1.7) immediately yields

$$\begin{aligned}\lim_n \sum_{\nu=0}^{n} A_\nu^\alpha |M_i| &= \lim_n (A_{n+i}^\beta)^{-1} \sum_{\nu=0}^{n} A_\nu^\alpha |B_i| \\ &= \lim_n O(1) \sum_{k=0}^{n} (n+1-k)^{\alpha-\beta} |\Delta^i \varepsilon_{n-k}| \, |A_k^{i+\beta-a-1}| = 0.\end{aligned}$$

2. Observe that (3.9) yields

$$M_i = N_i - (A_{n+i}^\beta)^{-1} D_i,$$

where

$$N_i = \left\{ (A_{n+i}^\beta)^{-1} A_{n-\nu}^{i+\beta-a-1} - \Delta^{a+1-i} 1 \right\} \Delta^{i+\alpha-a} \varepsilon_\nu.$$

By Lemma 4.3, for all $i = a-b+1, \ldots, a+1$, conditions (1.2) and (1.5) imply

$$\lim_n \sum_{\nu=0}^{n} A_\nu^\alpha |N_i| = 0,$$

and, in view of (4.7), also (5.2).

3. The remaining estimate for B_{a-b}, with $\alpha > a$ and $\beta > b$, was already proved when proving Theorem 1.2. Indeed, (1.2) and (1.5) imply (4.9) for $\beta - b + a - \alpha > 0$ and (4.10), while by (4.8) we see that (1.7) implies (4.9) also for $\beta - b + a - \alpha \le 0$.

Thus we proved that (5.2) for all $i \le a+1$ follows from (1.2), (1.5) and (1.7), and hence (5.1) holds for $0 \le \beta \le \alpha$.

Now let $\beta > \alpha \ge 0$. As above, (1.6) and (1.5) are sufficient for $\varepsilon \in (C_O^\alpha, C^\alpha)$, but by the inclusion $C^\alpha \subset C^\beta$ these conditions are also sufficient for $\varepsilon \in (C_O^\alpha, C^\beta)$. Since (1.6) implies (1.2), Theorem 1.3 is completely proved.

6. Proof of Theorem 1.4

For $\varepsilon \in (C_O^\alpha, |C^\beta|)$ and $\varepsilon \in (C^\alpha, |C^\beta|)$, where $\alpha, \beta \ge 0$, conditions (1.8), (1.9) and (1.5) are necessary by Lemma 2.5, while (2.25) and (2.13) are just (1.9) and (1.5), respectively.

To prove that these conditions are also sufficient, denote by σ_n^α and τ_n^α the C^α-means of the series (1.1) and of the sequence (nu_n), respectively. Since $C^\alpha \subset C^{\alpha+1}$ for $\alpha > -1$, formula (see [6], p. 204)

$$\tau_n^{\alpha+1} = (\alpha+1)(\sigma_n^\sigma - \sigma_n^{\alpha+1})$$

yields that the C^α-summability of the series (1.1) implies the $C^{\alpha+1}$-summability of the sequence (nu_n). Therefore, to prove Theorem 1.4 it suffices to prove that assuming (1.5), (1.8) and (1.9), we have that (2.22) with $y_n = \tau_n^{\alpha+1}$ follows from the convergence of $(\tau_n^{\alpha+1})$. Here by (2.5) with $C^{\alpha+1}$ in the sequence-to-sequence form and $B = C^\beta$ in the series-to-series form we have

$$g_{n\nu} = (nA_n^\beta)^{-1} A_\nu^{\alpha+1} \sum_{k=\nu}^{n} A_{k-\nu}^{-\alpha-2} A_{n-k}^{\beta-1} \varepsilon_k \quad (\nu \ge 1),$$

since $\tau_0^{\alpha+1} = 0$ and

$$a'_{k\nu} = A_\nu^{\alpha+1} A_{k-\nu}^{-\alpha-2},$$

where $|b_{00}\varepsilon_0 u_0| = |\varepsilon_0 u_0|$ is taken into account. Now we are going to prove the stronger condition

$$\sum_{\nu=1}^{\infty} \sum_{n=\nu}^{\infty} |g_{n\nu}| < \infty. \tag{6.1}$$

instead of (2.22). For this proof we need the following lemma of Bosanquet–Chow ([15], p. 79, [6], p. 198).

Lemma 6.1. *If* $\alpha > 0$, *then for each* $s = \alpha, \alpha - 1, \dots, \alpha - a$ *conditions* (1.5) *and* (1.8) *imply*

$$\sum (n+1)^{s-1} |\Delta^s \varepsilon_n| = O(1) \sum n^\alpha |\Delta^{\alpha+1} \varepsilon_n| < \infty$$

and $\Delta^{\alpha-a}\varepsilon_n = o(1)$.

Lemma 6.2. *If* $\sigma \geq -1$, $\alpha \geq 1$ *and* $\alpha + \sigma \geq 1$, *then conditions* (1.5) *and* (1.8) *imply the equation*

$$\Delta^\sigma(\Delta^\alpha \varepsilon_n) = \Delta^{\alpha+\sigma}\varepsilon_n.$$

Proof. Denoting $\mu_n = \Delta\varepsilon_n$, we first prove that $\mu_n = o(1)$. In fact, by (1.8) and Andersen's formula ([2], p. 22, [6], p. 79)

$$\sum_{\nu=k}^{n} |A^{\varkappa}_{\nu-k} A^{\lambda}_{n-\nu}| = O(1)[(n+1-k)^{\varkappa} + (n+1-k)^{\lambda} + (n+1-k)^{\varkappa+\lambda+1}] \quad (6.2)$$

(which is valid for any real $\varkappa$ and λ) we have

$$\Delta^{a-\alpha+1}(\Delta^{\alpha-a}\varepsilon_n) = \mu_n,$$

since the double series, arising from the definition of these differences, converges absolutely. Hence by Lemma 6.1, from (1.5) and (1.8), the estimate

$$\mu_n = \sum_{k=n}^{\infty} A^{\alpha-a-2}_{k-n} \Delta^{\alpha-a}\varepsilon_k = o(1) \sum_{k=n}^{\infty} |A^{\alpha-a-2}_{k-n}| = o(1)$$

follows. Therefore by Lemma 3.1 we obtain

$$\Delta^\sigma(\Delta^{\alpha-1}\mu_n) = \Delta^{\alpha+\sigma-1}\mu_n$$

if $\sigma \geq -1$, $\alpha - 1 \geq 0$ and $\alpha + \sigma - 1 \geq 0$. Now by (1.8) and (6.2) we obtain $\Delta^{\alpha-1}\mu_n = \Delta^\alpha\varepsilon_n$ and $\Delta^{\alpha+1-\sigma}\mu_n = \Delta^{\alpha+\sigma}\varepsilon_n$. □

First let $0 \leq \beta \leq \alpha + 1$. Because of (4.3) we get

$$\sum_{k=\nu}^{n} A^{-\alpha-2}_{k-\nu} A^{\beta-1}_{n-k} \varepsilon_k = \sum_{i=0}^{a+1} \binom{a+1}{i} \sum_{k=\nu}^{n-i} A^{a-\alpha-1}_{k-\nu} A^{i+\beta-a-2}_{n-i-k} \Delta^i \varepsilon_k. \quad (6.3)$$

Since by (1.9) we have

$$\sum_{\nu=1}^{\infty}\sum_{n=\nu}^{\nu+a+1}|g_{n\nu}| = O(1)\sum_{\nu=1}^{\infty}A_{\nu}^{\alpha+1}\sum_{n=\nu}^{\nu+a+1}n^{-\beta-1}\sum_{k=\nu}^{n}A_{n-k}^{\beta-1}|\varepsilon_k|$$
$$= O(1)\sum_{\nu=1}^{\infty}A_{\nu}^{\alpha+1}\sum_{k=\nu}^{\nu+a+1}|\varepsilon_k|\,k^{-\beta-1}$$
$$= O(1)\sum_{\nu=1}^{\infty}\sum_{k=\nu}^{\nu+a+1}k^{\alpha-\beta}|\varepsilon_k| < \infty,$$

we can apply (6.3) with substitute $n-i$ by n. Thus in view of (6.3) it suffices to show that (1.5) and (1.9) imply

$$\sum_{\nu=1}^{\infty}A_{\nu}^{\alpha+1}\sum_{n=\nu}^{\infty}n^{-\beta-1}|B_i'| < \infty,$$

where $i = 0, 1, \ldots, a+1$ and

$$B_i' = \sum_{k=\nu}^{n}A_{k-\nu}^{a-\alpha-1}A_{n-k}^{i+\beta-a-2}\Delta^i\varepsilon_k.$$

Consider three cases for i.

1. For all $i = 0, \ldots, a-b+1$ if $\beta = b$ and for all $i = 0, \ldots, a-b$ if $\beta > b$, it follows from (1.9) (since $i+\beta-a-2 \le -1$) that

$$\sum_{\nu=1}^{\infty}A_{\nu}^{\alpha+1}\sum_{n=\nu}^{\infty}n^{-\beta-1}|B_i'| = O(1)\sum_{\nu=1}^{\infty}A_{\nu}^{\alpha+1}\sum_{k=\nu}^{\infty}|A_{k-\nu}^{a-\alpha-1}|\,|\Delta^i\varepsilon_k|\,k^{-\beta-1}$$
$$= O(1)\sum_{k=1}^{\infty}k^{\alpha-\beta}|\Delta^i\varepsilon_k| < \infty.$$

2. For all $i = 1, \ldots, a+1$ we have as for (3.9) by (3.5) and Lemma 6.2, that (1.5) and (1.8) yield

$$B_i' = C_i' - D_i',$$

where

$$C_i' = A_{n-\nu}^{i+\beta-a-2}\Delta^{i+\alpha-a}\varepsilon_\nu,$$

$$D_i' = \begin{cases} \sum\limits_{\varkappa=0}^{n-\nu}A_{\varkappa}^{i+\beta-a-3}\sum\limits_{k=n-\varkappa+1}^{\infty}A_{k-\nu}^{a-\alpha-1}\Delta^i\varepsilon_k, & \text{if } \alpha > a,\\ 0, & \text{if } \alpha = a.\end{cases}$$

Therefore by (3.8) and Lemma 6.1, we obtain from (1.5) and (1.8) for all $i = a - b + 1, \ldots, a + 1$

$$\begin{aligned}\sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} \, |C_i'| &= O(1) \sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} |\Delta^{i+\alpha-a} \varepsilon_{\nu}| \sum_{n=\nu}^{\infty} A_{n-\nu}^{i+\beta-a-2} / A_n^{\beta+1} \\ &= O(1) \sum_{\nu=1}^{\infty} (\nu+1)^{i+\alpha-a-1} |\Delta^{i+\alpha-a} \varepsilon_{\nu}| < \infty.\end{aligned}$$

If $\alpha > a$, then to estimate D_i' observe that by Lemma 6.2 conditions (1.5) and (1.8) yield (4.5) and hence also (4.6) if $i \geq 1$. Putting (4.6) in D_i' by Lemma 6.1 with the help of (3.5) and (3.8) for any $i = a - b + 2, \ldots, a + 1$ we obtain that (1.5) and (1.8) imply

$$\begin{aligned}&\sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} |D_i'| = \\ &= O(1) \sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} \sum_{\varkappa=0}^{n-\nu} A_{\varkappa}^{i+\beta-a-3} A_{n-\nu-\varkappa}^{a-\alpha} \sum_{s=\nu}^{\infty} A_{s-\nu}^{\alpha-a-1} |\Delta^{i+\alpha-a} \varepsilon_s| \\ &= O(1) \sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} A_{n-\nu}^{i+\beta-\alpha-2} \sum_{s=\nu}^{\infty} A_{s-\nu}^{\alpha-a-1} |\Delta^{i+\alpha-a} \varepsilon_s| \\ &= O(1) \sum_{s=1}^{\infty} |\Delta^{i+\alpha-a} \varepsilon_s| \sum_{\nu=0}^{s} A_{s-\nu}^{\alpha-a-1} A_{\nu}^{i-1} \\ &= O(1) \sum_{s} (s+1)^{i+\alpha-a-1} |\Delta^{i+\alpha-a} \varepsilon_s| < \infty.\end{aligned}$$

3. It remains to estimate D_{a-b+1}' for $\alpha > a$ and $\beta > b$. It is more convenient to consider B_{a-b+1}'. By (3.10)

$$B_{a-b+1}' = E' + F',$$

where

$$\begin{aligned}E' &= A_{n-\nu}^{\beta-b+a-\alpha-1} \Delta^{a-b+1} \varepsilon_{n+1}, \\ F' &= \sum_{k=\nu}^{n} \Delta^{a+2-b} \varepsilon_k \cdot \sum_{\mu=\nu}^{k} A_{\mu-\nu}^{a-\alpha-1} A_{n-\mu}^{\beta-b-1}.\end{aligned}$$

Now if $\beta - b + a - \alpha \leq 0$, then by (1.9)

$$\sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} |E'| = O(1) \sum_{n} (n+1)^{\alpha-\beta} |\Delta^{a-b+1} \varepsilon_{n+1}| < \infty.$$

Even for $\beta - b + a - \alpha > 0$, by Lemma 6.1 and (3.5) conditions (1.5) and (1.8) yield

$$\sum_{\nu=1}^{\infty} A_\nu^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} |E'| = O(1) \sum_n (n+1)^{a-b} |\Delta^{a-b+1} \varepsilon_{n+1}| < \infty.$$

Further, by Lemma 6.2 the conditions (1.5) and (1.8) imply

$$\Delta^{a+2-b} \varepsilon_k = \sum_{s=k}^{\infty} A_{s-k}^{\alpha-a-2} \Delta^{\alpha+1-b} \varepsilon_s,$$

and hence from (3.11) we obtain

$$F' = C'_{a-b+1} + J' + H',$$

where

$$J' = \sum_{\varkappa=\nu}^{n-1} A_{n-\varkappa}^{\beta-b-2} \sum_{s=\varkappa+1}^{n} \Delta^{\alpha+1-b} \varepsilon_s \cdot \sum_{p=\nu}^{\varkappa} A_{s-p}^{\alpha-a-1} A_{p-\nu}^{a-\alpha-1},$$

$$H' = \sum_{s=n+1}^{\infty} \Delta^{\alpha+1-b} \varepsilon_s \cdot \sum_{k=\nu}^{n} A_{s-k}^{\alpha-a-2} \sum_{\varkappa=\nu}^{k} A_{\varkappa-\nu}^{a-\alpha-1} A_{n-\varkappa}^{\beta-b-1}.$$

Since, in view of (3.13), we have

$$\sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |J'| = O(1) \sum_{s=\nu}^{\infty} \frac{1}{A_s^{\beta+1}} A_{s-\nu}^{\beta-b-1} |\Delta^{\alpha+1-b} \varepsilon_s|,$$

then by (3.5) and Lemma 6.1 conditions (1.5) and (1.8) imply

$$\sum_{\nu=1}^{\infty} A_\nu^{\alpha+1} \sum_{n=\nu}^{\infty} \frac{1}{A_n^{\beta+1}} |J'| = O(1) \sum_{s=1}^{\infty} s^{-\beta-1} |\Delta^{\alpha+1-b} \varepsilon_s| \sum_{\nu=0}^{s} A_{s-\nu}^{\beta-b-1} A_\nu^{\alpha+1}$$
$$= O(1) \sum_{s=1}^{\infty} s^{\alpha-b} |\Delta^{\alpha+1-b} \varepsilon_s| < \infty.$$

Further, let $0 < \eta < \min\{\alpha - a,\ \beta - b\}$. Then

$$A_{s-k}^{\alpha-a-2} = O(1) A_{s-n}^{\eta-1} A_{n-k}^{\alpha-a-\eta-1}$$

since $-1 < \eta - 1,\ \alpha - a - \eta - 1 < 0$. Now by formulas (3.12) and (3.5)

$$\begin{aligned} H' &= O(1) A_{n-\nu}^{\beta-b-1} \sum_{s=n+1}^{\infty} |\Delta^{\alpha+1-b}\varepsilon_s| \sum_{k=\nu}^{n} |A_{s-k}^{\alpha-a-2}| A_{k-\nu}^{a-\alpha} \\ &= O(1) A_{n-\nu}^{\beta-b-\eta-1} \sum_{s=n}^{\infty} A_{s-n}^{\eta-1} |\Delta^{\alpha+1-b}\varepsilon_s|. \end{aligned}$$

By (3.5) and Lemma 6.1 conditions (1.5) and (1.8) imply

$$\begin{aligned} &\sum_{\nu=1}^{\infty} A_{\nu}^{\alpha+1} \sum_{n=\nu}^{\infty} n^{-\beta-1} |H'| \\ &\quad = O(1) \sum_{n=1}^{\infty} n^{-\beta-1} \sum_{s=n}^{\infty} A_{s-n}^{\eta-1} |\Delta^{\alpha+1-b}\varepsilon_s| \sum_{\nu=1}^{n} A_{\nu}^{\alpha+1} A_{n-\nu}^{\beta-b-\eta-1} \\ &\quad = O(1) \sum_{n=1}^{\infty} A_n^{\alpha+\beta-b-\eta-\beta} \sum_{s=n}^{\infty} A_{s-n}^{\eta-1} |\Delta^{\alpha+1-b}\varepsilon_s| \\ &\quad = O(1) \sum_{s=1}^{\infty} |\Delta^{\alpha+1-b}\varepsilon_s| \sum_{n=1}^{s} A_{s-n}^{\eta-1} A_n^{\alpha-b-\eta} \\ &\quad = O(1) \sum_{s} (s+1)^{\alpha-b} |\Delta^{\alpha+1-b}\varepsilon_s| < \infty. \end{aligned}$$

Thus Theorem 1.4 is proved for $\beta \le \alpha + 1$. For $\beta > \alpha + 1$, Theorem 1.4 follows from the inclusion $|C^{\beta}| \supset |C^{\alpha+1}|$, since (1.5) and (1.8) are sufficient for $\varepsilon \in (C_O^{\alpha}, |C^{\alpha+1}|)$.

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