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Counterexamples concerning topologization of spaces of strongly almost convergent sequences

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ABSTRACT. Let λ be a sequence space, f a modulus function, and $\mathcal{F} = (f_k)$ a sequence of moduli. We characterize the F-normability of the sequence space $\lambda(\mathcal{F})$ for $\lambda \subset \ell_\infty$. In the special case if λ is the space sac_0 of strongly almost convergent to zero sequences, we give two counterexamples concerning the topologization of various extensions of $sac_0(\mathcal{F})$ and $sac_0(f)$ considered by Nanda and others. We also correct a similar inaccuracy in a previous paper of the author.

1. Introduction

First, let us fix some terminology. By the term *sequence space*, we shall mean, as usual, any linear subspace of the vector space ω of all (real or complex) sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, \cdots\}$.

A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* (or simply a *modulus*) if

(i) $f(t) = 0$ if and only if $t = 0$,
(ii) $f(t + u) \leq f(t) + f(u)$,
(iii) f is increasing,
(iv) f is continuous from the right at 0.

Provided a modulus f and a sequence space λ, Ruckle [13], Maddox [10], and some other authors define a new sequence space $\lambda(f)$ by

$$\lambda(f) = \{(x_k) : \ (f(|x_k|)) \in \lambda\}.$$

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As an extension of the space $\lambda(f)$, the author [5, 7] considers, for a sequence of moduli $\mathcal{F} = (f_k)$, the set

$$\lambda(\mathcal{F}) = \{x = (x_k) : \mathcal{F}(|x|) \in \lambda\},$$

where $\mathcal{F}(|x|) = (f_k(|x_k|))$. It is not difficult to see that $\lambda(\mathcal{F})$ is a solid sequence space whenever the sequence space λ is *solid* (i.e. $(x_k) \in \lambda$ and $|y_k| \leq |x_k| \quad (k \in \mathbb{N})$ yield $(y_k) \in \lambda$).

Recall that an F-*seminorm* g on a vector space X is a functional $g : X \to \mathbb{R}$ satisfying, for all $x, y \in X$, the axioms

(N1) $g(0) = 0$,
(N2) $g(x + y) \leq g(x) + g(y)$,
(N3) $g(\alpha x) \leq g(x)$ for all scalars α with $|\alpha| \leq 1$,
(N4) $\lim_n g(\alpha_n x) = 0$ for every scalar sequence (α_n) with $\lim_n \alpha_n = 0$.

An F-seminorm g is called an F-*norm* if

(N5) $g(x) = 0 \implies x = 0$.

A *paranorm* on X is a functional $g : X \to \mathbb{R}$ satisfying (N1), (N2) and

(N6) $g(-x) = g(x)$,
(N7) $\lim_n g(\alpha_n x_n - \alpha x) = 0$ for every scalar sequence (α_n) with $\lim_n \alpha_n = \alpha$ and every sequence (x_n) with $\lim_n g(x_n - x) = 0 \quad (x_n, x \in X)$.

An F-seminorm (paranorm) g on a solid sequence space λ is said to be *absolutely monotone* if $g(x) \leq g(y)$ for all $x = (x_k), y = (y_k) \in \lambda$ with $|x_k| \leq |y_k| \quad (k \in \mathbb{N})$. An F-seminormed solid sequence space (λ, g) is called an AK-*space* if $x = \lim_n \sum_{k=1}^n x_k e^k$ for all $x = (x_k) \in \lambda$ (here $e^k = (\delta_{ik})_{i \in \mathbb{N}}$, where $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ otherwise).

If the sequence space λ is topologized by an F-seminorm (or paranorm) g, then, for the topologization of $\lambda(\mathcal{F})$, it is natural to consider the functional $g_{\mathcal{F}}$ defined by

$$g_{\mathcal{F}}(x) = g(\mathcal{F}(|x|)) \qquad (x \in \lambda(\mathcal{F})).$$

It is known (cf. [10], Theorem 8) that, in general, $g_{\mathcal{F}}$ may fail to be an F-seminorm on $\lambda(\mathcal{F})$. The author ([8], Theorem 2) proved

Theorem 1. *Let $\mathcal{F} = (f_k)$ be a sequence of moduli and let g be an absolutely monotone* F-*seminorm on a solid sequence space λ. If (λ, g) is an AK-space, then the functional $g_{\mathcal{F}}$ is an absolutely monotone* F-*seminorm on $\lambda(\mathcal{F})$. Moreover, $(\lambda(\mathcal{F}), g_{\mathcal{F}})$ is an* AK-*space.*

In Section 2, the F-normability of the space $\lambda(\mathcal{F})$ is characterized in the case $\lambda \subset \ell_\infty$ with some restrictions on λ and $\mathcal{F} = (f_k)$ (Theorem 2). If λ is the space sac_0 of strongly almost convergent to zero sequences, we give, in Section 3, two counterexamples (Corollaries 1 and 2) concerning the topologization of various extensions of the spaces $sac_0(f)$ and $sac_0(\mathcal{F})$, which have been considered by Nanda [11], Nuray and Savas [12], Esi [3, 4], and Bilgin [2]. We also correct a similar inaccuracy in the paper [8] of the author.

2. On the topologization of $\lambda(f)$ and $\lambda(\mathcal{F})$

Our main theorem deals with the topologization of $\lambda(\mathcal{F})$ if λ is a subspace of the Banach space ℓ_∞ of all bounded sequences equipped with the norm $\|x\|_\infty = \sup_k |x_k|$. For $g = \|\cdot\|_\infty$, we shall write $g_{\mathcal{F}}^\infty$ instead of $g_{\mathcal{F}}$, i.e.

$$g_{\mathcal{F}}^\infty(x) = \sup_k f_k(|x_k|) \qquad (x \in \lambda(\mathcal{F})).$$

Theorem 2. *Let λ be a solid subspace of the Banach space ℓ_∞ and let $\mathcal{F} = (f_k)$ be a sequence of moduli.*

(a) *If $\lambda(\mathcal{F}) \subset \ell_\infty$ and*

$$\lim_{t\to 0+} \sup_k f_k(t) = 0, \tag{1}$$

then $g_{\mathcal{F}}^\infty$ is an F*-norm on $\lambda(\mathcal{F})$.*

(b) *If $\lambda(\mathcal{F}) \not\subset \ell_\infty$ and*

$$\phi(t) = \inf_k f_k(t) > 0 \qquad (t > 0), \tag{2}$$

then $g_{\mathcal{F}}^\infty$ is not an F*-norm on $\lambda(\mathcal{F})$.*

Proof. (a). Suppose $\lambda(\mathcal{F}) \subset \ell_\infty$ and f satisfies (1). It is straightforward to verify that the functional $g_{\mathcal{F}}^\infty$ satisfies the axioms (N1)–(N3) and (N5). To prove the axiom (N4), let $x = (x_k) \in \lambda(\mathcal{F})$ and $\lim_n \alpha_n = 0$. The inclusion $\lambda(\mathcal{F}) \subset \ell_\infty$ yields the existence of a natural number N such that $|x_k| \leq N \quad (k \in \mathbb{N})$. Thus, for all $n \in \mathbb{N}$,

$$g_{\mathcal{F}}^\infty(\alpha_n x) = \sup_k f_k(|\alpha_n x_k|) \leq N \sup_k f_k(|\alpha_n|),$$

and from (1) it follows that $\lim_n g_{\mathcal{F}}^\infty(\alpha_n x) = 0$.

(b). Let $\mathcal{F}$ satisfy (2) and let $\lambda(\mathcal{F})$ contain an unbounded sequence $y = (y_k)$. We can choose an index sequence (n_i) so that $\lim_i |y_{n_i}| = \infty$. Defining

$$\alpha_n = \begin{cases} |y_{n_i}|^{-1} & \text{if } n = n_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases}$$

we have $\lim_n \alpha_n = 0$. But since

$$\sup_k f_k(|\alpha_{n_i} y_k|) \geq \phi(1) > 0 \qquad (i \in \mathbb{N}),$$

we get that $\lim_n g_{\mathcal{F}}^{\infty}(\alpha_n y) \neq 0$. Thus $g_{\mathcal{F}}^{\infty}$ fails the axiom (N4). □

Remark 1. It is easy to see (cf. [7], Theorem 3) that $\ell_\infty(\mathcal{F}) \subset \ell_\infty$ (and thus also $\lambda(\mathcal{F}) \subset \ell_\infty$ for any solid subspace $\lambda \subset \ell_\infty$) whenever

$$\lim_{t \to \infty} \inf_k f_k(t) = \infty. \tag{3}$$

A simple argument shows (cf. [6], Lemma 1) that (3) also implies (2).

Let f be a modulus function and let $\mathbf{p} = (p_k)$ be a sequence with $0 < p_k \leq 1$. For a solid sequence space λ, denote

$$\lambda^{\mathbf{p}}(f) = \{x = (x_k) : ((f(|x_k|))^{p_k}) \in \lambda\}.$$

The function $f_k^{\mathbf{p}}$ defined by

$$f_k^{\mathbf{p}}(t) = (f(t))^{p_k}$$

is clearly a modulus for every $k \in \mathbb{N}$. Denoting $\mathcal{F}^{\mathbf{p}} = (f_k^{\mathbf{p}})$, we may write

$$\lambda^{\mathbf{p}}(f) = \lambda(\mathcal{F}^{\mathbf{p}}).$$

Thus, provided an F-seminorm g on λ, it is natural to consider the functional $g_{\mathcal{F}^{\mathbf{p}}}$ for the topologization of $\lambda^{\mathbf{p}}(f)$. If λ is a subspace of ℓ_∞ and $g = \|\cdot\|_\infty$, then $g_{\mathcal{F}^{\mathbf{p}}}$ reduces to the functional $g_{f,\mathbf{p}}^{\infty}$ where

$$g_{f,\mathbf{p}}^{\infty}(x) = \sup_k (f(|x_k|))^{p_k} \qquad (x \in \lambda^{\mathbf{p}}(f)).$$

In the sequel, we shall use the following characteristic for λ:

(C) Every infinite sequence of indices (k_i) has a subsequence (l_i) such that λ contains the sequence (h_k) where

$$h_k = \begin{cases} 1 & \text{if } k = l_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Lemma 1. *Let f be a modulus, $\mathbf{p} = (p_k)$ a sequence with $0 < p_k \leq 1$, and λ a solid sequence space with property* (C). *Then $\lambda^{\mathbf{p}}(f) \not\subset \ell_\infty$ whenever*
(a) *f is bounded*
or
(b) $\inf_k p_k = 0$.

Proof. (a). Suppose f is bounded. Then there exists a constant $M \geq 1$ such that $f(t) \leq M \quad (t \geq 0)$. By property (C), there exists an index sequence (l_i) such that λ contains the sequence (h_k) defined as in (4). Defining

$$y_k = \begin{cases} i & \text{if } k = l_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

we obtain an unbounded sequence $(y_k) \in \lambda^{\mathbf{p}}(f)$ since

$$(f(|y_k|))^{p_k} \leq M h_k \qquad (k \in \mathbb{N})$$

and λ is solid.

(b). Suppose $\inf_k p_k = 0$. Then there exists an index sequence (k_i) such that (p_{k_i}) is decreasing and

$$(f(i))^{p_{k_i}} \leq 2.$$

By property (C), there exists a subsequence (l_i) of (k_i) such that λ contains the sequence (h_k) defined as in (4). The unbounded sequence $y = (y_k)$ defined by (5) belongs to $\lambda^{\mathbf{p}}(f)$ because

$$(f(|y_k|))^{p_k} \leq 2 h_k \qquad (k \in \mathbb{N})$$

and λ is solid. □

Applying Theorem 2 to $\lambda^{\mathbf{p}}(f)$, we get

Proposition 1. *Let f be a modulus, $\mathbf{p} = (p_k)$ a sequence with $0 < p_k \leq 1$, and λ a solid subspace of ℓ_∞ with the property* (C). *Then the functional $g^{\infty}_{f,\mathbf{p}}$ is an* F-*norm on $\lambda^{\mathbf{p}}(f)$ if and only if f is unbounded and* $\inf_k p_k > 0$.

Proof. Necessity. Note that the sequence $\mathcal{F}^{\mathbf{p}}$ of moduli satisfies (2) for any $\mathbf{p} = (p_k)$ with $0 < p_k \leq 1$ because

$$(f(t))^{p_k} \geq \min\{1,\ f(t)\} \qquad (t > 0).$$

Thus, Theorem 2(b) together with Lemma 1 yield that $g^{\infty}_{f,\mathbf{p}}$ is not an F-norm on $\lambda^{\mathbf{p}}(f)$ whenever f is bounded or $\inf_k p_k = 0$.

Sufficiency. Suppose that f is unbounded and $\inf_k p_k > 0$. It is straightforward to verify that the sequence $\mathcal{F}^{\mathbf{p}}$ of moduli satisfies (1) and (3). Theorem 2(a) with an appeal to Remark 1 now yield that $g^{\infty}_{f,\mathbf{p}}$ is an F-norm on $\lambda^{\mathbf{p}}(f)$. □

3. Consequences and counterexamples

Let c_0 be the space of all convergent to zero sequences. In the sequel, we shall consider the space of strongly almost convergent to zero sequences (cf. [9, 14])

$$sac_0^p = \{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p = 0 \quad \text{uniformly in } i\},$$

where $p > 0$. We shall write sac_0 instead of sac_0^1.

It is essential to note that, for $p \geq 1$, the natural norm

$$\|x\| = \sup_{n,i}(\frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p)^{1/p}$$

on sac_0^p coincides with the supremum-norm $\|\cdot\|_\infty$ since

$$|x_i| \leq \sup_n(\frac{1}{n} \sum_{k=i}^{i+n-1} |x_k|^p)^{1/p} \leq \sup_k |x_k| \quad (i \in \mathbb{N}).$$

Thus, for $p \geq 1$, we may consider sac_0^p as a solid subspace of the Banach space ℓ_∞. Since c_0 is the largest AK-subspace of $(\ell_\infty, \|\cdot\|_\infty)$ and

$$c_0 \subsetneq sac_0^p,$$

then we have that $(sac_0^p, \|\cdot\|_\infty)$ is not an AK-space in case $p \geq 1$. Hence Theorem 1 is inapplicable in the case $\lambda = sac_0$.

Our approach to the study of $sac_0(f)$ and also of the more general space

$$sac_0^{\mathbf{p}}(f) = \{x = (x_k) : \lim_n \frac{1}{n} \sum_{k=i}^{i+n-1} (f(|x_k|))^{p_k} = 0 \quad \text{uniformly in } i\}$$

is grounded on Proposition 1 since sac_0^p has property (C). Indeed, if $(k(i))$ is an index sequence, then sac_0^p contains, for example, the sequence (h_n) where $h_n = 1$ if $n = k(2^i)$ for some $i \in \mathbb{N}$, and $h_n = 0$ otherwise.

If, in the definition of $sac_0^{\mathbf{p}}(f)$, we allow the sequence $\mathbf{p} = (p_k)$ to be an arbitrary bounded sequence of positive numbers, then, denoting $r = \max\{1, \sup_k p_k\}$ and $\mathbf{q} = (p_k/r)$, we may write

$$sac_0^{\mathbf{p}}(f) = sac_0^r(\mathcal{F}^{\mathbf{q}}),$$

where $\mathcal{F}^{\mathbf{q}}$ is the sequence of moduli $f_k^{\mathbf{q}}$ defined by $f_k^{\mathbf{q}}(t) = (f(t))^{p_k/r}$. Thus from Proposition 1, for $\lambda = sac_0^r$, we get the following two corollaries.

Corollary 1. *Let f be a modulus and let $\mathbf{p} = (p_k)$ be a bounded sequence of positive numbers. Then the functional $g^{\infty}_{f,\mathbf{q}}$ defined by*

$$g^{\infty}_{f,\mathbf{q}}(x) = \sup_k (f(|x_k|))^{p_k/r} \qquad (x \in sac_0^{\mathbf{p}}(f))$$

is an F*-norm on $sac_0^{\mathbf{p}}(f)$ if and only if f is unbounded and* $\inf_k p_k > 0$.

Corollary 2. *The functional g^{∞}_f is an* F*-norm on $sac_0(f)$ if and only if f is unbounded.*

Subsequently, we shall interpret Corollaries 1 and 2 as counterexamples concerning the topologization of the various extensions of the spaces $sac_0(f)$ and $sac_0^{\mathbf{p}}(f)$ considered in [2, 3, 4, 8, 11, 12]. Several of these extensions are spaces of the type (cf. [2])

$$w_0(\mathcal{B}, f, \mathbf{p}) = \{x = (x_k : \lim_n \sum_k b_{nk}(i)(f(|x_k|))^{p_k} = 0 \text{ uniformly in } i\},$$

where $\mathbf{p} = (p_k)$ is a bounded sequence of positive numbers and $\mathcal{B}$ is a sequence of infinite non-negative matrices $B_i = (b_{nk}(i))$. The space $w_0(\mathcal{B}, f, \mathbf{p})$ reduces to $sac_0^{\mathbf{p}}(f)$ if $\mathcal{B} = \mathcal{B}_1$ where $\mathcal{B}_1 = (b^1_{nk}(i))$ with

$$b^1_{nk}(i) = \begin{cases} 1/n & \text{if } i \le k < i+n \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = (a_{nk})$ be an infinite non-negative matrix. Proposition 3 of Nanda [11] asserts that, for every bounded sequence $\mathbf{p} = (p_k)$ of positive numbers, the space $w_0(\mathcal{B}, f, \mathbf{p})$ with

$$b_{nk}(i) = \frac{1}{n+1} \sum_{j=0}^{n} a_{i+j,k}$$

can be paranormed by the functional $g^{\mathbf{p}}_{\mathcal{B},f}$ where

$$g^{\mathbf{p}}_{\mathcal{B},f}(x) = \sup_{n,i} \Big(\sum_k b_{nk}(i)(f(|x_k|))^{p_k}\Big)^{1/r}.$$

Since, for A being the unit matrix, this space is exactly the space $sac_0^{\mathbf{p}}(f)$ with $g^{\infty}_{f,\mathbf{q}} = g^{\mathbf{p}}_{\mathcal{B},f}$, Corollary 1 shows that Proposition 3 of [11] is not true for every $\mathbf{p} = (p_k)$ and every f. The same inaccuracy is contained in Theorem 2 of Bilgin [2].

A generalization of the space $sac_0(f)$ is related to invariant means (or σ-means). Consider a one-to-one mapping σ of $\mathbb{N}$ into itself such that

$\sigma^k(n) \neq n$ for all $n, k \in \mathbb{N}$, where $\sigma^k(n)$ denotes the iterate of order k of the mapping σ at n. Nuray and Savas [12] introduced the space

$$w_0(A_\sigma, f) = \{x = (x_k) : \lim_n \sum_k a_{nk} f(|x_{\sigma^k(i)}|) = 0 \text{ uniformly in } i\}.$$

Theorem 3 of [12] claims that $w_0(A_\sigma, f)$ can be topologized by the paranorm

$$g(x) = \sup_{n,i} \sum_k a_{nk} f(|x_{\sigma^k(i)}|)$$

for an arbitrary modulus function f. But our Corollary 2 shows that this is not true for all modulus functions, since $w_0(A_\sigma, f)$ reduces to $sac_0(f)$ if $\sigma(n) = n + 1$ and $A = C_1$ is the matrix of arithmetical means. A similar correction is needed in some results of Esi [3, 4].

It should be noted that an inaccuracy of the same type is contained also in Proposition 6 of the author [8] that deals with the F-seminormability of the more general space

$$w_0^p(\mathcal{B}, \mathcal{F}) = \{x = (x_k) : \lim_n \sum_k b_{nk}(i)(f_k(|x_k|))^p = 0 \text{ uniformly in } i\},$$

where $p \geq 1$. This proposition asserts that the space $w_0^p(\mathcal{B}, \mathcal{F})$ is a complete F-seminormed AK-space with the F-seminorm

$$g^p_{\mathcal{B},\mathcal{F}}(x) = \sup_{n,i} (\sum_k b_{nk}(i)(f_k(|x_k|))^p)^{1/p}$$

for an arbitrary sequence of modulus functions $\mathcal{F} = (f_k)$. But this is not true in general. For example, if a bounded sequence $\mathbf{p} = (p_k)$ of positive numbers is such that $\inf_k p_k = 0$, then, in view of the equality

$$w_0^r(\mathcal{B}_1, \mathcal{F}^{\mathbf{q}}) = sac_0^{\mathbf{p}}(f),$$

the functional $g^r_{\mathcal{B}_1,\mathcal{F}^{\mathbf{q}}}$ is not an F-seminorm on $w_0^r(\mathcal{B}_1, \mathcal{F}^{\mathbf{q}})$ by Corollary 1.

The mentioned inaccuracy arises from Proposition 2 of [8] which asserts that, for an arbitrary matrix sequence $\mathcal{B}$, the space

$$w_0^p(\mathcal{B}) = \{x = (x_k) : \lim_n \sum_k b_{nk}(i)|x_k|^p = 0 \text{ uniformly in } i\}$$

with $p \geq 1$ is a complete seminormed AK-space with respect to the seminorm

$$g^p_{\mathcal{B}}(x) = \sup_{n,i} (\sum_k b_{nk}(i)|x_k|^p)^{1/p}.$$

In fact, $(w_0^p(\mathcal{B}), g_{\mathcal{B}}^p)$ is not an AK-space in general, since it reduces to $(sac_0^p, \| \|_\infty)$ if $\mathcal{B} = \mathcal{B}_1$. A simple argument shows that $(w_0^p(\mathcal{B}), g_{\mathcal{B}}^p)$ is a seminormed AK-space whenever

$$\lim_m \sup_{n,i} \sum_{k=m+1}^{\infty} b_{nk}(i)|x_k|^p = 0 \qquad (x \in w_0^p(\mathcal{B})). \tag{6}$$

Thus, in [8], Propositions 2 and 6 must be reworded, respectively, in the following way.

Proposition 2. *Let $\mathcal{B}$ be a sequence of non-negative matrices. Then $(w_0^p(\mathcal{B}), g_{\mathcal{B}}^p)$ is a complete seminormed space, it is a* BK*-space if $\mathcal{B}$ is column-positive. If* (6) *holds then $(w_0^p(\mathcal{B}), g_{\mathcal{B}}^p)$ is an* AK*-space.*

Proposition 3. *Let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. Then $(w_0^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is a complete* F*-seminormed space, where $g_{\mathcal{B},\mathcal{F}}^p$ is absolutely monotone and has the property* (K). *If $\mathcal{B}$ is column-positive then $(w_0^p(\mathcal{B}, \mathcal{F}), g_{\mathcal{B},\mathcal{F}}^p)$ is an* FK*-space. It is an* AK*-space if* (6) *is satisfied.*

Analogous corrections are needed in Corollaries 3 and 8 of [8].

Finally, using that condition (6) holds for any constant sequence $\mathcal{B} = (A)$, from Proposition 3 we immediately get

Corollary 3. *Let $p \geq 1$, $A = (a_{nk})$ a non-negative matrix, and $\mathcal{F} = (f_k)$ a sequence of modulus functions. Then the space*

$$w_0^p(A, \mathcal{F}) = \{x = (x_k) : \lim_n \sum_k a_{nk}(f_k(|x_k|))^p = 0\}$$

is a complete F*-seminormed* AK*-space with respect to the absolutely monotone* F*-seminorm*

$$g_{A,\mathcal{F}}^p(x) = \sup_n (\sum_k a_{nk}(f_k(|x_k|))^p)^{1/p}.$$

Corollary 3 partially extends Theorem 1 of Bilgin [1].

References

1. T. Bilgin, *On strong A-summability defined by a modulus*, Chinese J. Math. **24** (1996), 159–166.
2. T. Bilgin, *Spaces of strongly A-summable sequences*, Acta et Comment. Univ. Tartuensis Math. 1 (1996), 75–80.
3. A. Esi, *Some new sequence spaces defined by a modulus function*, J. Inst. Math. Comput. Sci. Math. Ser. **8** (1995), 81–86.
4. A. Esi, *Some new sequence spaces defined by a sequence of moduli*, Tr. J. of Mathematics **21** (1997), 61–68.

5. E. Kolk, *Sequence spaces defined by a sequence of moduli*, Problems of Pure and Applied Mathematics. Abstracts of conference (Tartu, September 21–22, 1990), Tartu, 1990, pp. 131–134.
6. E. Kolk, *On strong boudedness and summability with respect to a sequence of moduli*, Tartu Ül. Toimetised **960** (1993), 41–50.
7. E. Kolk, *Inclusion theorems for some sequence spaces defined by a sequence of moduli*, Tartu Ül. Toimetised **970** (1994), 65–72.
8. E. Kolk, *F-seminormed sequence spaces defined by a sequence of modulus functions and strong summability*, Indian J. Pure Appl. Math. **28** (1997), 1547–1566.
9. I. J. Maddox, *A new type of convergence*, Math. Proc. Cambridge Philos. Soc. **83** (1978), 61–64.
10. I. J. Maddox, *Sequence spaces defined by a modulus*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 161–166.
11. S. Nanda, *Some sequence spaces and almost convergence*, J. Austral. Math. Soc. **22** (1976), 446–455.
12. F. Nuray and E. Savas, *Some new sequence spaces defined by a modulus function*, Indian J Pure Appl. Math. **24** (1993), 657–663.
13. W. H. Ruckle, *FK spaces in which the sequence of coordinate vectors is bounded*, Canad. J. Math. **25** (1973), 973–978.
14. J. Swetits, *Strongly almost convergent sequences*, Publ. Inst. Math. **22** (1977), 259–265.

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