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Sequence spaces defined by a sequence of modulus functions and $\mathcal{X}$-nearly convergence

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ABSTRACT. Let E be a sequence space and let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. For $x = (\xi_k) \in E$ define $\mathcal{F}(x) = (f_k(|\xi_k|))$. The purpose of this paper is to investigate relations between the inclusion $\mathcal{F}(x) \in E$ and the $\mathcal{X}$-nearly convergence of the sequences x and $\mathcal{F}(x)$.

1. Introduction

The notions of zero-classes and $\mathcal{X}$-nearly convergence were introduced by Freedman and Sember [3].

Let $\mathbb{N}$ denote the set of positive integers. A class $\mathcal{X}$ of the subsets of $\mathbb{N}$ is called a *zero-class* if the following conditions hold:

(1) A finite $\Rightarrow A \in \mathcal{X}$,
(2) $A, B \in \mathcal{X} \Rightarrow A \cup B \in \mathcal{X}$,
(3) $A \subset B,\ B \in \mathcal{X} \Rightarrow A \in \mathcal{X}$,
(4) $\mathbb{N} \notin \mathcal{X}$.

Definition. Let $\mathcal{X}$ be a zero-class. A number sequence $x = (\xi_k)$ is called *$\mathcal{X}$-nearly convergent* to l if there exists a set $Z \in \mathcal{X}$ such that

$$\lim_{k \in \mathbb{N}\setminus Z} \xi_k = l.$$

The sets of all bounded real $\mathcal{X}$-nearly convergent and $\mathcal{X}$-nearly convergent to zero sequences are denoted respectively by $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{X}}^0$. Let c, c_0 and ℓ_∞ be respectively the spaces of convergent, convergent to zero and bounded real sequences. Then $c \subset \omega_{\mathcal{X}}$, $c_0 \subset \omega_{\mathcal{X}}^0$ and $\omega_{\mathcal{X}}$, $\omega_{\mathcal{X}}^0$ are linear subspaces of ℓ_∞ (cf. [3]).

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Definition. A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* if f is strictly increasing and continuous on $[0, \infty)$, $f(t + u) \leq f(t) + f(u)$ and $f(0) = 0$.

Let E be a sequence space of real sequences and $\mathcal{F} = (f_k)$ be a sequence of modulus functions. The space $E(\mathcal{F})$ is defined as follows:

$$E(\mathcal{F}) = \{x = (\xi_k) : \ \mathcal{F}(x) = (f_k(|\xi_k|)) \in E\}.$$

The spaces of this kind were introduced by Ruckle [12] and Maddox [10] for $\mathcal{F} = (f)$, this definition was extended by Kolk [5] to non-constant sequences $\mathcal{F} = (f_k)$.

In this paper we investigate relations between $E(\mathcal{F}) \cap \ell_\infty$ and $\omega^0_{\mathcal{X}}$ for certain zero-classes $\mathcal{X}$.

2. Some preliminary results

Let E be a sequence space such that

(5) $c_0 \subset E$,

(6) E is solid, i.e. $\ell_\infty \cdot E \subset E$,

(7) $e \notin E$, where $e = (1, 1, ...)$.

We denote the characteristic sequence of a set $Z \subset \mathbb{N}$ by $\varphi_Z = (\varphi^Z_k)$, i.e.

$$\varphi^Z_k = \begin{cases} 1 & \text{if } k \in Z, \\ 0 & \text{if } k \notin Z. \end{cases}$$

It is easy to check that if a sequence space E satisfies conditions (5)–(7), then the system

$$\mathcal{X}_E = \{Z \subset \mathbb{N} : \ \varphi_Z \in E\}$$

is a zero-class. For $\mathcal{X} = \mathcal{X}_E$ we further denote $\omega_{\mathcal{X}_E} = \omega_E$ and $\omega^0_{\mathcal{X}_E} = \omega^0_E$.

Proposition 1. *Let E be a sequence space which satisfies conditions* (5)–(7). *Then*

$$\omega^0_E \subset E \cap \ell_\infty \subset \overline{\omega^0_E},$$

where the closure is taken with respect to the norm on ℓ_∞.

Proof. 1) Let $x = (\xi_k) \in \omega^0_E$, i.e. there exits a set $Z \subset \mathbb{N}$ such that $\lim_{k \in \mathbb{N} \setminus Z} \xi_k = 0$ and $\varphi_Z \in E$. We may write

$$x = \varphi_{\mathbb{N} \setminus Z} x + \varphi_Z x.$$

Then $\varphi_{\mathbb{N} \setminus Z} x \in c_0 \subset E$. Since E is solid, $x \in \omega^0_E \subset \ell_\infty$ and $\varphi_Z \in E$, we also have $\varphi_Z x \in E \cap \ell_\infty$. Therefore $x \in E \cap \ell_\infty$ for each $x \in \omega^0_E$.

2) Let now $x = (\xi_k) \in E \cap \ell_\infty$. Define

$$Z_n = \{k \in \mathbb{N} : \ |\xi_k| > n^{-1}\}$$

and take $y = (\eta_k)$ such that

$$\eta_k = \begin{cases} 1/\xi_k & \text{for } k \in Z_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then $y \in \ell_\infty$ for each fixed n and so $\varphi_{Z_n} = y \cdot x \in E$, because E is solid. Now define $y_n = (\eta_k^n)$ by

$$\eta_k^n = \begin{cases} \xi_k & \text{for } k < n, \\ \xi_k & \text{for } k > n \text{ and } k \in Z_n, \\ 0, & \text{otherwise.} \end{cases}$$

Then it follows from the representation of y_n that $y_n \in \omega_E^0$ for each $n \in \mathbb{N}$. Since $|\eta_k^n - \xi_k| < 1/n$ for each $k \in \mathbb{N}$, the sequence y_n converges to x in ℓ_∞ and therefore $x \in \overline{\omega_E^0}$. □

3. The space $E(\mathcal{F}) \cap \ell_\infty$ and $\mathcal{X}$-nearly convergence

Further we use the following characteristics for a sequence $\mathcal{F} = (f_k)$ of modulus functions:

(8) $\sup_k f_k(t) < \infty$ for each $t > 0$,
(9) $\lim_{t \to 0+} \sup_k f_k(t) = 0$,
(10) $\inf_k f_k(t) > 0$ for each $t > 0$.

Kolk (cf. [6] and [7]) proved that (8) $\Leftrightarrow \ell_\infty \subset \ell_\infty(\mathcal{F})$, (9) $\Leftrightarrow c_0 \subset c_0(\mathcal{F})$ and (10) $\Leftrightarrow c_0(\mathcal{F}) \subset c_0$.

Proposition 2. *If a sequence space E satisfies conditions* (5)–(7) *and a sequence $\mathcal{F} = (f_k)$ of modulus functions satisfies conditions* (8)–(10), *then*

$$\omega_E^0 = \omega_E^0(\mathcal{F}) \cap \ell_\infty.$$

Proof. Let $x \in \omega_E^0$, then $\varphi_{\mathbb{N}\setminus Z} \cdot x \in c_0$ and $\varphi_Z \in E$ for a certain set $Z \subset \mathbb{N}$. Then it follows from (9) that also $\varphi_{\mathbb{N}\setminus Z} \cdot \mathcal{F}(x) \in c_0$, i.e. $\mathcal{F}(x)$ is $\mathcal{X}_E$-nearly convergent to zero. As $\omega_E^0 \subset \ell_\infty$, we have $x \in \ell_\infty$ and by (8) we have $\mathcal{F}(x) \in \ell_\infty$. Therefore, $\mathcal{F}(x) \in \omega_E^0$ and $x \in \omega_E^0(\mathcal{F}) \cap \ell_\infty$. In the same manner we can show that if (10) holds, then $\omega_E^0(\mathcal{F}) \cap \ell_\infty \subset \omega_E^0$. □

Lemma 1. *Let $B \subset \ell_\infty$ be a solid sequence space and let $\mathcal{F} = (f_k)$ be a sequence of modulus functions. If*

(11) *for each interval $I_b = [0, b]$, $b > 0$, there exists $H > 0$ so that for all $u_1, u_2 \in I_b$ and $k \in \mathbb{N}$*

$$| f_k^{-1}(u_2) - f_k^{-1}(u_1) | \leq H | u_2 - u_1 |,$$

then $\overline{B}(\mathcal{F}) \subset \overline{B}$.

Proof. Take $x = (\xi_k) \in \overline{B}(\mathcal{F})$, i.e. $\mathcal{F}(x) = (f_k(|\xi_k|)) \in \overline{B}$. Then there exists a sequence $y_n = (\eta_k^n) \in B$, $\eta_k^n \geq 0$ so that y_n converges to $\mathcal{F}(x)$ in ℓ_∞. It means that for arbitrary $\varepsilon > 0$ there exists $N_\varepsilon > 0$ such that

$$|\eta_k^n - f_k(|\xi_k|)| < \varepsilon H^{-1} \tag{3.1}$$

for all $n > N_\varepsilon$ and $k \in \mathbb{N}$.

Let $x_n = (\xi_k^n)$, $\xi_k^n \geq 0$, be the sequence such that $f_k(\xi_k^n) = \eta_k^n$. Take

$$\widetilde{\xi}_k^n = \begin{cases} \xi_k^n (sgn \xi_k)^{-1} & \text{if } \xi_k \neq 0, \\ 0 & \text{if } \xi_k = 0. \end{cases}$$

Then we have by assumption (11) and by (3.1) that the following estimations are true (on a certain finite interval):

$$\begin{aligned} |\widetilde{\xi}_k^n - \xi_k| &= |\xi_k^n - |\xi_k|| \\ &= |f_k^{-1}[f_k(\xi_k^n)] - f_k^{-1}[f_k(|\xi_k|)]| \\ &\leq H|\eta_k^n - f_k(|\xi_k|)| < \varepsilon \end{aligned}$$

for all $n > N_\varepsilon$, $k \in \mathbb{N}$. Hence the sequence $\widetilde{x}_n = (\widetilde{\xi}_k^n)$ converges to $x = (\xi_k)$ in ℓ_∞.

We show now that $\widetilde{x}_n \in B$, $n \in \mathbb{N}$. Take in (11) $u_2 = \eta_k^n$ and $u_1 = 0$. We have $0 \leq \xi_k^n = f_k^{-1}(\eta_k^n) < H\eta_k^n$ and, as B is solid, $x_n \in B$ and also $\widetilde{x}_n \in B$.□

Corollary 1. *If a sequence space E satisfies conditions (5)–(7) and a sequence $\mathcal{F} = (f_k)$ of modulus functions satisfies condition (11), then*

$$\overline{\omega_E^0}(\mathcal{F}) \subset \overline{\omega_E^0}.$$

Proof. Take $B = \omega_E^0$ in Lemma 1. □

Example 1. If the functions f_k^{-1} are differentiable on $[0,\infty)$ and the sequence of derivatives of f_k^{-1} is uniformly bounded on each interval $[0,b]$, then applying Lagrangian theorem it is easy to show that condition (11) holds. In the particular case when $f_k(t)=t^{p_k}$, $0<p_k\le 1$, we have that

$$\frac{d}{du}f_k^{-1}(u)=\frac{1}{p_k}u^{1/p_k-1}$$

and condition (11) holds if and only if $\inf_k p_k>0$.

Example 2. Let $\mathcal{F}=(f)$, where f is an unbounded modulus function. Then condition (11) is the Lipschitz condition for the inverse function f^{-1}. As the inverse function f^{-1} of an unbounded modulus function f is convex in each interval $(0,b)$, $b>0$, it satisfies Lipschitz condition (cf. [8], Lemma 1.3) and (11) is fulfilled.

Lemma 2. *If condition* (11) *holds, then* $\ell_\infty(\mathcal{F})\subset\ell_\infty$ *and* $c_0(\mathcal{F})\subset c_0$ (*i.e.* (10) *is fulfilled*).

Proof. Let $(f_k(|\xi_k|))=(\eta_k)\in\ell_\infty$, then $0\le\eta_k\le b$ for a certain $b>0$. Take $u_2=\eta_k$, $u_1=0$ in (11), then we have:

$$|\xi_k|\le Hb,$$

i.e. $x=(\xi_k)\in\ell_\infty$.

In the same manner we can show that $c_0(\mathcal{F})\subset c_0$. □

Proposition 3. *Let a sequence space E satisfy conditions* (5)–(7) *and a sequence $\mathcal{F}=(f_k)$ of modulus functions satisfy conditions* (8), (9) *and* (11). *Then*

$$\omega_E^0\subset E(\mathcal{F})\cap\ell_\infty\subset\overline{\omega_E^0}. \tag{3.2}$$

Proof. It follows immediately from Proposition 1 that

$$\omega_E^0(\mathcal{F})\subset(E\cap\ell_\infty)(\mathcal{F})\subset\overline{\omega_E^0}(\mathcal{F}).$$

Then, applying Lemma 2 and (8), it is not difficult to show that

$$(E\cap\ell_\infty)(\mathcal{F})=E(\mathcal{F})\cap\ell_\infty$$

and (3.2) follows immediately from Proposition 2 (by Lemma 2, (10) is satisfied) and Corollary 1. □

Proposition 4. *Let a sequence space E satisfy conditions* (5)–(7) *and* $E\cap\ell_\infty=\omega_E^0$. *If a sequence $\mathcal{F}=(f_k)$ of modulus functions satisfies conditions* (8)–(10), *then*

$$E(\mathcal{F})\cap\ell_\infty=E\cap\ell_\infty.$$

Proof. It follows immediately from $E\cap\ell_\infty=\omega_E^0$ that $E(\mathcal{F})\cap\ell_\infty(\mathcal{F})=\omega_E^0(\mathcal{F})$. Since $\ell_\infty\subset\ell_\infty(\mathcal{F})$ by (8), we may write $E(\mathcal{F})\cap\ell_\infty=\omega_E^0(\mathcal{F})\cap\ell_\infty$. Applying also Proposition 2 we have $E(\mathcal{F})\cap\ell_\infty=\omega_E^0=E\cap\ell_\infty$. □

Remark. If $\mathcal{F}=(f)$, then conditions (8)–(10) are fulfilled.

4. Some sequence spaces related to $\mathcal{X}$-nearly convergence

4.1. The space $[c_A]_0(\mathcal{F})$. Let $A = (a_{nk})$ be a regular matrix method with $a_{nk} \geq 0$ and

$$[c_A]_0 = \{x = (\xi_k) : \lim_n \sum_k a_{nk}|\xi_k| = 0\},$$

i.e. $[c_A]_0$ is the set of strongly A-summable to zero sequences.

Take $E = [c_A]_0$, then it is easy to check that conditions (5)–(7) are fulfilled. By results of Hill and Sledd [4] and Sember and Freedman [3] we have

$$E \cap \ell_\infty = \omega_E^0 = \overline{\omega_E^0}. \tag{4.1}$$

Therefore, by Proposition 4, we can state that

$$[c_A]_0(\mathcal{F}) \cap \ell_\infty = [c_A]_0 \cap \ell_\infty \tag{4.2}$$

for each sequence $\mathcal{F} = (f_k)$ of modulus functions which satisfies conditions (8)–(10).

Example 3. Let $E = [c_A]_0$ and $f_k(t) = t^{p_k}$, $0 < p_k \leq 1$. In this case $[c_A]_0(\mathcal{F}) = [c_A(p)]_0$, the space of sequences that are strongly A summable to zero with exponent $p = (p_k)$. Then conditions (8) and (9) are fulfilled and condition (10) is fulfilled if and only if $\inf p_k > 0$ (cf. [7]). Hence for $0 < \inf p_k \leq p_k \leq 1$

$$[c_A(p)]_0 \cap \ell_\infty = [c_A]_0 \cap \ell_\infty.$$

This result is well known if $A = (C, 1)$ and $p_k = \widetilde{p}$, $0 < \widetilde{p} < 1$ (cf. [9]).

Further (cf. Proposition 5) we show that a stronger result than (4.2), namely the equality $[c_A]_0(\mathcal{F}) = [c_A]_0$ is not true in general.

Let $\widetilde{f}(t) = \sup_k f_k(t)$, then condition (9) guarantees that $\widetilde{f}$ is also a modulus function (cf. [5]). Let

$$w_0 = \{x = (\xi_k) : \lim_n (n+1)^{-1} \sum_{k=o}^{n} |\xi_k| = 0\},$$

i.e. $w_0 = [c_A]_0$ for $A = (C, 1)$. It is clear that $E(\widetilde{f}) \subset E(\mathcal{F})$ for each solid space E and thus applying also (4.2) for $\widetilde{f}$ we have

$$w_0 \cap \ell_\infty = w_0(\widetilde{f}) \cap \ell_\infty \subset w_0(\mathcal{F}) \cap \ell_\infty. \tag{4.3}$$

Proposition 5. *If the sequence $\mathcal{F} = (f_k)$ of modulus functions satisfies condition* (9) *and* $\lim_{t\to\infty} \widetilde{f}(t)/t = 0$, *then there exists an unbounded sequence* $z \in w_0(\mathcal{F})\backslash w_0$.

Proof. By the version of Kuttner's theorem proved by Maddox [11] the condition $\lim_{t\to\infty} \widetilde{f}(t)/t = 0$ implies that for each locally convex FK-space X

$$X \supset w_0(\widetilde{f}) \Rightarrow X \supset \ell_\infty \tag{4.4}$$

(for an arbitrary modulus function $\widetilde{f}$). Let us take $X = w_0$, then $\ell_\infty \not\subset w_0$ and by (4.4) we have that $w_0 \not\supset w_0(\widetilde{f})$. Therefore, by (4.3), there exists an unbounded sequence $z \in w_0(\widetilde{f})\backslash w_0$ and also $z \in w_0(\mathcal{F})\backslash w_0$. □

Remark. The condition $\lim_{t\to\infty} f(t)/t = 0$ is fulfilled if the modulus function f is the inverse function of any Orlicz function (cf. [8]).

Remark. Proposition 5 has an extension if we consider the strong summability field determined by a lacunary sequence (cf. [14], Theorem 9) instead of w_0.

4.2. The space $[c_\alpha]_0(\mathcal{F})$. Let $\alpha = (A_i)$ be a sequence of matrices $A_i = (a_{nik})$ with $a_{nik} \geq 0$. Define

$$[c_\alpha]_0 = \{x = (\xi_k) : \lim_n \sum_k a_{nik}|\xi_k| = 0 \textit{ uniformly in } i\},$$

i.e. $[c_\alpha]_0$ is the set of sequences which are strongly summable to zero by a sequential method α.

Take $E = [c_\alpha]_0$. Then it is clear that E is solid, i.e. condition (6) holds. Let the sequential method α be regular, i.e. $\lim_n \sum_k a_{nik}\xi_k = \lim_k \xi_k$ (uniformly in i) for each $x = (\xi_k) \in c$. It is not difficult to show that in this case $c_0 \subset E$ and $e \notin E$. Therefore conditions (5) and (7) also hold.

Let now $f_k(t) = t$, $k \in \mathbb{N}$. Then $E(\mathcal{F}) = E$ and by Proposition 1 we have

$$\omega_E^0 \subset E \cap \ell_\infty \subset \overline{\omega_E^0}.$$

If α is regular, then A_i are regular matrix methods. Then, by Section 4.1, the sets $[c_{A_i}]_0 \cap \ell_\infty$ are closed in ℓ_∞. Moreover (cf. [13])

$$E = \cap_{T \in U}[c_T]_0,$$

where U is the family of matrices $T = (t_{nk})$ such that $t_{nk} = a_{nik}$ for some i. Therefore the set $E \cap \ell_\infty$ is also closed and

$$E \cap \ell_\infty = \overline{\omega_E^0}. \tag{4.5}$$

The following example shows that there exists a sequential method α so that $\omega_E^0 \subsetneqq E \cap \ell_\infty$ (cf. (4.1)).

Example 4. Take

$$a_{nik} = \begin{cases} (n+1)^{-1} & \text{if } i \le k \le i+n, \\ 0 & \text{otherwise.} \end{cases}$$

Then $[c_\alpha]_0 = sac_0$, the space of strongly almost convergent to zero sequences, and (cf. [3])

$$\omega_E^0 \subsetneq sac_0 = \overline{\omega_E^0}.$$

4.3. $\mathcal{X}$-nearly convergence and statistical convergence. Let ω denote the space of all real sequences. For any $x = (\xi_k) \in \omega$ and $\varepsilon > 0$ define

$$N_\varepsilon(x) = \{y = (\eta_k) \in \omega : \sup_k |\xi_k - \eta_k| < \varepsilon\}.$$

Then the class $\{N_\varepsilon(x) : x \in \omega, \varepsilon > 0\}$ forms a base for the topology τ_∞ of uniform convergence on ω. On the space ℓ_∞ this is the usual "sup-norm" topology.

For an arbitrary zero class $\mathcal{X}$ Chun and Freedman [1] defined

$$V_{\mathcal{X}}^0 = \{x = (\xi_k) \in \omega : \forall \alpha > 0,\ Z_\alpha = \{k : |\xi_k| > \alpha\} \in \mathcal{X}\}$$

and proved that $V_{\mathcal{X}}^0$ is the closure (with respect to the topology τ_∞) of the space of $\mathcal{X}$-nearly convergent to zero sequences.

If $A = (a_{nk})$ is a regular matrix method with $a_{nk} \ge 0$ and

$$\mathcal{X} = \{Z \subset \mathbb{N} : \varphi_Z \in [c_A]_0\},$$

then $V_{\mathcal{X}}^0 = st_A^0$, where st_A^0 denotes the set of A-statistically convergent to zero sequences. About A-statistically convergence see, for example, [2] and [7]. If $E = [c_A]_0$, then $\overline{\omega_E^0} = st_A \cap \ell_\infty$ and therefore (4.1) and (4.5) also follow from Theorem 4.4 of [7].

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