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An alternative way to derive the geodesic deviation equation for rapidly diverging geodesics

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ABSTRACT. We present a derivation of the equation of geodesic deviation under the assumption that the geodesics are adjacent in some neigbourhood, but their rate of separation is arbitrary. The resulting modified equation of geodesic deviation is nonlinear, it reduces to the ordinary linear geodesic deviation equation when the changes of position of corresponding points on the two geodesics as well as the changes of directions of the corresponding tangents are small. Our derivation is straightforward but shorter and more lucid than the earlier ones. Some of the consequences of the modified equation are also discussed.

1. Introduction

Let us consider two geodesics L and $\bar{L}$ on a Riemannian manifold and denote their unit tangent vectors as $\vec{u}$ and $\vec{\bar{u}}$, respectively. In the tangent vector space T_P to the manifold, where P is a point on the geodesic L, we define a vector $\vec{\eta}$ so that the exponential image of its endpoint is a point $\bar{P}$ on the second geodesic $\bar{L}$. This vector is called *the geodesic deviation vector*. Its components with respect to the coordinate basis at the point P are, in the first approximation, the differences of the corresponding curvilinear coordinates x^κ and $\bar{x}^\kappa$ of the two points P and $\bar{P}$:

$$\eta^\kappa = \bar{x}^\kappa - x^\kappa + \mathcal{O}_2,$$

where Landau's symbol $\mathcal{O}_2 := \mathcal{O}(|\eta^\kappa|^2)$ denotes the second order small quantities with respect to $|\eta^\kappa|$.

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The evolution of the deviation vector $\vec{\eta}$ can be calculated from the geodesic deviation equation, which relates the second absolute derivatives of its components with respect to the parameter s of the fiducial geodesic to the Riemann curvature tensor. In the case of parallel geodesics, when the rate of separation of geodesics $\frac{D\vec{\eta}}{ds}$ is of the same order of smallness than the deviation vector $\vec{\eta}$, we can use the ordinary geodesic deviation equation, derived by Levi-Civita and Synge in 1926 (see [4, 7]), which has the form

$$\frac{D^2\eta^\kappa}{ds^2} = -R^\kappa_{\lambda\mu\nu}u^\lambda\eta^\mu u^\nu. \tag{1}$$

Here u^λ and $R^\kappa_{\lambda\mu\nu}$ denote, respectively, the components of the tangent vector $\vec{u}$ to the first geodesic line and of the Riemann curvature tensor at the point P. A generalized equation, derived by Hodgkinson in 1974 (see [3] and also [1, 5]), holds in the case of non-parallel geodesics whose rate of separation is arbitrary, and has the form

$$\frac{D^2\eta^\kappa}{ds^2} = -R^\kappa_{\lambda\mu\nu}u^\lambda\eta^\mu u^\nu - 2R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu u^\nu - \frac{2}{3}R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu\frac{D\eta^\nu}{ds}. \tag{2}$$

If the rate of separation of the geodesics is small, this equation reduces, naturally, to the ordinary geodesic deviation equation (1). The ordinary and Hodgkinson deviation equations of the forms (1) and (2) describe the behavior of the deviation vector in the case of the so-called natural correspondence, when the vector $\vec{\eta}$ connects a pair of corresponding points on the geodesics L and $\bar{L}$ with the same value of the affine parameters $s_P = \bar{s}_{\bar{P}}$.

Because the derivation of the formula (2), as given by Hodgkinson, is rather capacious and mathematically complicated, the aim of the present paper is to give an alternative, more "transparent" and shorter derivation. Surprisingly enough, the present straightforward derivation turns out to be not only more lucid but also shorter than the original one by Hodgkinson.

2. Deviation of rapidly diverging geodesics

Let us consider on a Riemannian manifold two geodesic lines, the first, fiducial geodesic $L : x^\kappa = \phi^\kappa(s)$ with an affine parameter s and the second, displaced geodesic $\bar{L} : x^\kappa = \bar{\phi}^\kappa(\bar{s})$ with an affine parameter $\bar{s}$. It is assumed that the geodesics L and $\bar{L}$ are adjacent some time in their history, namely in the vicinity of the points corresponding to the zero values of their parameters $s = \bar{s} = 0$. Let Λ be a mapping from the geodesic line L to the geodesic line $\bar{L}$:

$$\Lambda : L \to \bar{L}.$$

Assume that the pairs of corresponding points of these two geodesics are connected by a family of geodesic lines $\Gamma : x^\kappa = \psi^\kappa(\rho)$, where ρ is the natural parameter along the connecting geodesics, and that each geodesic from the family Γ is uniquely determined by a given pair of points P and $\bar{P}$. Thus Λ is a bijection. Let us consider a pair of corresponding points P and $\bar{P}$ under the bijection Λ. At the point P of the intersection of geodesics Γ and L we have

$$\psi^\kappa(0) = \phi^\kappa(s_P).$$

Denote the geodesic arc length of Γ between the points P and $\bar{P}$ by $\Delta\rho$, then

$$\psi^\kappa(\Delta\rho) = \bar{\phi}^\kappa(\bar{s}_{\bar{P}})$$

are the coordinates of the point of intersection of geodesics Γ and $\bar{L}$. We also define the unit tangent vector to the connecting geodesic Γ at the point P with components

$$n^\kappa = \left.\frac{d\psi^\kappa}{d\rho}\right|_{\rho=0}.$$

Then the components of the deviation vector are

$$\eta^\kappa = n^\kappa \Delta\rho = \left.\frac{d\psi^\kappa}{d\rho}\right|_{\rho=0} \Delta\rho. \tag{3}$$

At the next stage our aim is to express, using Taylor's expansion, the coordinates of the second point $\bar{\phi}^\kappa(\bar{s})$ in terms of the coordinates of the first point $\phi^\kappa(s)$, and the components of the deviation vector. Because the components of the deviation vector are supposed to be infinitely small, but their derivatives with respect to $\bar{s}$ are not, and the second derivatives of the coordinates are needed, we have to write the series including the third order terms. Thus we have

$$\begin{aligned}\bar{\phi}^\kappa(\bar{s}) &= \phi^\kappa(s) + \left.\frac{d\psi^\kappa(\rho)}{d\rho}\right|_{\rho=0} \Delta\rho + \frac{1}{2}\left.\frac{d^2\psi^\kappa(\rho)}{d\rho^2}\right|_{\rho=0} \Delta\rho^2 \\ &+ \frac{1}{6}\left.\frac{d^3\psi^\kappa(\rho)}{d\rho^3}\right|_{\rho=0} \Delta\rho^3 + \mathcal{O}_4 = \\ &= \phi^\kappa(s) + n^\kappa\Delta\rho + \frac{1}{2}\frac{dn^\kappa}{d\rho}\Delta\rho^2 + \frac{1}{6}\frac{d^2n^\kappa}{d\rho^2}\Delta\rho^3 + \mathcal{O}_4.\end{aligned}$$

The differential equation of the connecting geodesic Γ

$$\frac{d^2\psi^\kappa(\rho)}{d\rho^2} = -\Gamma^\kappa_{\lambda\mu}\frac{d\psi^\lambda(\rho)}{d\rho}\frac{d\psi^\mu(\rho)}{d\rho}$$

can be written with the help of relation (3) as

$$\frac{dn^\kappa}{d\rho} = -\Gamma^\kappa_{\lambda\mu}n^\lambda n^\mu.$$

Taking into account the last formula and replacing the quantities $n^\kappa \Delta\rho$ with the components η^κ from relation (3), we obtain for the coordinates of the point $\bar{P}$ on the displaced geodesic the following expression (cf. [2]):

$$\bar{\phi}^\kappa(\bar{s}) = \phi^\kappa(s) + \eta^\kappa - \frac{1}{2}\Gamma^\kappa_{\lambda\mu}\eta^\lambda\eta^\mu - \frac{1}{6}\Gamma^\kappa_{\lambda\mu,\nu}\eta^\lambda\eta^\mu\eta^\nu + \frac{1}{3}\Gamma^\kappa_{\pi\nu}\Gamma^\pi_{\lambda\mu}\eta^\lambda\eta^\mu\eta^\nu + \mathcal{O}_4.$$

Taking now the first ordinary derivative of the last expression with respect to the parameter $\bar{s}$, the affine parameter of the second geodesic, we obtain

$$\begin{aligned} \frac{d\bar{\phi}^\kappa(\bar{s})}{d\bar{s}} = \Big(u^\kappa &+ \frac{d\eta^\kappa}{ds} - \frac{1}{2}\Gamma^\kappa_{\lambda\mu,\nu}\eta^\lambda\eta^\mu u^\nu - \Gamma^\kappa_{\lambda\mu}\frac{d\eta^\lambda}{ds}\eta^\mu \\ &- \frac{1}{3}\Gamma^\kappa_{\lambda\mu,\nu}\eta^\lambda\frac{d\eta^\mu}{ds}\eta^\nu - \frac{1}{6}\Gamma^\kappa_{\lambda\mu,\nu}\eta^\lambda\eta^\mu\frac{d\eta^\nu}{ds} \\ &+ \frac{2}{3}\Gamma^\kappa_{\pi\nu}\Gamma^\pi_{\lambda\mu}\eta^\lambda\frac{d\eta^\mu}{ds}\eta^\nu + \frac{1}{3}\Gamma^\kappa_{\pi\nu}\Gamma^\pi_{\lambda\mu}\eta^\lambda\eta^\mu\frac{d\eta^\nu}{ds}\Big)\frac{ds}{d\bar{s}} + \mathcal{O}_3. \end{aligned}$$

Differentiating the last equation and taking into account that L and $\bar{L}$ are geodesic lines, we obtain the following equation:

$$\begin{aligned} &\frac{d^2\eta^\kappa}{ds^2} + 2\Gamma^\kappa_{\lambda\mu}u^\lambda\frac{d\eta^\mu}{ds} - \Gamma^\kappa_{\lambda\mu}\frac{d^2\eta^\lambda}{ds}\eta^\mu + \Gamma^\kappa_{\lambda\mu,\nu}u^\lambda u^\mu\eta^\nu \\ &-2\Gamma^\kappa_{\lambda\mu,\nu}\frac{d\eta^\lambda}{ds}\eta^\mu u^\nu + 2\Gamma^\kappa_{\lambda\mu,\nu}\frac{d\eta^\lambda}{ds}u^\mu\eta^\nu + \frac{2}{3}R^\kappa_{\lambda\mu\nu}\frac{d\eta^\lambda}{ds}\eta^\mu\frac{d\eta^\nu}{ds} \\ &+\left(u^\kappa + \frac{d\eta^\kappa}{ds} - \Gamma^\kappa_{\lambda\mu}\frac{d\eta^\lambda}{ds}\eta^\mu\right)\frac{d^2 s}{d\bar{s}}\left(\frac{d\bar{s}}{ds}\right)^2 + \mathcal{O}_2 = 0. \end{aligned} \tag{4}$$

As the next step we substitute the first and second ordinary derivatives of the deviation vector by the corresponding covariant derivatives from the following definitions:

$$\frac{d\eta^\kappa}{ds} = \frac{D\eta^\kappa}{ds} - \Gamma^\kappa_{\lambda\mu}u^\lambda\eta^\mu,$$

$$\frac{d^2\eta^\kappa}{ds^2} = \frac{D^2\eta^\kappa}{ds^2} - \Gamma^\kappa_{\lambda\mu,\nu}u^\lambda\eta^\mu u^\nu - \Gamma^\kappa_{\lambda\mu}\frac{du^\lambda}{ds}\eta^\nu - 2\Gamma^\kappa_{\lambda\mu}u^\lambda\frac{d\eta^\mu}{ds} + \Gamma^\kappa_{\pi\nu}\Gamma^\pi_{\lambda\mu}u^\lambda\eta^\mu u^\nu.$$

After substitution of these expressions into the equation (4) and some simplification we obtain

$$\begin{aligned} \frac{D^2\eta^\kappa}{ds^2} &= -R^\kappa_{\lambda\mu\nu}u^\lambda\eta^\mu u^\nu - 2R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu u^\nu \\ &-\frac{2}{3}R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu\frac{D\eta^\nu}{ds} - \Gamma^\kappa_{\lambda\mu}\frac{D^2\eta^\lambda}{ds^2}\eta^\mu \\ &-\left(u^\kappa + \frac{D\eta^\kappa}{ds} - \Gamma^\kappa_{\lambda\mu}u^\lambda\eta^\mu - \Gamma^\kappa_{\lambda\mu}\frac{D\eta^\lambda}{ds}\eta^\mu\right)\frac{d^2 s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 + \mathcal{O}_2. \end{aligned} \tag{5}$$

Thus far we have assumed η^κ to be a first order small quantity and $\dfrac{D\eta^\kappa}{ds}$ to be finite, while no assumption about the order of magnitude of $\dfrac{D^2\eta^\kappa}{ds^2}$ has been made. Next we will consider the following two particular cases: (i)when $\dfrac{D^2\eta^\kappa}{ds^2}$ is finite and (ii) when $\dfrac{D^2\eta^\kappa}{ds^2}$ is a small quantity of the same order as $|\eta^\kappa|$.

(i) Assuming now for the time being $\dfrac{D^2\eta^\kappa}{ds^2}$ to be finite and collecting the zeroth-order terms (these do not contain the components η^κ), we obtain the equation

$$\frac{D^2\eta^\kappa}{ds^2}\left(\frac{ds}{d\bar{s}}\right)^2 + \left(u^\kappa + \frac{D\eta^\kappa}{ds}\right)\frac{d^2 s}{d\bar{s}} + \mathcal{O}_1 = 0,$$

which can be written as

$$\frac{D}{d\bar{s}}\left[\frac{ds}{d\bar{s}}\left(u^\kappa + \frac{D\eta^\kappa}{ds}\right)\right] + \mathcal{O}_1 = 0.$$

The expression in parentheses is the tangent vector of the displaced geodesic in zeroth approximation. So we get in this case the equation $\dfrac{D\bar{u}^\kappa}{d\bar{s}} = 0$ which is the differential equation of the second geodesic in the zeroth-order approximation. This produces little of interest and we proceed to consider the second particular case.

(ii) Assuming the second derivative $\dfrac{D^2\eta^\kappa}{ds^2}$ to be an infinitesimal of the same order as η^κ, it follows now from the equation (5) that $\dfrac{d^2 s}{d\bar{s}^2}$ is a first order small quantity, and the equation (5) takes the final form

$$\begin{aligned}\frac{D^2\eta^\kappa}{ds^2} = &-R^\kappa_{\lambda\mu\nu}u^\lambda\eta^\mu u^\nu - 2R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu u^\nu - \frac{2}{3}R^\kappa_{\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu\frac{D\eta^\nu}{ds} \\ &-\left(u^\kappa + \frac{D\eta^\kappa}{ds}\right)\frac{d^2 s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 + \mathcal{O}_2.\end{aligned} \tag{6}$$

This equation describes the behavior of the geodesic deviation vector in case of rapidly diverging geodesics in a neighbourhood where the separation of the geodesics is infinitesimal but the magnitude of the rate of separation is arbitrary. If the rate of separation of the geodesics is small, the derived equation reduces to the ordinary geodesic deviation equation (1).

3. Discussion

In this concluding section we consider some particular ways of establishing a correspondence between the points of two geodesics in the $(n+1)$-dimensional pseudo-Riemannian spacetime, and present some general remarks.

At first one should mention the case of so called *natural correspondence*, when the deviating geodesics are parametrized by their natural parameters, and the connecting geodesics Γ join such pairs of points on these geodesics whose values of natural parameters are equal. (In physical terms the natural correspondence corresponds to the case when the initially synchronized clocks, comoving with the freely falling point masses, show the same readings.) Now the second derivative $\dfrac{d^2s}{d\bar{s}^2}$ is zero and we obtain, from (6), the Hodgkinson equation in its mathematically simplest and shortest form (2).

Next let us consider the case of so called *orthogonal correspondence*. Then the corresponding points on the neighbouring geodesics are chosen so that Γ and L are orthogonal at every point of L. Construct an orthonormal basis $(\vec{e_0}, \vec{e_1}, ..., \vec{e_n})$ at a certain point of L. The zeroth basis vector $\vec{e_0} = \vec{u}$ is supposed to be the tangent vector $\vec{u}$ to the fiducial geodesic. Propagate the basis parallelly along the fiducial geodesic. In the comoving coordinate system, where $u^0 = 1$ and $u^i = 0$ for all $i = 1, ..., n$, the deviation vector $\vec{\eta}$ has only spatial components, i.e. $\eta^0 \equiv 0$, and the deviation equation (6) takes the form

$$\frac{D^2\eta^i}{ds^2} = -R^i_{0j0}\eta^j - 2R^i_{jk0}\frac{D\eta^j}{ds}\eta^k - \frac{2}{3}R^i_{jkl}\frac{D\eta^j}{ds}\eta^k\frac{D\eta^l}{ds} - \frac{D\eta^i}{ds}\frac{d^2s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2. \quad (7)$$

To determine the quantity $\dfrac{d^2s}{d\bar{s}^2}\left(\dfrac{d\bar{s}}{ds}\right)^2$, we return to the formula (6) and contract it with the covector $g_{\kappa\pi}u^\pi$:

$$g_{\kappa\pi}u^\kappa\frac{D^2\eta^\pi}{ds^2} = -u^\kappa R_{\kappa\lambda\mu\nu}u^\lambda\eta^\mu u^\nu - 2u^\kappa R_{\kappa\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu u^\nu$$
$$-\frac{2}{3}u^\kappa R_{\kappa\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu\frac{D\eta^\nu}{ds} - \left(g_{\kappa\lambda}u^\kappa u^\lambda + u^\kappa\frac{D\eta^\pi}{ds}g_{\kappa\pi}\right)\frac{d^2s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2.$$

The first term on the right hand side vanishes, because the components of the curvature tensor are antisymmetric. The zeroth component of the deviation vector is zero at every point on L in the constructed moving basis in the case of orthogonal correspondence. This means that its covariant derivatives $\dfrac{D\eta^0}{ds}$ and $\dfrac{D^2\eta^0}{ds^2}$ must also be zero. Consequently, the left hand side of the last equation and the last term in parentheses vanish. The first term in parentheses equals unity, so we can write:

$$\frac{d^2 s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 = -2u^\kappa R_{\kappa\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu u^\nu - \frac{2}{3}u^\kappa R_{\kappa\lambda\mu\nu}\frac{D\eta^\lambda}{ds}\eta^\mu\frac{D\eta^\nu}{ds}.$$

In the orthonormal moving basis we get

$$\frac{d^2 s}{d\bar{s}^2}\left(\frac{d\bar{s}}{ds}\right)^2 = -2R^0_{ij0}\frac{D\eta^i}{ds}\eta^j - \frac{2}{3}R^0_{ijk}\frac{D\eta^i}{ds}\eta^j\frac{D\eta^\kappa}{ds}.$$

As the last step we substitute this result into equation (7). Then we will obtain the geodesic deviation equation for the case of orthonormal correspondence in the moving tetrad in the following form:

$$\frac{D^2\eta^i}{ds^2} = -R^i_{0j0}\eta^j - 2R^i_{jk0}\frac{D\eta^j}{ds}\eta^k - \frac{2}{3}R^i_{jkl}\frac{D\eta^j}{ds}\eta^k\frac{D\eta^l}{ds}$$
$$+2\left(R^0_{kj0}\frac{D\eta^k}{ds}\eta^j + \frac{1}{3}R^0_{ljk}\frac{D\eta^l}{ds}\eta^j\frac{D\eta^k}{ds}\right)\frac{D\eta^i}{ds}.$$

The last equation, depending cubically on the rate of separation vector, is mathematically even more complicated than the deviaton equation (6) which is also a non-linear differential equation with respect to the deviation vector and its covariant derivative. However, from the physical point of view the last equation is conceptually more simple than the deviation equation (6), as it gives in physical terminology for an observer co-moving with the first point mass the components of the relative spatial acceleration of two instantaneously neighbouring, freely falling point masses which are departing with a relativistic relative velocity. One should emphasize that due to its non-linearity the geodesic deviation equation for rapidly diverging geodesics differs considerably from geodesic deviation equation for almost parallel geodesics as the displacements in the orthogonal directions are no longer independent. The non-linearity causes the absence of the principle of superposition of displacements valid, and makes this generalized geodesic deviation equation (6) mathematically very complicated to solve.

Some applications of the modified geodesic deviation equation (6) are tackled in our paper [6].

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