ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 4, 2000

On sequence spaces defined by a regularly varying modulus

ENNO KOLK

ABSTRACT. Let λ be an F-seminormed solid sequence space. We characterize the F-seminormability of the sequence space $\lambda(f) = \{(x_k) : (f(|x_k|)) \in \lambda\}$ for a regularly varying modulus function f.

1. Introduction

By the term *sequence space* we shall mean, as usual, any linear subspace of the vector space ω of all (real or complex) sequences $x = (x_k) = (x_k)_{k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, \ldots\}$. A sequence space λ is called *solid* if $(x_k) \in \lambda$ and $|y_k| \leq |x_k| \quad (k \in \mathbb{N})$ yield $(y_k) \in \lambda$.

A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* (or simply a *modulus*) if

(M1) $f(t) = 0$ if and only if $t = 0$,
(M2) $f(t + u) \leq f(t) + f(u)$,
(M3) f is increasing,
(M4) f is continuous from the right at 0.

Provided a modulus f and a sequence spaces λ, Ruckle [4], Maddox [3] and some other authors define a new sequence space $\lambda(f)$ by

$$\lambda(f) = \{(x_k) \in \omega : \ (f(|x_k|)) \in \lambda\}.$$

It is not difficult to see that $\lambda(f)$ is a solid sequence space whenever the sequence space λ is solid.

Received October 31, 2000.

2000 *Mathematics Subject Classification.* 46A45, 26A12, 40F05.

Key words and phrases. Modulus function, F-seminorm, regularly varying function, slowly varying function.

This research was in part supported by Estonian Scientific Foundation Grant 3991.

A positive, finite and measurable function f, defined on $[a, \infty)$ for some $a > 0$, is said to be *regularly varying at infinity* (see [1]) if the limit

$$\lim_{t\to\infty} \frac{f(ut)}{f(t)} = \mu(u)$$

is positive and finite for each $u > 0$. The function $\mu(u)$ is called *index function* of regularly varying function f. It is known ([1], Theorem 1.4.1) that the index function of a regularly varying function f is necessarily in the form

$$\mu(u) = u^\rho$$

for some $\rho \in \mathbb{R}$ and for each $u > 0$. Here the number ρ is called *index* of f. Thus f varies regularly with index ρ at infinity if for each $u > 0$,

$$\lim_{t\to\infty} \frac{f(ut)}{f(t)} = u^\rho. \tag{1}$$

A positive, finite and measurable function f, defined on $(0, b]$ for some $b > 0$, is said to be *regularly varying of index $\sigma \in \mathbb{R}$ at the origin* if

$$\lim_{t\to 0+} \frac{f(ut)}{f(t)} = u^\sigma \tag{2}$$

for each $u > 0$. This is equivalent to saying that the function $f(1/t)$ varies regularly with index $-\sigma$ at infinity.

Regularly varying function of index $\rho = 0$ $(\sigma = 0)$ is said to be *slowly varying at infinity (at the origin)*.

In this note we describe the F-seminormability of $\lambda(f)$ assuming that λ is an F-seminormed solid sequence space and the modulus function f is regularly varying at infinity and at the origin.

2. On topologization of $\lambda(f)$

Recall that an F-*seminorm* g on a vector space X is a functional $g : X \to \mathbb{R}$ satisfying for all $x, y \in X$ the axioms

(N1) $g(0) = 0$,
(N2) $g(x + y) \le g(x) + g(y)$,
(N3) $g(\alpha x) \le g(x)$ for all scalars α with $|\alpha| \le 1$,
(N4) $\lim_n g(\alpha_n x) = 0$ for every scalar sequence (α_n) with $\lim_n \alpha_n = 0$.

An F-seminorm on a solid sequence space λ is said to be *absolutely monotone* if $g(x) \le g(y)$ for all $x = (x_k)$, $y = (y_k)$ from λ with $|x_k| \le |y_k|$ $(k \in \mathbb{N})$.

If the sequence space λ is topologized by an F-seminorm (or paranorm) g then for the topologization of $\lambda(f)$ it is natural to consider the functional g_f defined by

$$g_f(x) = g(f(|x|)) \qquad (x \in \lambda(f)).$$

It is known (cf. [3], Theorem 8) that g_f may not be an F-seminorm on $\lambda(f)$ in general. From some results of Soomer ([5], Theorem 3) and the author ([2], Theorem 1) we immediately get

Proposition 1. *Let f be a modulus and let g be an absolutely monotone* F*-seminorm on a solid sequence space λ. The functional g_f is an absolutely monotone* F*-seminorm on $\lambda(f)$ if f satisfies one of following two equivalent conditions:*

(M5) *There exists a function ν with $f(ut) \le \nu(u)f(t)$ $\quad (0 < u \le 1,\ t > 0)$ and $\lim_{u\to 0+} \nu(u) = 0$;*

(M6) $\displaystyle\lim_{u\to 0+} \sup_{t>0} \frac{f(ut)}{f(t)} = 0.$

First we characterize indices of regularly varying moduli.

Proposition 2. *Any modulus function which is regularly varying at infinity (or at the origin) has index from the interval* $[0, 1]$.

Proof. Let f be a modulus function. By monotonicity and subadditivity of f we have

$$f(ut) \le f(([u] + 1)t) \le ([u] + 1)f(t),$$

where $[u]$ denotes the integer part of u. Hence for all $u, t > 0$,

$$0 < \frac{f(ut)}{f(t)} \le u + 1. \tag{3}$$

If f is also regularly varying at infinity of index ρ, then (1) holds, and by (3) we see that $0 \le \rho \le 1$.

If f is regularly varying at the origin of index σ, then from (2) and (3) we similarly get $0 \le \sigma \le 1$. □

In the following we give some examples of regularly and slowly varying modulus functions.

Example 1. Every bounded modulus f is slowly varying at infinity by

$$\lim_{t\to\infty} \frac{f(ut)}{f(t)} = \frac{\sup_{t>0} f(ut)}{\sup_{t>0} f(t)} = 1.$$

Example 2. The unbounded modulus $f(t) = t^p \quad (0 < p \leq 1)$ is a regularly varying function of index p because of $f(ut)/f(t) = u^p$.

Example 3. The unbounded modulus $f(t) = \ln(1+t)$ is slowly varying at infinity by $\lim_{t\to\infty} \ln(1+ut)/\ln(1+t) = 1$ and regularly varying of index 1 at the origin by $\lim_{t\to 0} \ln(1+ut)/\ln(1+t) = u$.

Example 4. The unbounded modulus $f(t) = t/\ln(t+e^2)$, considered by Maddox (see [3], p. 164), varies regularly with index 1 at infinity and at the origin.

Now we are ready to prove our main result.

Theorem. *Let g be an absolutely monotone* F*-seminorm on a solid sequence space λ. If a modulus f varies regularly with index ρ at infinity and with index σ at the origin, then the functional g_f is an absolutely monotone* F*-seminorm on $\lambda(f)$ whenever* $\min\{\rho,\ \sigma\} > 0$. *If f is slowly varying at infinity or at the origin, then conditions* (M5) *and* (M6) *are not satisfied.*

Proof. Let f be a modulus which varies regularly with index $\rho > 0$ at infinity and with index $\sigma > 0$ at the origin. It is known ([1], Theorem 1.5.2) that the equalities (1) and (2) hold uniformly in u on each interval $(0, b]$ with $b > 0$. Thus for an arbitrary number $\varepsilon > 0$ we can find a number $t_1 > 0$ and a natural number $t_2 > t_1$ such that

$$\frac{f(ut)}{f(t)} < u^\sigma + \frac{\varepsilon}{2} \quad (0 < t < t_1), \quad \frac{f(ut)}{f(t)} < u^\rho + \frac{\varepsilon}{2} \quad (t_2 < t < \infty)$$

for all $u \in (0, 1]$. If $t \in [t_1, t_2]$, then by monotonicity and subadditivity of f we have

$$\frac{f(ut)}{f(t)} \leq \frac{t_2}{f(t_1)} f(u).$$

Since u^σ, u^ρ and $f(u)$ tend to zero as $u \to 0+$, there exists $\delta > 0$ such that

$$max\{u^\sigma,\ u^\rho\} < \frac{\varepsilon}{2}, \quad f(u) < \frac{f(t_1)}{t_2}\varepsilon$$

for $0 < u < \delta$. Consequently, we get

$$\sup_{t>0} \frac{f(ut)}{f(t)} \leq \varepsilon \qquad (0 < u < \delta),$$

which shows that the condition (M6), and hence the equivalent condition (M5), of Proposition 1 hold.

If the modulus f varies slowly at infinity or at the origin, then

$$\lim_{t\to\infty} \frac{f(ut)}{f(t)} = 1 \quad \textit{or} \quad \lim_{t\to 0+} \frac{f(ut)}{f(t)} = 1.$$

Since for $0 < u \leq 1$ we have

$$0 < \frac{f(ut)}{f(t)} \leq 1,$$

then clearly

$$\sup_{t>0} \frac{f(ut)}{f(t)} = 1.$$

Thus (M6), and hence (M5), are not satisfied in this case. □

Examples 1–4 show that the conditions (M5) and (M6) of Proposition 1 hold for moduli $f(t) = t^p \quad (0 < p \leq 1)$ and $f(t) = t/\ln(t + e^2)$, but are not satisfied for bounded moduli and for unbounded modulus $f(t) = \ln(1 + t)$.

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Institute of Pure Mathematics, University of Tartu, 50090 Tartu, Estonia
E-mail address: ekolk@math.ut.ee