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On r-convex sequence spaces defined by modulus functions

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ABSTRACT. Let E be a sequence space and let $\mathcal{F} = (f_k)$ be a sequence of r-convex modulus functions. The purpose of this paper is to study some properties of the spaces $E(\mathcal{F}) = \{(\xi_k) | (f_k(|\xi_k|)) \in E\}$.

1. Introduction

Let E be a real linear space and $r > 0$.

Definition 1. A set $K \subseteq E$ is called *r-convex* in E if $x, y \in K$ and $\alpha, \beta \geq 0$ with $\alpha^r + \beta^r = 1$ imply $\alpha x + \beta y \in K$.

Remark 1. If $r = 1$, then the above definition gives the concept of a convex set in a linear space.

Definition 2. A functional φ on an r-convex set $K \subseteq E$ is called *r-convex* if for all $x, y \in K$ and $\alpha, \beta \geq 0$ with $\alpha^r + \beta^r = 1$ one has the inequality

$$\varphi(\alpha x + \beta y) \leq \alpha^r \varphi(x) + \beta^r \varphi(y).$$

Definition 3. A function $f : [0, \infty) \to [0, \infty)$ is called a *modulus function* if f is strictly increasing and continuous on $[0, \infty)$, $f(t+u) \leq f(t) + f(u)$ for all $u, t \geq 0$ and $f(0) = 0$.

Let E be a sequence space of real sequences and $\mathcal{F} = (f_k)$ be a sequence of modulus functions. The space $E(\mathcal{F})$ is defined as follows:

$$E(\mathcal{F}) = \{x = (\xi_k) |\ \mathcal{F}(x) = (f_k(|\xi_k|)) \in E\}.$$

In this paper we investigate some properties of the spaces $E(\mathcal{F})$ defined by r-convex $(0 < r \leq 1)$ modulus functions.

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2. Preliminaries

A sequence space E is called *solid* (or *normal*) if from $(\eta_k) \in E$ and $|\xi_k| \leq |\eta_k|$, it follows that $(\xi_k) \in E$.

A real function g on a linear space E is called an F-*seminorm*, if

(i) $g(0) = 0$,

(ii) $|\alpha| \leq 1$ $(\alpha \in \mathbb{K}) \Rightarrow$ $g(\alpha x) \leq g(x)$ for all $x, y \in E$,

(iii) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in E$,

(iv) $\lim_n \alpha_n = 0$ $(\alpha_n \in \mathbb{K})$, $x \in E \Rightarrow \lim_n g(\alpha_n x) = 0$.

An F-*space* is defined as a complete F-seminormed space. An F-seminorm g in a sequence space E is called *absolutely monotone* if $|\xi_k| \leq |\eta_k|$ implies $g(x) \leq g(y)$ for all $x = (\xi_k), y = (\eta_k)$ in E.

Let $g_{\mathcal{F}}(x) = g(\mathcal{F}(x))$. According to the results of Kolk ([3], Theorem 1) and the author ([6], Theorem 3) we immediately get:

Theorem 1. *Let $\mathcal{F} = (f_k)$ be a sequence of moduli and let g be an absolutely monotone* F-*seminorm on a solid sequence space E. The functional $g_{\mathcal{F}}$ defines an absolutely monotone* F-*seminorm on $E(\mathcal{F})$ if one of the following two equivalent conditions holds:*

(F1) *There exists a function ν such that $f_k(ut) \leq \nu(u) f_k(t), 0 < u \leq 1$, $t \geq 0$, and $\lim_{u \to 0+} \nu(u) = 0$;*

(F2) $\lim_{u \to 0+} \sup_k \sup_{t>0} f_k(ut)/f_k(t) = 0$.

Remark 2. It is easy to check that condition (F1) holds for each r-convex $(0 < r \leq 1)$ modulus function.

We give some examples.

Example 1. The function $f(x) = t^p$, $0 < p \leq 1$, is a p-convex modulus function.

Example 2. Let $f(t) = \ln(1 + t)$, then f is a modulus function, but f is not r-convex.

3. On r-convexity of the space $E(\mathcal{F})$

We start with the following definitions.

Definition 4. For $r > 0$ a non-empty subset K in a linear space E is called *absolutely r-convex* in E if $x, y \in K$ and $|\alpha|^r + |\beta|^r \leq 1$ imply that $\alpha x + \beta y \in K$ (or equivalently $x_1, ..., x_n \in K$, $\sum_{k=1}^n |\alpha_k|^r \leq 1$ imply that $\sum_{k=1}^n \alpha_k x_k \in K$).

It is easy to check that every absolutely r-convex set is absolutely s-convex whenever $0 < s < r \leq 1$.

Definition 5. A linear topological space is called *r-convex* if there is a neighbourhood base of zero that consists of absolutely r-convex sets.

It is clear that the 1-convexity of E means that E is locally convex in the ordinary sense. For $r > 1$ Maddox and Roles [4] have proved that a topological linear space E is r-convex if and only if E is the only neighbourhood of zero.

Theorem 2. *Let $\mathcal{F} = (f_k)$ be a sequence of s_k-convex modulus functions with $0 < \inf_k s_k = s \leq 1$ and let g be a τ-convex $(0 < \tau \leq 1)$ absolutely monotone* F*-seminorm on a solid sequence space E. Then $E(\mathcal{F})$ is an r-convex $(r = \tau s)$ sequence space with the* F*-seminorm $g_{\mathcal{F}}$.*

Proof. By Theorem 1 and Remark 2 $g_{\mathcal{F}}$ is an F-seminorm. It is sufficient to show that the set

$$V_\delta = \{x = (\xi_k) | g_{\mathcal{F}}(x) < \delta\}$$

is an absolutely τs-convex set.

Indeed, by the assumptions for g and $\mathcal{F}$, we have that for each $x, y \in V_\delta$ and $|\alpha|^r + |\beta|^r \leq 1$, $r = \tau s$, the following estimates are true

$$\begin{aligned} g_{\mathcal{F}}(\alpha x + \beta y) &= g(\mathcal{F}(\alpha x + \beta y)) = g[(f_k(|\alpha \xi_k + \beta \eta_k|))] \\ &\leq g[(|\alpha|^{s_k} f(|\xi_k| + |\beta|^{s_k} f(|\eta_k|))] \leq g(|\alpha|^s \mathcal{F}(x) + |\beta|^s \mathcal{F}(y)) \\ &\leq (|\alpha|^r + |\beta|^r)\delta \leq \delta, \end{aligned}$$

so that $\alpha x + \beta y \in V_\delta$. Thus V_δ is absolutely r-convex. □

Remark 3. If the function f is s_1-convex, then it is s_2-convex for each $0 < s_2 < s_1$. Therefore the s_k-convex modulus functions f_k are s-convex if $s = \inf_k s_k > 0$.

Further we will apply Theorem 2 to the investigation of the r-convexity of the space $[m_A]^p$.

Let $A = (a_{nk})$ be an infinite matrix with $a_{nk} \geq 0$, $n, k \in \mathbb{N}$, and let

$$[m_A] = \{x = (\xi_k) | \sup_n \sum_{k=1}^{\infty} a_{nk} |\xi_k| < \infty\},$$

i.e. $[m_A]$ is the space of *strongly A-bounded sequences.* The space $[m_A]$ is a locally convex space with absolutely monotone seminorm g, where

$$g(x) = \sup_n \sum_n a_{nk} |\xi_k|.$$

If we take $f_k(t) = t^{p_k}$ $(t \geq 0)$ with $1 > p_k \geq \inf_k p_k = \tilde{p} > 0$, then the sequence $\mathcal{F} = (f_k)$ satisfies the assumptions of Theorem 2. And by Theorem 2 we get that the space

$$[m_A](\mathcal{F}) = [m_A]^p = \{x = (\xi_k) | \sup_n \sum_k a_{nk} |\xi_k|^{p_k} < \infty\}, \quad p = (p_k),$$

is $\tilde{p}$-convex.

In the case when

$$a_{nk} = \begin{cases} 1, & k \leq n, \\ 0, & k > n, \end{cases}$$

we get that $[m_A] = \ell$ and therefore the space

$$\ell(p) = \{x = (\xi_k) | \sum_{k=1}^{\infty} |\xi_k|^{p_k} < \infty\}$$

is $\tilde{p}$-convex for $0 < \tilde{p} = \inf_k p_k \leq p_k \leq 1$. This result was proved by Landsberg [1] and Simons [5].

Note that for $\inf p_k < \tilde{p} < 1$ the space $\ell(p)$ is not $\tilde{p}$-convex (cf. [5]). The spaces $\ell(p)$, $p = (p_k)$, $0 < p_k < 1$, were first investigated in order to find a linear topological space which is r-convex for some $r < 1$ but not locally convex.

A topological linear space E is called *locally bounded* if there exists a bounded neighbourhood of zero. We recall that $K \subset E$ is bounded if and only if $(x_n) \subset K$ and $\lambda_n \to 0$ $(\lambda_n \in \mathbb{R},\ n \in \mathbb{N})$ imply that $\lambda_n x_n \to 0$, $n \to \infty$. Using this criteria of boundedness it is easy to show that the sets $V_\delta = \{x | g(x) < \delta\}$ and $V_\delta(\mathcal{F}) = \{x | g_{\mathcal{F}}(x) < \delta\}$ are bounded in topological linear spaces E and $E(\mathcal{F})$ respectively if the conditions of Theorem 2 are satisfied. Therefore these spaces are locally bounded.

It is known (cf. [2]) that every locally bounded space is r-convex for some $r > 0$.

4. On r-convexity of the space $\ell(\mathcal{F})$

We denote ℓ_p for $\ell(p)$ if $p_k = p$ for all $k \in \mathbb{N}$. For ℓ_1 we write ℓ as usual. If we take $E = \ell$, then we have the space

$$\ell(\mathcal{F}) = \{x = (\xi_k) | \sum_{k=1}^{\infty} f_k(|\xi_k|) < \infty\}.$$

Theorem 3. *Let $\mathcal{F} = (f_k)$ be a sequence of unbounded modulus functions such that*

(i) $f_k(tu) \geq C\ f_k(t) f_k(u)$ *for some* $C > 0$ *and for all* $t, u \geq 0, k \in \mathbb{N}$,
(ii) $\inf_k f_k(t) > 0$ *for all* $t > 0$.

If the space $\ell(\mathcal{F})$ is r-convex, then $\ell_r \subset \ell(\mathcal{F})$.

Proof. The space $\ell(\mathcal{F})$ is an F-seminormed space with the F-seminorm $g_{\mathcal{F}}(x) = \sum_{k=1}^{\infty} f_k(|\xi_k|)$ for an arbitrary sequence $\mathcal{F} = (f_k)$ of modulus functions. Indeed, the assertion of Theorem 1 is true for an arbitrary $\mathcal{F} = (f_k)$ if E is an AK-space (see [3], Theorem 2). Let $V_\varepsilon = \{x | g_{\mathcal{F}}(x) < \varepsilon\}$, then there exist $0 < \delta < 1$ and an absolutely convex set U such that

$$V_\delta \subset U \subset V_1.$$

If we take $\tilde{x}_k = f_k^{-1}(\delta) e_k$ where $e_k = (\delta_{\nu k})$, then $g_{\mathcal{F}}(\tilde{x}_k) = f_k f_k^{-1}(\delta) = \delta < 1$ and thus $\tilde{x}_k \in U$ for all $k \in \mathbb{N}$. Therefore for each $m \in \mathbb{N}$ the inequality $\sum_{k=1}^{m} |\alpha_k|^r \leq 1$ implies that $\sum_{k=1}^{m} \alpha_k \tilde{x}_k \in U \subset V_1$, so that

$$g_{\mathcal{F}}\Big(\sum_{k=1}^{m} \alpha_k \tilde{x}_k\Big) = \sum_{k=1}^{m} f_k(|\alpha_k| f_k^{-1}(\delta)) < 1. \tag{1}$$

If now $x = (\xi_k) \in l_r$ and $S = \sum_{k=1}^{\infty} |\xi_k|^r$, then $\sum_{k=1}^{m} \frac{1}{S} |\xi_k|^r \leq 1$. Thus, if we take $\alpha_k = \frac{1}{S^{1/r}} |\xi_k|$ in (1), then we have that

$$\sum_{k=1}^{m} f_k\Big(\frac{|\xi_k|}{S^{1/r}} f_k^{-1}(\delta)\Big) < 1. \tag{2}$$

Now applying conditions (i) and (ii) of the present theorem we have that

$$f_k\Big(\frac{|\xi_k|}{S^{1/r}} f_k^{-1}(\delta)\Big) \geq C^2 \delta s f_k(|\xi_k|),$$

where $s = \inf_k f_k(\frac{1}{S^{1/r}}) > 0$. It follows from condition (2) that

$$\sum_{k=1}^{m} f_k(|\xi_k|) \leq \frac{1}{C^2 \delta s}$$

for all $m \in \mathbb{N}$ and thus $x = (\xi_k) \in \ell(\mathcal{F})$. □

Example 3. Take $f_k(t) = \lambda_k t^{r_k}$, where $\inf_k \lambda_k > 0$, $\sup_k \lambda_k < \infty$, $r = \inf r_k > 0$. Then the space $\ell(\mathcal{F})$ is r-convex by Theorem 2 and conditions (i) and (ii) of Theorem 3 are fulfilled.

Remark 4. If $0 < p_k \leq 1$, $0 < r < 1$ and $f_k(t) = t^{p_k}$, then $\ell_r \subseteq \ell(\mathcal{F})$ implies that $\ell(\mathcal{F})$ is r-convex (cf. [5]).

Let us take now $f_k = f$ for each $k \in \mathbb{N}$ and write $\ell(f)$ instead of $\ell(\mathcal{F})$.

Theorem 4. *Let an unbounded modulus function f satisfy the following conditions:*

(i) $f(tu) \geq Cf(t)f(u)$ *for some* $C > 0$ *and for all* $t, u \geq 0$,
(ii) $\lim_{t\to 0+} f(t^{1/r})/t = \infty$,

then the space $\ell(f)$ *is not* r*-convex.*

Proof. If we suppose that the space $\ell(f)$ is r-convex, then there exist an absolutely r-convex neighbourhood of zero U and $\delta > 0$, such that

$$V_\delta \subset U \subset V_1,$$

where $V_\varepsilon = \{x = (\xi_k) | g_{\mathcal{F}}(x) = \sum_k f(|\xi_k|) < \varepsilon\}$.

Let us take $x_k = f^{-1}(\frac{\delta}{2})e_k$, $k \in \mathbb{N}$; then $g_{\mathcal{F}}(x_k) = \frac{\delta}{2}$ and thus $x_k \in V_\delta \subset U$. By r-convexity of the set U the inequality $\sum_{k=1}^n |\alpha_k|^r \leq 1$ implies that $\sum_{k=1}^n \alpha_k x_k \in U$. But for $\alpha_k = n^{-(1/r)}$ we have by the assumption (i) that

$$g_{\mathcal{F}}(\sum_{k=1}^n \frac{1}{n^{1/r}} x_k) = \sum_{k=1}^n f(\frac{1}{n^{1/r}} f^{-1}(\frac{\delta}{2})) \geq C\frac{\delta}{2} n f(\frac{1}{n^{1/r}}).$$

Now it follows from (ii) that

$$g_{\mathcal{F}}(\sum_{k=1}^n \frac{1}{n^{1/r}} x_k) > 1$$

for sufficiently large n. Therefore the space $\ell(f)$ is not r-convex if the modulus function f satisfies the assumptions (i) and (ii). □

Remark 5. If we take $f(t) = t^p$ in Theorem 4, then we get the well-known result: the space ℓ_p is not r-convex for $0 < p < r$.

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