ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 4, 2000

Semi-parallel and parallel symplectic surfaces in the four-dimensional symplectic space

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ABSTRACT. Parallel and semi-parallel symplectic surfaces in the symplectic space Sp_4 are studied. It is proved that a symplectic surface is semi-parallel if and only if its tangent connection is flat. The parallel symplectic surfaces are characterized and the existence of semi-parallel symplectic surfaces which are not parallel is exhibited. The study is conducted by the method of moving frame.

1. The concept of the symplectic space

Let us consider the four-dimensional symplectic space Sp_4. This is an affine space whose direction space is a symplectic vector space, which we shall denote by $\overrightarrow{Sp}_4$. The latter is a vector space with regular skew-symmetric scalar product, i.e. for each $\overrightarrow{x}, \overrightarrow{y} \in \overrightarrow{Sp}_4$ there holds the equality $\langle \overrightarrow{x}, \overrightarrow{y} \rangle = -\langle \overrightarrow{y}, \overrightarrow{x} \rangle$. As it follows from here that $|\overrightarrow{x}|^2 = \langle \overrightarrow{x}, \overrightarrow{x} \rangle = 0$, all vectors in this space have length zero.

We denote a moving frame in Sp_4 by $\{X; \overrightarrow{e}_I\}$, where the indices $I, J, \dots$ acquire the values 1, 2, 3, 4. Instead of the origin $X \in Sp_4$ of the frame we often use its position vector $\overrightarrow{x} = \overrightarrow{OX}$ relative to an optional fixed point $O \in Sp_4$. The motion of the frame is described by the differentiation formulae

$$d\overrightarrow{x} = \omega^L \overrightarrow{e}_L, \qquad d\overrightarrow{e}_I = \omega_I^L \overrightarrow{e}_L, \tag{1.1}$$

where the 1-forms ω^L and ω_I^L satisfy the conditions of complete integrability (the structure equations)

$$d\omega^I = \omega^L \wedge \omega_L^I, \qquad d\omega_K^I = \omega_K^L \wedge \omega_L^I. \tag{1.2}$$

Received June 26, 1999; revised December 14, 2000.
2000 *Mathematics Subject Classification.* 53D05.
Key words and phrases. Symplectic space, parallel and semi-parallel symplectic surfaces.

The matrix $G = \| g_{IK} \|$ formed by scalar products $g_{IK} = \langle \overrightarrow{e_I}, \overrightarrow{e_K} \rangle$ of the basis vectors is regular and skew-symmetric. Differentiation of the scalar products g_{IK} with use of (1.1) gives

$$dg_{IK} = g_{LK}\omega_I^L + g_{IL}\omega_K^L, \tag{1.3}$$

which is often useful in the matrix notation

$$dG = -(G\omega)^T + G\omega, \tag{1.4}$$

where $\omega = \| \omega_K^I \|$. In what follows, for all matrices, the upper index is the row index.

2. The symplectic surface

Let us consider the surfaces with symplectic tangent planes in the symplectic space Sp_4. Any such surface M_2 will be called a *symplectic surface.* The study of a symplectic surface can be reduced to study of a suitable moving frame $\{X; \overrightarrow{e_i}, \overrightarrow{e_\alpha}\}$ connected with that symplectic surface. Here and in what follows we take $i, j, \ldots = 1, 2$ and $\alpha, \beta, \ldots = 3, 4$.

We require that the symplectic surface should be described by the origin X of the moving frame ($X \in M_2$). The basis vectors $\overrightarrow{e_i}$ are in the vector space $\overrightarrow{T}_X M_2$ (determined by the tangent plane $T_X M_2$) and the basis vectors $\overrightarrow{e_\alpha}$ are in the vector space $\overrightarrow{T}_X^{\perp} M_2$ (determined by the normal plane $T_X^{\perp} M_2$). Accordingly

$$T_X M_2 = X + L(\overrightarrow{e_1}, \overrightarrow{e_2}), \quad T_X^{\perp} M_2 = X + L(\overrightarrow{e_3}, \overrightarrow{e_4}).$$

It needs to be pointed out that the symplecity of the symplectic surface M_2 assures the uniqueness of the orthogonal complement $\overrightarrow{T}_X^{\perp} M_2$ of $\overrightarrow{T}_X M_2$, and

$$\overrightarrow{T}_X M_2 + \overrightarrow{T}_X^{\perp} M_2 = \overrightarrow{T}_X M_2 \oplus \overrightarrow{T}_X^{\perp} M_2 = \overrightarrow{S}p_4.$$

We shall call the selected frame the *adapted frame* of the symplectic surface. By this choice of frame the condition $\langle \overrightarrow{y}, \overrightarrow{z} \rangle = 0$ holds for each $\overrightarrow{y} \in \overrightarrow{T}_X M_2$ and $\overrightarrow{z} \in \overrightarrow{T}_X^{\perp} M_2$, hence $g_{i\alpha} = \langle \overrightarrow{e_i}, \overrightarrow{e_\alpha} \rangle = 0$. So, in turn, $dg_{i\alpha} = 0$, and the equations (1.3) become

$$dg_{ik} = g_{sk}\omega_i^s + g_{is}\omega_k^s, \quad dg_{\alpha\beta} = g_{\gamma\beta}\omega_\alpha^\gamma + g_{\alpha\gamma}\omega_\beta^\gamma$$

and

$$\omega_\alpha^k = -g^{ki}\omega_i^\gamma g_{\gamma\alpha}. \tag{2.1}$$

Here g^{ki} are the elements of the matrix G_1^{-1}, which is the inverse of the skew-symmetric matrix $G_1 = \| g_{ik} \|$ and is also skew-symmetric.

It may be remarked that in an analogous way we may get the matrices $G_2 = \| g_{\alpha\beta} \|$ and $G_2^{-1} = \| g^{\alpha\beta} \|$, where

$$G = G_1 \oplus G_2.$$

Let us now return to studying the symplectic surface M_2. For the adapted frame the differential $d\vec{x}$ of the position vector $\vec{x}$ of an arbitrary point X of M_2 belongs to the subspace $L(\vec{e}_1, \vec{e}_2)$, in consequence of which the first of the equations (1.1) takes the form $d\vec{x} = \omega^s \vec{e}_s$, or

$$\omega^\alpha = 0. \tag{2.2}$$

The equations (2.2) are called the *differential equations* of the symplectic surface. By double extension of these equations (that is, by exterior differentiation and use of the Cartan lemma) we get

$$\omega_i^\alpha = h^\alpha_{is}\omega^s, \quad h^\alpha_{is} = h^\alpha_{si}, \tag{2.3}$$

$$dh^\alpha_{ij} - h^\alpha_{sj}\omega^s_i - h^\alpha_{is}\omega^s_j + h^\beta_{ij}\omega^\alpha_\beta = h^\alpha_{ijs}\omega^s, \quad h^\alpha_{ijs} = h^\alpha_{isj}. \tag{2.4}$$

In the equations (2.4) the functions h^α_{ijk} are symmetric not only with respect to the last pair of lower indices, as required by the Cartan lemma, but also with respect to all the lower indices, as can be checked directly from (2.4).

Let us construct the vectors $\vec{h}_{ij} = h^\alpha_{ij}\vec{e}_\alpha$, $\vec{h}_{ij} \in \vec{T}^\perp_X M_2$, which determine a subspace $N_X M_2$ of the normal plane $T^\perp_X M_2$,

$$N_X M_2 = X + L(\vec{h}_{11}, \vec{h}_{12}, \vec{h}_{22}).$$

This is called the *(first) normal plane* of the symplectic surface M_2 at the point $X \in M_2$. Here we took note of the fact that $\vec{h}_{21} = \vec{h}_{12}$. Because of $dim N_X M_2 = rank\{\vec{h}_{11}, \vec{h}_{12}, \vec{h}_{22}\}$, the dimension of this normal plane is 2, 1 or 0. In the first case $N_X M_2 = T^\perp_X M_2$. It is worth noting that if we have a symplectic space Sp_{2n} $(n > 2)$ with higher dimension, instead of Sp_4, then it is possible to achieve $dim N_X M_2 = 3$.

The basis vectors $\vec{e}_i$ and $\vec{e}_\alpha$ of the moving frame $\{X; \vec{e}_i, \vec{e}_\alpha\}$ at a point X of the symplectic surface M_2 are not unique. It is only known that $\vec{e}_i \in \vec{T}_X M_2$ and $\vec{e}_\alpha \in \vec{T}^\perp_X M_2$. Hence at each point $X \in M_2$ it is permissible to effect basis transformations

$$\vec{e}_i\,' = A_i^j \vec{e}_j, \quad \vec{e}_\alpha\,' = A_\alpha^\beta \vec{e}_\beta, \tag{2.5}$$

where $A_1 = \| A_i^j \|$ and $A_2 = \| A_\alpha^\beta \|$ are regular matrices. The equations (2.5) describe the arbitrariness of the adapted frame at the point X of a symplectic surface. The functions h^α_{ij} are transformed by the equations

$$'h^\alpha_{ij} = A^s_i A^t_j h^\beta_{st} \tilde{A}^\alpha_\beta, \tag{2.6}$$

where $\tilde{A}^{\alpha}_{\beta}$ are the elements of the inverse matrix A_2^{-1}.

By means of the equations (2.2) we get from the structure equations (1.2) of Sp_4 the *structure equations of the symplectic surface* M_2

$$d\omega^i = \omega^s \wedge \omega^i_s, \quad d\omega^j_i = \omega^s_i \wedge \omega^j_s + \Omega^j_i \tag{2.7}$$

and the *structure equations for the manifold of normal vector spaces* $\vec{T}^{\perp} M_2 = \bigcup_{X \in M_2} \vec{T}^{\perp}_X M_2$

$$d\omega^{\beta}_{\alpha} = \omega^{\gamma}_{\alpha} \wedge \omega^{\beta}_{\gamma} + \Omega^{\beta}_{\alpha}. \tag{2.8}$$

In the equations (2.7) and (2.8) the 2-forms Ω^j_i and Ω^{β}_{α} are the curvature forms of the tangential connection ∇ of the symplectic surface M_2 and of the normal connection $\nabla^{\perp}$, respectively. In particular,

$$\Omega^j_i = R^j_{ist}\omega^s \wedge \omega^t, \quad \Omega^{\beta}_{\alpha} = R^{\beta}_{\alpha st}\omega^s \wedge \omega^t,$$

where, taking into account (2.1) and (2.3), the functions R^j_{ist} and $R^{\beta}_{\alpha st}$ may be expressed as

$$R^j_{ist} = g^{jk} h^{\alpha}_{i[s} h^{\beta}_{|k|t]} g_{\alpha\beta}, \qquad R^{\beta}_{\alpha st} = g_{\alpha\gamma} h^{\gamma}_{i[s} h^{\beta}_{|k|t]} g^{ki}. \tag{2.9}$$

Here the bracketed indices are alternated except for the indices between the vertical lines $|\ |$. The functions R^j_{ist} and $R^{\beta}_{\alpha st}$ are the curvature tensors of the tangential and normal connections, respectively. In order to bring to light a property of these tensors it is useful to define

$$R_{ijst} = g_{iu} R^u_{jst}, \quad R_{\alpha\beta st} = g_{\alpha\tau} R^{\tau}_{\beta st}, \tag{2.10}$$

which will also be called curvature tensors. They are symmetric with respect to the first pair of indices, i.e.

$$R_{ijst} = R_{jist}, \quad R_{\alpha\beta st} = R_{\beta\alpha st}.$$

The property alluded to follows from the equations

$$R_{ijst} = g_{\alpha\beta} h^{\alpha}_{j[s} h^{\beta}_{|i|t]}, \quad R_{\alpha\beta st} = g_{\alpha\tau} g_{\beta\gamma} h^{\gamma}_{i[s} h^{\tau}_{|k|t]} g^{ki}, \tag{2.11}$$

which result from (2.10) with the help of the equations (2.9). For the curvature tensors R^j_{ist} and $R^{\beta}_{\alpha st}$, or R_{ijst} and $R_{\alpha\beta st}$, only the elements R^j_{i12} and $R^{\beta}_{\alpha 12}$, or R_{ij12} and $R_{\alpha\beta 12}$, may be non-zero. Henceforth we denote these elements as follows:

$$R^j_i = R^j_{i12}, \quad R^{\beta}_{\alpha} = R^{\beta}_{\alpha 12}, \quad R_{ij} = R_{ij12}, \quad R_{\alpha\beta} = R_{\alpha\beta 12}. \tag{2.12}$$

As it is well known, the curvature tensors transform under the basis transformation (2.5) in accordance with the tensor law. So, for example, in the case of (2.11)

$$R'_{ijst} = A_i^k A_j^l A_s^u A_t^v R_{kluv}, \quad R'_{\alpha\beta st} = A_\alpha^\gamma A_\beta^\delta A_s^u A_t^v R_{\gamma\delta uv}.$$

Using these relations we get the transformation equations for R_{ij} and $R_{\alpha\beta}$:

$$R'_{ij} = \mid A_1 \mid A_i^k A_j^l R_{kl}, \quad R'_{\alpha\beta} = \mid A_1 \mid A_\alpha^\gamma A_\beta^\delta R_{\gamma\delta}. \tag{2.13}$$

Here $\mid A_1 \mid$ is the determinant of matrix A_1.

Let us suppose from now on that the adapted frame $\{X; \vec{e_i}, \vec{e_\alpha}\}$ of a symplectic surface is symplectic. This means that the matrix G has the form

$$G = \begin{Vmatrix} I & 0 \\ 0 & I \end{Vmatrix}, \quad I = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \quad 0 = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}.$$

Now, as $dG = 0$, the equation (1.4) leads to $(G\omega)^T = G\omega$, from which it follows that

$$\omega_2^2 = -\omega_1^1, \ \omega_4^4 = -\omega_3^3, \ \omega_3^1 = -\omega_2^4, \ \omega_3^2 = \omega_1^4, \ \omega_4^1 = \omega_2^3, \ \omega_4^3 = -\omega_1^3. \tag{2.14}$$

The freedom of choice of the adapted symplectic frame for each point $X \in M_2$ may be inferred from the equations (2.5). In these equations A_1 and A_2 must be such that the matrices $G_1 = G_2 = I$ would not change under transition from one adapted frame to another at any point $X \in M_2$, i.e. $G'_1 = G_1$ and $G'_2 = G_2$, or equivalently,

$$A_1^T I A_1 = I, \quad A_2^T I A_2 = I.$$

From here we see that the only condition which must hold in the choice of the matrices A_1 and A_2 is that $\mid A_1 \mid = 1$ and $\mid A_2 \mid = 1$. Now the equations (2.13) simplify

$$R'_{ij} = A_i^k A_j^l R_{kl}, \quad R'_{\alpha\beta} = A_\alpha^\gamma A_\beta^\delta R_{\gamma\delta}.$$

Thus the transformations follow the tensor law. With the help of the equations (2.12) and (2.11) we may express R_{ij} and $R_{\alpha\beta}$ in terms of the functions h_{ij}^α :

$$R_{11} = \begin{vmatrix} h_{11}^3 & h_{12}^3 \\ h_{11}^4 & h_{12}^4 \end{vmatrix}, \quad R_{12} = R_{21} = \tfrac{1}{2} \begin{vmatrix} h_{11}^3 & h_{22}^3 \\ h_{11}^4 & h_{22}^4 \end{vmatrix}, \quad R_{22} = \begin{vmatrix} h_{12}^3 & h_{22}^3 \\ h_{12}^4 & h_{22}^4 \end{vmatrix} \tag{2.15}$$

and

$$R_{33} = \begin{vmatrix} h_{11}^4 & h_{12}^4 \\ h_{12}^4 & h_{22}^4 \end{vmatrix}, \quad R_{44} = \begin{vmatrix} h_{11}^3 & h_{12}^3 \\ h_{12}^3 & h_{22}^3 \end{vmatrix},$$

$$R_{34} = R_{43} = \tfrac{1}{2}(2h_{12}^3 h_{12}^4 - h_{11}^3 h_{22}^4 - h_{11}^4 h_{22}^3). \tag{2.16}$$

To get analogous expressions for R^i_j and R^β_α we must take into account the equations

$$R^2_2 = -R^1_1 = R_{12}, \quad R^1_2 = -R_{22}, \quad R^2_1 = R_{11};$$
$$R^4_4 = -R^3_3 = R_{34}, \quad R^3_4 = -R_{44}, \quad R^4_3 = R_{33}.$$

As it is shown in [6], the symplectic surfaces fall into five classes $\mathcal{A}_1, \ldots, \mathcal{A}_5$. Using a suitable symplectic frame the symplectic surfaces in the various classes are characterized as in Table 1.

Table 1.

Class of symplectic surfaces	Expression of tensors h^α_{ij}, R^j_i and R^β_α
$\mathcal{A}_1 \quad (\varepsilon = 1)$ $\mathcal{A}_2 \quad (\varepsilon = -1)$	$h^\alpha_{22} = \varepsilon h^\alpha_{11}, \ h^4_{12} = -\varepsilon h^3_{11}, \ h^4_{11} = -h^3_{12},$ $R^2_1 = \varepsilon R^4_3 = (h^3_{12})^2 - \varepsilon (h^3_{11})^2 \neq 0,$ $R^1_1 = R^2_2 = R^3_3 = R^4_4 = 0,$ $R^1_2 = -\varepsilon R^2_1, \quad R^3_4 = -\varepsilon R^4_3$
$\mathcal{A}_3$	$h^\alpha_{22} = h^3_{12} = 0, \quad h^3_{11} \neq 0, \quad h^4_{22} \neq 0,$ $R^2_1 = 2h^3_{11} h^4_{12} \neq 0, \quad R^4_3 = 2(h^4_{12})^2 \neq 0,$ $R^1_1 = R^2_2 = R^1_2 = R^3_3 = R^4_4 = R^3_4 = 0$
$\mathcal{A}_4$	$h^3_{ij} = 0, \quad R^i_j = R^3_3 = R^4_4 = R^3_4 = 0,$ $R^4_3 = h^4_{11} h^4_{22} - (h^4_{12})^2 \neq 0$
$\mathcal{A}_5$	$h^3_{ij} = \lambda h^4_{ij}, \quad (h^\alpha_{12})^2 = h^\alpha_{11} h^\alpha_{22}, \ R^i_j = R^\beta_\alpha = 0$

3. Parallel and semi-parallel symplectic surfaces

In this section we consider a pair of closely connected classes of symplectic surfaces in the symplectic space Sp_4 – the parallel and semi-parallel symplectic surfaces. Parallel and semi-parallel submanifolds in the Euclidean space and the space with a constant curvature have been studied by several authors (e.g. Deprez [1], Ferus [2], Lumiste [4], Vilms [7]). Parallel and semi-parallel symplectic submanifolds in the symplectic space have been studied in [5].

To present the concepts of such symplectic surfaces, let $\overline{\nabla} = (\nabla, \nabla^{\perp})$ denote the van der Waerden-Bortolotti connection of a symplectic surface. The covariant differentials of the functions h^{α}_{ij} relative to the connection $\overline{\nabla}$

$$\overline{\nabla} h^{\alpha}_{ij} = dh^{\alpha}_{ij} - h^{\alpha}_{sj}\omega^{s}_{i} - h^{\alpha}_{is}\omega^{s}_{j} + h^{\beta}_{ij}\omega^{\alpha}_{\beta},$$

may be expressed as

$$\overline{\nabla} h^{\alpha}_{ij} = h^{\alpha}_{ijs}\omega^{s}, \tag{3.1}$$

where the functions h^{α}_{ijs} are the ones used earlier in the equations (2.4). The covariant differentials of the functions h^{α}_{ijk}

$$\overline{\nabla} h^{\alpha}_{ijk} = dh^{\alpha}_{ijk} - h^{\alpha}_{sjk}\omega^{s}_{i} - h^{\alpha}_{isk}\omega^{s}_{j} - h^{\alpha}_{ijs}\omega^{s}_{k} + h^{\beta}_{ijk}\omega^{\alpha}_{\beta} \tag{3.2}$$

may be expressed as

$$\overline{\nabla} h^{\alpha}_{ijk} = h^{\alpha}_{ijks}\omega^{s}.$$

Definition (see [5]). A symplectic surface in the symplectic space Sp_4 will be called *parallel* if $\overline{\nabla} h^{\alpha}_{ij} = 0$.

From the equations (3.1) we see that the condition of parallelism is equivalent to

$$h^{\alpha}_{ijs} = 0. \tag{3.3}$$

To establish the concept of a semi-parallel symplectic surface we write out the exterior differential for the expression (3.1):

$$\overline{\nabla} h^{\alpha}_{ijk} \wedge \omega^{k} = h^{\beta}_{ij}\Omega^{\alpha}_{\beta} - h^{\alpha}_{sj}\Omega^{s}_{i} - h^{\alpha}_{is}\Omega^{s}_{j}. \tag{3.4}$$

The formulae (2.7) and (2.8) were used here to substitute for $d\omega^{s}_{i}$ and $d\omega^{\beta}_{\alpha}$.

Definition (see [5]). A symplectic surface in the symplectic space Sp_4 will be called *semi-parallel* if

$$h^{\beta}_{ij}\Omega^{\alpha}_{\beta} - h^{\alpha}_{sj}\Omega^{s}_{i} - h^{\alpha}_{is}\Omega^{s}_{j} = 0. \tag{3.5}$$

It is seen that every parallel symplectic surface is also semi-parallel. Indeed, using (3.3) in (3.2), we get $\overline{\nabla} h^{\alpha}_{ijk} = 0$, and the use of (3.4) gives (3.5). This indicates that it is useful to study the semi-parallel symplectic surfaces first and only thereafter the parallel symplectic surfaces.

So let us begin with semi-parallel symplectic surfaces. As we have divided the symplectic surfaces in the preceding into five classes $\mathcal{A}_1, \ldots, \mathcal{A}_5$, we need to consider the semi-parallel symplectic surfaces likewise in accordance with these classes. First of all we note that the condition (3.5) is by reason of (2.12) equivalent to

$$h^{\beta}_{ij} R^{\alpha}_{\beta} - h^{\alpha}_{sj} R^{s}_{i} - h^{\alpha}_{is} R^{s}_{j} = 0. \tag{3.6}$$

When writing this out in detail, we keep in mind that, for all the surface classes $\mathcal{A}_1, \ldots, \mathcal{A}_5$, the frame of the symplectic surface is assumed to be canonized so that $R^1_1 = R^2_2 = 0$ and $R^3_3 = R^4_4 = 0$ (see Table 1), and the equations (3.6) read

$$\begin{aligned} h^4_{11} R^3_4 - 2h^3_{12} R^2_1 &= 0, \quad & h^3_{11} R^4_3 - 2h^4_{12} R^2_1 &= 0, \\ h^4_{12} R^3_4 - h^3_{22} R^2_1 - h^3_{11} R^1_2 &= 0, \quad & h^3_{12} R^4_3 - h^4_{22} R^2_1 - h^4_{11} R^1_2 &= 0, \\ h^4_{22} R^3_4 - 2h^3_{12} R^1_2 &= 0, \quad & h^3_{22} R^4_3 - 2h^4_{12} R^1_2 &= 0. \end{aligned}$$

Using Table 1 we see that classes $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ do not contain semi-parallel symplectic surfaces, but *every symplectic surface in the classes $\mathcal{A}_4$ and $\mathcal{A}_5$ is semi-parallel.* To formulate these conclusions into a theorem, we note first that if the symplectic surface belongs to one of the classes $\mathcal{A}_1, \mathcal{A}_2$ or $\mathcal{A}_3$, then its tangent connection is not flat, whereas the tangent connection of a symplectic surface of class $\mathcal{A}_4$ or $\mathcal{A}_5$ is flat.

Now we can summarize the above in the following theorem.

Theorem 1. *A symplectic surface is semi-parallel if and only if its tangent connection ∇ is flat.*

The final part of this section is devoted to a study of parallel symplectic surfaces of the symplectic space Sp_4. As a parallel symplectic surface is also semi-parallel, we may say that the classes $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ do not contain parallel symplectic surfaces. Parallel surfaces must therefore be sought from the classes $\mathcal{A}_4$ and $\mathcal{A}_5$.

Now we must specify the symplectic frame. To this end we write out (2.4) in detail, taking into account the equalities (2.14). We have

$$\begin{aligned} dh^3_{11} - h^3_{11}(2\omega^1_1 - \omega^3_3) - 2h^3_{12}\omega^2_1 + h^4_{11}\omega^3_4 &= h^3_{111}\omega^1 + h^3_{112}\omega^2, \\ dh^3_{12} - h^3_{11}\omega^1_2 + h^3_{12}\omega^3_3 - h^3_{22}\omega^2_1 + h^4_{12}\omega^3_4 &= h^3_{112}\omega^1 + h^3_{122}\omega^2, \\ dh^3_{22} + h^3_{22}(2\omega^1_1 + \omega^3_3) - 2h^3_{12}\omega^1_2 + h^4_{22}\omega^3_4 &= h^3_{122}\omega^1 + h^3_{222}\omega^2 \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} dh^4_{11} - h^4_{11}(2\omega^1_1 + \omega^3_3) - 2h^4_{12}\omega^2_1 + h^3_{11}\omega^4_3 &= h^4_{111}\omega^1 + h^4_{112}\omega^2, \\ dh^4_{12} - h^4_{11}\omega^1_2 - h^4_{12}\omega^3_3 - h^4_{22}\omega^2_1 + h^3_{12}\omega^4_3 &= h^4_{112}\omega^1 + h^4_{122}\omega^2, \\ dh^4_{22} + h^4_{22}(2\omega^1_1 - \omega^3_3) - 2h^4_{12}\omega^1_2 + h^3_{22}\omega^4_3 &= h^4_{122}\omega^1 + h^4_{222}\omega^2. \end{aligned} \tag{3.8}$$

Let us consider a symplectic surface of class $\mathcal{A}_4$. From Table 1 we get

$$h^3_{ij} = 0, \quad h^4_{11}h^4_{22} - (h^4_{12})^2 \neq 0. \tag{3.9}$$

Hence h^4_{11} and h^4_{12} and likewise h^4_{22} and h^4_{12} are not zero at the same time. With the help of the basis transformations which preserve the conditions (3.9) it is possible to achieve $h^4_{11} > 0$. Indeed, from (2.6) it follows that the basis transformation

$$\vec{e_1}\,' = \vec{e_2}, \quad \vec{e_2}\,' = -\vec{e_1}, \quad \vec{e_\alpha}\,' = \vec{e_\alpha}$$

gives $'h^4_{11} = h^4_{22}$, hence we may confine attention to the case where h^4_{11} and h^4_{12} are not simultaneously zero. Let us suppose $h^4_{11} \neq 0$. If $h^4_{11} = 0$, then $h^4_{22} = 0$ and $h^4_{12} \neq 0$. With the help of the basis transformation

$$\vec{e_1}\,' = \vec{e_1} + \vec{e_2}, \quad \vec{e_2}\,' = \vec{e_2}, \quad \vec{e_\alpha}\,' = \vec{e_\alpha}$$

we get from (2.6) that $'h^4_{11} = 2h^4_{12} \neq 0$. To obtain $h^4_{11} > 0$ we make the basis transformation

$$\vec{e_i}\,' = \vec{e_i}, \quad \vec{e_\alpha}\,' = -\vec{e_\alpha}$$

which gives $'h^4_{11} = -h^4_{11}$. From the equations (3.8) it now follows that, with the help of the 1-forms ω^1_1, ω^1_2 and ω^3_3, it is possible to get $h^4_{11} = 1$, $h^4_{12} = 0$ and $h^4_{22} = \varepsilon$, where $\varepsilon = \pm 1$. Thus canonization has led to the following results:

$$h^3_{ij} = 0, \quad h^4_{11} = 1, \quad h^4_{12} = 0, \quad h^4_{22} = \varepsilon, \quad R_{33} = \varepsilon, \quad \varepsilon = \pm 1. \tag{3.10}$$

The forms ω^α_i and ω^i_α now appear as

$$\begin{aligned} &\omega^3_1 = 0, \quad \omega^3_2 = 0, \quad \omega^4_1 = \omega^1, \quad \omega^4_2 = \varepsilon\omega^2, \\ &\omega^2_4 = 0, \quad \omega^1_4 = 0, \quad \omega^2_3 = \omega^1, \quad \omega^1_3 = -\varepsilon\omega^2. \end{aligned} \tag{3.11}$$

As to the remaining forms, by means of (3.7) and (3.8) with the use of (3.10), we may say that

$$\begin{aligned} &\omega^1_1 = \tfrac{1}{4}(-h^4_{111} + \varepsilon h^4_{122})\omega^1 + \tfrac{1}{4}(-h^4_{112} + \varepsilon h^4_{222})\omega^2, \\ &\omega^2_1 = -\varepsilon\omega^1_2 - \varepsilon h^4_{112}\omega^1 - \varepsilon h^4_{122}\omega^2, \\ &\omega^3_3 = -\tfrac{1}{2}(h^4_{111} + \varepsilon h^4_{112})\omega^1 - \tfrac{1}{2}(h^4_{112} + \varepsilon h^4_{122})\omega^2, \\ &\omega^3_4 = 0. \end{aligned} \tag{3.12}$$

The forms ω_2^1 and ω_3^4 are arbitrary. In addition we have $h_{111}^3 = h_{112}^3 = h_{122}^3 = h_{222}^3 = 0$.

Let us consider a symplectic surface of class $\mathcal{A}_5$. From Table 1 we get

$$h_{ij}^4 = \lambda h_{ij}^3, \tag{3.13}$$

and the conditions $R_{ij} = 0$ and $R_{\alpha\beta} = 0$ must also hold. From (2.15) it follows that the condition $R_{ij} = 0$ is an identity and thus does not restrict the functions h_{ij}^α. From the condition $R_{\alpha\beta} = 0$ we get for the functions h_{ij}^α the equation

$$R_{44} = 0 \quad \Leftrightarrow \quad \begin{vmatrix} h_{11}^3 & h_{12}^3 \\ h_{12}^3 & h_{22}^3 \end{vmatrix} = 0, \tag{3.14}$$

because, with the help of (2.16), we see that

$$R_{33} = \lambda^2 R_{44}, \quad R_{34} = R_{43} = \lambda R_{44}.$$

If $dim N_X M_2 = 1$, then we may assume, without loss of generality, that $h_{11}^3 \neq 0$. Here is the explanation. If $h_{11}^3 = 0$, but $h_{22}^3 \neq 0$, we select the basis transformation $\overrightarrow{e_1}' = \overrightarrow{e_2}$ and $\overrightarrow{e_2}' = -\overrightarrow{e_1}$. In this case, by reason of (2.6), $'h_{11}^\alpha = h_{22}^\alpha$, $'h_{22}^\alpha = h_{11}^\alpha$ and $'h_{12}^\alpha = -h_{12}^\alpha$, whence $'h_{11}^3 = h_{22}^3 \neq 0$. If $h_{11}^3 = 0$ and $h_{22}^3 = 0$, then $h_{12}^3 \neq 0$. Now we make a change of basis $\overrightarrow{e_1}' = \overrightarrow{e_1} + \overrightarrow{e_2}$ and $\overrightarrow{e_2}' = \overrightarrow{e_2}$. The equations (2.6) then give

$$'h_{11}^\alpha = h_{11}^\alpha + 2h_{12}^\alpha + h_{22}^\alpha, \quad 'h_{12}^\alpha = h_{12}^\alpha + h_{22}^\alpha, \quad 'h_{22}^\alpha = h_{22}^\alpha.$$

Hence, when $\alpha = 3$, we get $'h_{11}^3 = 2h_{12}^3 \neq 0$.

On account of this, (3.14) becomes

$$h_{12}^3 = \mu h_{11}^3, \quad h_{22}^3 = \mu h_{12}^3 = \mu^2 h_{11}^3, \tag{3.15}$$

and due to the latter result the relation (3.13) now appears as

$$h_{11}^4 = \lambda h_{11}^3, \quad h_{12}^4 = \lambda\mu h_{11}^3, \quad h_{22}^4 = \lambda\mu^2 h_{11}^3. \tag{3.16}$$

It turns out that it is possible to choose such an adapted symplectic frame for which $\mu = 0$. Here is an explanation of the details. If $\mu \neq 0$ we transform the basis by $\overrightarrow{e_1}' = \overrightarrow{e_1}$ and $\overrightarrow{e_2}' = -\mu\overrightarrow{e_1} + e_2$. Then

$$'h_{11}^\alpha = h_{11}^\alpha, \quad 'h_{12}^\alpha = -\mu h_{11}^\alpha + h_{12}^\alpha, \quad 'h_{22}^\alpha = \mu^2 h_{11}^\alpha - 2\mu h_{12}^\alpha + h_{22}^\alpha,$$

and as $'h_{11}^3 = h_{11}^3$, the condition $'h_{11}^3 \neq 0$ remains intact. In addition we get from these equations

$$'h_{22}^3 = 0, \quad 'h_{12}^4 = 0, \quad 'h_{22}^4 = 0$$

as a result of which (3.15) and (3.16) show that μ has become zero. Thus we have

$$h^3_{11} \neq 0, \quad h^3_{12} = 0, \quad h^3_{22} = 0, \quad h^4_{11} = \lambda h^3_{11}, \quad h^4_{12} = 0, \quad h^4_{22} = 0. \quad (3.17)$$

To complete the canonization we recall the differential equations (3.7) and (3.8). We obtain

$$\begin{aligned} dh^3_{11} + h^3_{11}(-2\omega^1_1 + \omega^3_3) + \lambda h^3_{11}\omega^3_4 &= h^3_{111}\omega^1 + h^3_{112}\omega^2, \\ -h^3_{11}\omega^1_2 &= h^3_{112}\omega^1 + h^3_{122}\omega^2, \\ 0 &= h^3_{122}\omega^1 + h^3_{222}\omega^2, \\ h^3_{11}d\lambda + \lambda dh^3_{11} - \lambda h^3_{11}(2\omega^1_1 + \omega^3_3) + h^3_{11}\omega^4_3 &= h^4_{111}\omega^1 + h^4_{112}\omega^2, \\ -\lambda h^3_{11}\omega^1_2 &= h^4_{112}\omega^1 + h^4_{122}\omega^2, \\ 0 &= h^4_{122}\omega^1 + h^4_{222}\omega^2. \end{aligned} \quad (3.18)$$

From the first of these it is seen that, depending on the sign of h^3_{11}, we may replace h^3_{11} by $+1$ or -1. Therefore $h^3_{11} = \varepsilon$, where $\varepsilon = \pm 1$. Furthermore, the fourth equation of the system (3.18) allows, with the help of the form ω^4_3, to replace λ by zero.

In summary, we have obtained the following.

If $dim N_X M_2 = 1$, then from (3.17)

$$h^3_{11} = \varepsilon, \quad h^3_{12} = 0, \quad h^3_{22} = 0, \quad h^4_{ij} = 0.$$

The forms ω^α_i and ω^i_α are given by

$$\begin{aligned} &\omega^3_1 = \varepsilon\omega^1, \quad \omega^3_2 = 0, \quad \omega^4_1 = 0, \quad \omega^4_2 = 0, \\ &\omega^2_4 = -\varepsilon\omega^1, \quad \omega^1_4 = 0, \quad \omega^2_3 = 0, \quad \omega^1_3 = 0. \end{aligned} \quad (3.19)$$

The relations (3.18) give

$$-2\omega^1_1 + \omega^3_3 = \varepsilon h^3_{111}\omega^1 + \varepsilon h^3_{112}\omega^2, \quad \omega^1_2 = -\varepsilon h^3_{112}\omega^1, \quad \omega^4_3 = \varepsilon h^4_{111}\omega^1. \quad (3.20)$$

If $dim N_X M_2 = 0$, then $h^\alpha_{ij} = 0$.

Let us find the differential equations of a parallel symplectic surface of class $\mathcal{A}_4$. These equations are given by the formulae of infinitesimal displacement of an adapted frame $\{X; \vec{e}_i, \vec{e}_\alpha\}$ of the particular surface which must be reduced on account of the condition $h^\alpha_{ijk} = 0$. From the equations (3.12) we therefore obtain

$$\omega^1_1 = 0, \quad \omega^2_1 = -\varepsilon\omega^1_2, \quad \omega^3_3 = 0, \quad \omega^3_4 = 0.$$

Taking also into account (3.11), we find that the differential equations of a parallel symplectic surface are

$$d\vec{x} = \omega^1\vec{e}_1 + \omega^2\vec{e}_2, \quad d\vec{e}_1 = -\varepsilon\omega_2^1\vec{e}_2 + \omega^1\vec{e}_4, \quad d\vec{e}_2 = \omega_2^1\vec{e}_1 + \varepsilon\omega^2\vec{e}_4,$$
$$d\vec{e}_3 = -\varepsilon\omega^2\vec{e}_1 + \omega^1\vec{e}_2 + \omega_3^4\vec{e}_4, \quad d\vec{e}_4 = \vec{0}.$$

It is seen that a parallel symplectic surface is located in a three-dimensional subspace of the symplectic space Sp_4 whose direction space is the linear cover $L(\vec{e}_1, \vec{e}_2, \vec{e}_4)$. Since $d\vec{e}_4 = \vec{0}$, the subspace $L(\vec{e}_1, \vec{e}_2, \vec{e}_4)$, in turn, contains the one-dimensional fixed subspace $L(\vec{e}_4)$. So the differential equations of a parallel symplectic surface become

$$\begin{aligned} d\vec{x} &= \omega^1\vec{e}_1 + \omega^2\vec{e}_2, \quad d\vec{e}_1 = -\varepsilon\omega_2^1\vec{e}_2 + \omega^1\vec{e}_4, \\ d\vec{e}_2 &= \omega_2^1\vec{e}_1 + \varepsilon\omega^2\vec{e}_4, \quad d\vec{e}_4 = \vec{0}, \end{aligned} \tag{3.21}$$

where we have used the relation $\omega_2^1 + \varepsilon\omega_2^1 = 0$. The presence of the form ω_1^2 shows that, on the tangent plane $X + L(\vec{e}_1, \vec{e}_2)$, it is permissible to effect a basis transformation $\{\vec{e}_i\} \to \{\vec{e}_i{}'\}$ given by the equations

$$\begin{aligned} \vec{e}_1{}' &= (\cos\phi)\vec{e}_1 + (\sin\phi)\vec{e}_2, \ \vec{e}_2{}' = (-\sin\phi)\vec{e}_1 + (\cos\phi)\vec{e}_2 \quad (\varepsilon = 1), \\ \vec{e}_1{}' &= (\mathrm{ch}\phi)\vec{e}_1 + (\mathrm{sh}\phi)\vec{e}_2, \quad \vec{e}_2{}' = (\mathrm{sh}\phi)\vec{e}_1 + (\mathrm{ch}\phi)\vec{e}_2 \quad (\varepsilon = -1). \end{aligned} \tag{3.22}$$

Let us fix the position of the basis vectors $\{\vec{e}_i\}$ at the point X of the symplectic surface. As in such case $\omega_2^1 = \omega_1^2 = 0$, we get from the equations (3.21)

$$d\vec{x} = \omega^1\vec{e}_1 + \omega^2\vec{e}_2, \quad d\vec{e}_1 = \omega^1\vec{e}_4, \quad d\vec{e}_2 = \varepsilon\omega^2\vec{e}_4, \quad d\vec{e}_4 = \vec{0}. \tag{3.23}$$

These formulae retain their form for each position of the basis vectors $\{\vec{e}_i\}$ which is admissible by the equations (3.22). Consequently the sections of the parallel symplectic surface with the planes $x_4 = const$ are "circles" for $\varepsilon = 1$ and "hyperbolae" for $\varepsilon = -1$.

Now we may describe the structure of the parallel symplectic surface with the help of the coordinate lines $\omega^2 = 0$ and $\omega^1 = 0$ on it.

For the coordinate line $\omega^2 = 0$ we have $d\omega^1 = 0$ whence ω^1 is the total differential, i.e. $\omega^1 = dt$. From the formulae (3.23) the differential equations of this coordinate line are

$$\frac{d\vec{x}}{dt} = \vec{e}_1, \quad \frac{d\vec{e}_1}{dt} = \vec{e}_4, \quad \frac{d\vec{e}_4}{dt} = \vec{0},$$

from which we get

$$\vec{e}_4 = \vec{e}_4^{\,(0)}, \quad \vec{e}_1 = t\vec{e}_4^{\,(0)} + \vec{e}_1^{\,(0)}, \quad \vec{x} = \tfrac{1}{2}t^2\vec{e}_4^{\,(0)} + t\vec{e}_1^{\,(0)}.$$

We see that the coordinate line is a parabola with axis $X_0 + L(\overrightarrow{e_4}^{(0)})$, where $X = X_0$ is the point on the parallel symplectic surface from which we begin drawing the coordinate line (parabola) $\omega^2 = 0$, and $\overrightarrow{e_1}^{(0)}, \overrightarrow{e_4}^{(0)}$ are the basis vectors $\overrightarrow{e_1}, \overrightarrow{e_4}$ fixed at the point X_0. Here, of course, the vector $\overrightarrow{e_4}$ is invariable at all points of the symplectic surface, seeing that $d\overrightarrow{e_4} = \overrightarrow{0}$.

For the coordinate line $\omega^1 = 0$ we have $d\omega^2 = 0$, i.e. $\omega^2 = dt$. From the formulae (3.23) the differential equations of this coordinate line are

$$\frac{d\overrightarrow{x}}{dt} = \overrightarrow{e_2}, \quad \frac{d\overrightarrow{e_2}}{dt} = \varepsilon \overrightarrow{e_4}, \quad \frac{d\overrightarrow{e_4}}{dt} = \overrightarrow{0},$$

from which we get

$$\overrightarrow{e_4} = \overrightarrow{e_4}^{(0)}, \quad \overrightarrow{e_2} = \varepsilon t \overrightarrow{e_4}^{(0)} + \overrightarrow{e_2}^{(0)}, \quad \overrightarrow{x} = 2^{-1}\varepsilon t^2 \overrightarrow{e_4}^{(0)} + t\overrightarrow{e_2}^{(0)}.$$

We see that the coordinate line is a parabola with axis $X_0 + L(\overrightarrow{e_4}^{(0)})$. Summing up, we may say that the parallel symplectic surface is an elliptic paraboloid for $\varepsilon = 1$ and a hyperbolic paraboloid for $\varepsilon = -1$.

Let us next find the differential equations for a parallel symplectic surface of class $\mathcal{A}_5$. As the canonization of the frame in this class leads to two distinct cases, we must consider both of them.

In the first case we must reduce the formulae (3.20) in view of the condition $h^{\alpha}_{ijk} = 0$. We get $\omega^3_3 = 2\omega^1_1$, $\omega^1_2 = 0$ and $\omega^4_3 = 0$. Taking also into account (3.19), we find the differential equations of the parallel symplectic surface to be

$$\begin{gathered} d\overrightarrow{x} = \omega^1 \overrightarrow{e_1} + \omega^2 \overrightarrow{e_2}, \quad d\overrightarrow{e_1} = \omega^1_1 \overrightarrow{e_1} + \omega^2_1 \overrightarrow{e_2} + \varepsilon\omega^1 \overrightarrow{e_3}, \\ d\overrightarrow{e_2} = -\,\omega^1_1 \overrightarrow{e_2}, \quad d\overrightarrow{e_3} = 2\omega^1_1 \overrightarrow{e_3}. \end{gathered} \tag{3.24}$$

The equation $d\overrightarrow{e_4} = -\varepsilon\omega^1 \overrightarrow{e_2} + \omega^3_4 \overrightarrow{e_3} - 2\omega^1_1 \overrightarrow{e_4}$ is superfluous because the symplectic surface is located in a three-dimensional subspace of Sp_4, as can be seen from the equations (3.24). Its direction space is $L(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$. In this subspace there are two one-dimensional fixed subspaces $L(\overrightarrow{e_2})$ and $L(\overrightarrow{e_3})$. Consequently at each point X of the parallel symplectic surface there are determined a pair of straight lines $X + L(\overrightarrow{e_2})$ and $X + L(\overrightarrow{e_3})$. If the point X is now allowed to vary on the parallel symplectic surface, we get two families $\{X + L(\overrightarrow{e_2}) \mid X \in M_2\}$ and $\{X + L(\overrightarrow{e_3}) \mid X \in M_2\}$ of parallel straight lines. Every straight line $X + L(\overrightarrow{e_2})$ belongs to the tangent plane $T_X(M_2)$ and, furthermore, also to the symplectic surface for every $X \in M_2$. Hence a parallel symplectic surface is in this case a cylindrical symplectic surface the direction of whose generators is determined by the vector $\overrightarrow{e_2}$. The expression for $d\overrightarrow{e_1}$ shows that the cylinder is parabolic.

In the second case we have the condition $h^{\alpha}_{ij} = 0$. The equations (3.1) then give $h^{\alpha}_{ijk} = 0$. So there exist parallel symplectic surfaces. The differential equations of symplectic surfaces

$$d\vec{x} = \omega^i \vec{e}_i, \quad d\vec{e}_i = \omega^s_i \vec{e}_s$$

show that that each symplectic surface is a plane.

We sum up these results in the following theorem.

Theorem 2. *A parallel symplectic surface M_2 in the symplectic space Sp_4 is an elliptic paraboloid, a hyperbolic paraboloid or a parabolic cylinder of a three-dimensional space or else a symplectic plane.*

Finally it is natural to investigate whether the classes $\mathcal{A}_4$ and $\mathcal{A}_5$ of semi-parallel symplectic surfaces include some non-parallel symplectic surfaces.

A semi-parallel symplectic surface is not parallel if in the equations (3.12) the functions h^4_{111}, h^4_{112}, h^4_{122} and h^4_{222} are not simultaneously zero. It will be demonstrated that this is possible and hence a non-parallel symplectic surface exists. For this purpose we use the notation and methods presented in book [3].

We have to find the Cartan number Q, which is expressed in terms of "characters" as

$$Q = s_1 + 2s_2 + \ldots + ns_n,$$

where n is the dimension of the manifold sought for; in our case it equals two. Thus

$$Q = s_1 + 2s_2.$$

As the characters satisfy the inequality $s_n < s_{n-1} < \ldots < s_1$, we have $s_2 < s_1$. It is known that $q = s_1 + s_2$, where q is the number of independent exterior differentials $d\omega^\alpha$ of the left sides of the equations $\omega^\alpha = 0$ of a symplectic surface M_2,

$$d\omega^\alpha \Longleftrightarrow \omega^\alpha_i \wedge d\omega^i \Longleftrightarrow \omega^3_1 \wedge \omega^1 + \omega^3_2 \wedge \omega^2, \; \omega^4_1 \wedge \omega^1 + \omega^4_2 \wedge \omega^2. \tag{3.25}$$

We see that $q = 2$, and so $s_1 + s_2 = 2$. The character s_1 is the rank of the new system of forms $\{\omega^3_1, \omega^3_2, \omega^4_1, \omega^4_2\}$ in equation (3.25); consequently (3.12) yields $s_1 = 2$, $s_2 = 0$ and the Cartan number $Q = 2$.

In the end, we need the number of nonzero coefficients N in the expressions of the forms $\{\omega^3_1, \omega^3_2, \omega^4_1, \omega^4_2\}$. From the equations (3.12) we conclude that these coefficients are 1 and ε, hence $N = 2$. In the case where N and Q are equal (here $N = Q = 2$), a semi-parallel symplectic surface exists for arbitrary choise of any pair of the functions h^4_{111}, h^4_{112}, h^4_{122} and h^4_{222}. Hence in the case of symplectic surface $M_2 \in \mathcal{A}_4$ a semi-parallel symplectic surface exists which is not parallel.

Let us consider the analogous problem of a semi-parallel symplectic surface $M_2 \in \mathcal{A}_5$. This surface may be non-parallel if $dim N_X(M_2) = 1$. Again the existence of a symplectic surface does not depend on the choice of the functions h^3_{111} and h^3_{112} in the equations (3.20). As here also $n = 2$, we have two characters s_1 and s_2. The relations (3.25) contain only a single significant equation ($q = 1$), seeing that for this class $h^4_{ij} = 0$, that is $\omega^4_1 = 0$ and $\omega^4_2 = 0$. From the equations (3.10) we get $s_1 = 1$, consequently $s_2 = 0$. In this case $N = Q = 1$ and semi-parallel symplectic surfaces exist. Therefore a semi-parallel symplectic surface exists which is not parallel.

Theorem 3. *In the symplectic space Sp_4 semi-parallel symplectic surfaces exist which are not parallel.*

Acknowledgement

The author is grateful to the referee for useful remarks.

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