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Matrix transformations of double sequences

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ABSTRACT. We study the space $\mathcal{C}_e$ of double sequences (x_{kl}) satisfying $\lim_l \overline{\lim}_k |x_{kl} - a| = 0$ for some number a, and its subspace $\mathcal{C}_{be}$ of elements with bounded columns. Using the gliding hump method, we find necessary and sufficient conditions for a 4-dimensional matrix to transform a space X into a space Y, where $X, Y \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$.

1. Introduction and preliminaries

In [1] Boos, Leiger and Zeller introduced and investigated the notion of $\mathcal{C}_e$-convergence for double sequence spaces, which is essentially weaker than the well-known Pringsheim convergence. Recall that a double sequence (x_{kl}) is said to *converge to the limit a in Pringsheim's sense* if

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}:\ k, l > N\ \Rightarrow |\, x_{kl} - a\,| < \varepsilon.$$

Thus in this limiting process of double sequences the row-index k and the column-index l tend to infinity independently from each other. In contrast, the $\mathcal{C}_e$-convergence is characterized by the dependence of the row-index k on the column-index l in tending to infinity. More precisely, the space of all $\mathcal{C}_e$-convergent double sequences is defined as

$$\begin{aligned}\mathcal{C}_e := \big\{x \in \Omega \mid \exists a \in \mathbb{K}\ \forall \varepsilon > 0\ \exists l_0 \in \mathbb{N}\ \forall l \geq l_0\ \exists k_l \in \mathbb{N}:&\\ k \geq k_l\ \Rightarrow\ |x_{kl} - a| \leq \varepsilon\big\}&\\ = \Big\{x \in \Omega \mid \exists a \in \mathbb{K}:\ \lim_l \overline{\lim_k} |x_{kl} - a| = 0\Big\},&\end{aligned}$$

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where Ω denotes the linear space of all complex (or real) double sequences and $\mathbb{K}$ is the field of all complex (or real) numbers. We will be also concerned with the subspace

$$\mathcal{C}_{be} := \Big\{x \in \mathcal{C}_e |\ \forall l \in \mathbb{N} : \sup_k |x_{kl}| < \infty\Big\}$$

of $\mathcal{C}_e$. Note that in [1] the notation $\widehat{\mathcal{C}}$ was used instead of $\mathcal{C}_{be}$. Both the spaces $\mathcal{C}_e$ and $\mathcal{C}_{be}$ contain the subspace Φ =span $\{\mathbf{e^{kl}} |\ k,\ l \in \mathbb{N}\}$, where $\mathbf{e}^{\mathbf{kl}}_{ij} := 1$ if $(k,l) = (i,j)$ and $\mathbf{e}^{\mathbf{kl}}_{ij} := 0$ otherwise. The double sequences $\mathbf{e^l} := \sum_k \mathbf{e^{kl}}$ $(l \in \mathbb{N})$, $\mathbf{e_k} := \sum_l \mathbf{e^{kl}}$ $(k \in \mathbb{N})$, $\mathbf{e} := \sum_{k,l} \mathbf{e^{kl}}$ (pointwise sums) are also contained in these spaces.

In Section 2 we consider the matrix transformation

$$Bx := \Big(\sum_k b_{mnk} x_k\Big)_{m,n}$$

defined by a 3-dimensional matrix $B = (b_{mnk})$, and find necessary and sufficient conditions for B to map the space of all sequences ω into $\mathcal{C}_e$ or $\mathcal{C}_{be}$. Note that in [2] we already established the conditions for B to map the spaces of all convergent and bounded sequences into $\mathcal{C}_e$ or $\mathcal{C}_{be}$.

Let $\mathcal{V}$ be the space of double sequences, converging with respect to some linear convergence rule $\mathcal{V}$-lim : $\mathcal{V} \to \mathbb{K}$. The sum of a double series $\sum_{k,l} u_{kl}$ with respect to this rule will be defined by

$$\mathcal{V}\text{-}\sum_{k,l} u_{kl} := \mathcal{V}\text{-}\lim_{m,n} \sum_{k=1}^{m}\sum_{l=1}^{n} u_{kl}.$$

For a subset $M \subset \Omega$, its *β-dual* will be specified as

$$M^{\beta(\mathcal{V})} := \Big\{u \in \Omega |\ \forall x \in M :\ \Big(\sum_{l=1}^{n}\sum_{k=1}^{m} u_{kl}x_{kl}\Big)_{m,n} \in \mathcal{V}\Big\}.$$

We write $\beta(e)$ and $\beta(be)$ instead of $\beta(\mathcal{C}_e)$ and $\beta(\mathcal{C}_{be})$, respectively.

Given any 4-dimensional scalar matrix $A = (a_{mnkl})$ and two spaces $\mathcal{V}_1$, $\mathcal{V}_2$ of double sequences, converging with respect to linear convergence rules $\mathcal{V}_1$-lim and $\mathcal{V}_2$-lim, respectively, we denote

$$(\mathcal{V}_1)^{(\mathcal{V}_2)}_A := \Big\{x \in \Omega |\ Ax := \Big(\mathcal{V}_2\text{-}\sum_{k,l} a_{mnkl}x_{kl}\Big)_{m,n} \text{ exists and } Ax \in \mathcal{V}_1\Big\}. \quad (1)$$

The main results of this paper (Section 3) give necessary and sufficient conditions for a 4-dimensional matrix $A = (a_{mnkl})$ to map a space X into a space Y, where $X, Y \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$, and the convergence of the double series $\sum_{k,l} a_{mnkl} x_{kl}$ is understood either in the $\mathcal{C}_e$- or $\mathcal{C}_{be}$-sense. In other words, we will find conditions for A to satisfy $\mathcal{V}_3 \subset (\mathcal{V}_1)_A^{(\mathcal{V}_2)}$, where $\mathcal{V}_i \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$ $(i = 1, 2, 3)$. Evidently, for any such matrix map A we get $(a_{mnkl})_{k,l} \in \mathcal{V}_3^{\beta(\mathcal{V}_2)}$ for every $m, n \in \mathbb{N}$.

We will use the notation $\mathfrak{m}$, ℓ, and φ for the spaces of all bounded, absolutely summable, and finite sequences, respectively.

2. Three-dimensional matrix maps

In [2] we established necessary and sufficient conditions for a 3-dimensional matrix to map $\mathfrak{m}$ into $\mathcal{C}_e$ and $\mathcal{C}_{be}$:

Proposition 2.1 ([2], Proposition 3.1). *A 3-dimensional matrix $B = (b_{mnk})$ maps $\mathfrak{m}$ into $\mathcal{C}_e$ if and only if the following conditions hold:*

(i) *for every $k \in \mathbb{N}$ the limit $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ exists,*

(ii) $\sum_k |b_{mnk}| < \infty$ *for every $m, n \in \mathbb{N}$,*

(iii) *there exists $N \in \mathbb{N}$ such that* $\sup_{n \geq N} \overline{\lim}_m \sum_k |b_{mnk}| < \infty$,

(iv) $\lim_n \overline{\lim}_m \sum_k |b_{mnk} - b_k| = 0$.

Under these circumstances, $(b_k) \in \ell$ and

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

Proposition 2.2 ([2], Proposition 3.5). *A 3-dimensional matrix $B = (b_{mnk})$ maps $\mathfrak{m}$ into $\mathcal{C}_{be}$ if and only if B satisfies* (iv) *of Proposition 2.1,*

(i′) *for every $k \in \mathbb{N}$ the limit $b_k := \mathcal{C}_{be}\text{-}\lim_{m,n} b_{mnk}$ exists,*

(ii′) $\sup_n \overline{\lim}_m \sum_k |b_{mnk}| < \infty$.

Under these circumstances, $(b_k) \in \ell$ and

$$\lim_B x = \sum_k b_k x_k \quad (x \in \mathfrak{m}).$$

To investigate conditions for a matrix $B = (b_{mnk})$ to map ω into $\mathcal{C}_e$ and $\mathcal{C}_{be}$, we introduce some notation.

Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection having $\varphi[(1,1)] = 1$, $\varphi[(1,2)] = 2$, $\varphi[(2,1)] = 3, \ldots, \varphi[(1,n)] = (n-1)n/2 + 1$, $\varphi[(2, n-1)] = (n-1)n/2 + 2, \ldots, \varphi[(n,1)] = n(n+1)/2, \ldots$. Let π_1 and π_2 be operators from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ having $\pi_1 : (a,b) \mapsto a$, $\pi_2 : (a,b) \mapsto b$. We put $\lambda_i := \pi_i \varphi^{-1}$ $(i = 1, 2)$.

Remark 2.3. If we have a double sequence of subsets $I_{ij} = \{k_{ij}, k_{ij} + 1, \ldots, l_{ij}\} \subset \mathbb{N}$ such that $k_{i,j+1} > l_{ij}$ $(i, j \in \mathbb{N})$, then passing to a subsequence of $(I_{ij})_j$ if required $(i \in \mathbb{N})$, we can assume that the subsets are arranged in such a way that $k_{\varphi^{-1}(i+1)} > l_{\varphi^{-1}(i)}$ (note that in this case $(\bigcup_j I_{i_1 j}) \cap (\bigcup_j I_{i_2 j}) = \emptyset$ for every $i_1 \neq i_2$). To verify it we will construct a double sequence (m_{ij}) such that $m_{i,j+1} > m_{ij}$ $(i,j \in \mathbb{N})$ and $k_{\lambda_1(s+1)m_{\varphi^{-1}(s+1)}} > l_{\lambda_1(s)m_{\varphi^{-1}(s)}}$ $(s \in \mathbb{N})$.

On the 1-st step set $m_{11} := 1$. On the second step set $m_{12} := 2$ and choose $m_{21} \in \mathbb{N}$ such that $k_{2m_{21}} > l_{1m_{12}}$. On the j-th step choose $m_{1j} > m_{1,j-1}$ such that $k_{1m_{1j}} > l_{j-1,m_{j-1,1}}$, then choose $m_{2,j-1} > m_{2,j-2}$ such that $k_{2m_{2,j-1}} > l_{1m_{1j}}$ and so on until $m_{j1} \in \mathbb{N}$ such that $k_{jm_{j1}} > l_{j-1,m_{j-1,2}}$.

Proposition 2.4. *A 3-dimensional matrix $B = (b_{mnk})$ maps ω into $\mathcal{C}_e$ if and only if the following conditions hold:*

(i) *for every $k \in \mathbb{N}$ the limit $b_k := \mathcal{C}_e\text{-}\lim_{m,n} b_{mnk}$ exists,*

(ii) *$b^{(m,n)} := (b_{mnk})_k \in \varphi$ for every $m, n \in \mathbb{N}$,*

(iii) *for every sequence (ε_k) of positive numbers there exist $N, K \in \mathbb{N}$ such that*

$$\forall n > N \; \exists m_n : \; m > m_n \; \Rightarrow \; |b_{mnk}| < \varepsilon_k \; (k > K).$$

Under these circumstances, $b := (b_k) \in \varphi$ and

$$\mathcal{C}_e\text{-}\lim_{m,n} (Bz)_{mn} = \sum_k b_k z_k \quad (z \in \omega).$$

Proof. Necessity of **(i)** and **(ii)** is evident. **(iii)** By (ii), $b^{(m,n)} \in \varphi$ for every $m, n \in \mathbb{N}$. We will verify that starting with some index $n_0 \in \mathbb{N}$ there exists $K(n) \in \mathbb{N}$ with $(b_{mnk})_k = \sum_{k=1}^{K(n)} b^{(m,n)} e^k$ $(m \in \mathbb{N})$.

We suppose the contrary that there exist an increasing sequence (n_i) and double sequences (m_{ij}), (k_{ij}) such that $m_{i,j+1} > m_{ij}$, $k_{\varphi^{-1}(i+1)} > k_{\varphi^{-1}(i)}$ (cf. Remark 2.3), $b_{m_{ij} n_i k_{ij}} \neq 0$, and $b_{m_{ij} n_i k} = 0$ for $k > k_{ij}$ $(i,j \in \mathbb{N})$. Passing to subsequences of (n_i) and $(m_{ij})_j$ $(i \in \mathbb{N})$ on need, we can suppose that $\Re(b_{m_{ij} n_i k_{ij}}) > 0 \quad (i,j \in \mathbb{N})$.

We put $z_{k_{ij}} := \big(i - \Re(\sum_{s=1}^{\varphi(i,j)-1} b_{m_{ij} n_i k_{\varphi^{-1}(s)}} z_{k_{\varphi^{-1}(s)}})\big)/\Re(b_{m_{ij} n_i k_{ij}})$ $(i,j \in \mathbb{N})$ and $z_k := 0$ for $k \notin \{k_{ij} |\; i,j \in \mathbb{N}\}$. Then

$$\Re\big((Bz)_{m_{ij} n_i}\big) = \Re\big(b_{m_{ij} n_i k_{ij}} z_{k_{ij}}\big) + \Re\Big(\sum_{s=1}^{\varphi(i,j)-1} b_{m_{ij} n_i k_{\varphi^{-1}(s)}} z_{k_{\varphi^{-1}(s)}}\Big) = i$$

for every $i, j \in \mathbb{N}$, yielding a contradiction.

Now if (iii) fails, then there exists a sequence (ε_k) with $\varepsilon_k > 0$, an increasing sequence (n_i) and double sequences (m_{ij}), (k_{ij}) such that $m_{i,j+1} > m_{ij}$, $k_{ij} > i$, and $|b_{m_{ij} n_i k_{ij}}| \geq \varepsilon_{k_{ij}}$ $(i,j \in \mathbb{N})$. Keeping in mind (i) and the first part of the proof and passing to subsequences (n_i) and $(m_{ij})_j$ $(i \in \mathbb{N})$ on need, we can suppose that $\Re(b_{m_{ij}n_ik_{ij}}) \geq \varepsilon_{k_{ij}}/2$ $(i,j \in \mathbb{N})$, $k_{ij} =: k_i$ is constant for every $i \in \mathbb{N}$, $k_{i+1} > K(n_i)$, and $\sum_{s=1}^{i} |b_{m_{i+1,j}n_{i+1}k_s}| \leq 2\sum_{s=1}^{i} |b_{k_s}|$ $(i,j \in \mathbb{N})$. We put $z_k := 0$ for $k \notin \{k_i |\ i \geq 2\}$ and $z_{k_i} := 2\big(i + 2\sum_{s=1}^{i-1} |b_{k_s}| \max\{z_{k_s} |\ s = 1,\dots,i-1\}\big)/\varepsilon_{k_i}$ $(i \geq 2)$. Then

$$\big|(Bz)_{m_{ij}n_i}\big| \geq \Re\big(b_{m_{ij}n_ik_i}z_{k_i}\big) - 2\sum_{s=1}^{i-1}|b_{k_s}| \max_{s=1,\dots,i-1} z_{k_s} \geq i \quad (i,j \in \mathbb{N};\ i \geq 2),$$

yielding a contradiction.

Sufficiency. First we will verify that $(b_k) \in \varphi$. Suppose the contrary, that there exists an index sequence (k_i) such that $b_{k_i} \neq 0$ $(i \in \mathbb{N})$. By (iii), we can find $N, i_0 \in \mathbb{N}$ and a sequence (m_n) such that $|b_{mnk_i}| < |b_{k_i}|/2$ for $n \geq N$, $m \geq m_n$ and $i \geq i_0$. By (i), we can also find $N_1 > N$ and $m'_n > m_n$ $(n > N_1)$ such that $|b_{mnk_{i_0}} - b_{k_{i_0}}| < |b_{k_{i_0}}|/2$ for $n \geq N_1$, $m \geq m'_n$. Hence $|b_{k_{i_0}}| < |b_{k_{i_0}}|$, yielding a contradiction.

Now let $z \in \omega$ and $\varepsilon > 0$ be fixed and let $k_0 \in \mathbb{N}$ be such that $b_k = 0$ for $k > k_0$. We can suppose that $z_k \neq 0$ $(k \in \mathbb{N})$. By (iii), there exists $K > k_0$, $N_1 \in \mathbb{N}$ and a sequence (m_n) such that $|b_{mnk}| < \varepsilon/(2^k|z_k|)$ for $k > K$, $m > m_n$, $n > N_1$. By (i), there exists $N_2 > N_1$ and a sequence (m'_n) such that $m'_n > m_n$ $(n > N_2)$ and $|b_{mnk} - b_k| < \varepsilon/(2^k|z_k|)$ for $k = 1,\dots,K$, $m > m'_n$, $n > N_2$. Now for $n > N_2$, $m > m'_n$ we obtain

$$\Big|\sum_k b_{mnk}z_k - \sum_{k=1}^{K} b_kz_k\Big| \leq \sum_{k=1}^{K} |b_{mnk} - b_k||z_k| + \sum_{k=K+1}^{\infty} |b_{mnk}||z_k| \leq \sum_k \frac{\varepsilon}{2^k} < \varepsilon.$$

Hence $\mathcal{C}_e\text{-}\lim_{m,n}(Bz)_{mn} = \sum_{k=1}^{K} b_kz_k$. □

Recall that a matrix $D = (d_{nk})$ maps ω into $\mathfrak{m}$ if and only if (i) there exists $N \in \mathbb{N}$ such that $d_{nk} = 0$ for $k > N$ and every $n \in \mathbb{N}$; and (ii) $\sup_n \sum_k |d_{nk}| < \infty$.

Proposition 2.5. *A 3-dimensional matrix $B = (b_{mnk})$ maps ω into $\mathcal{C}_{be}$ if and only if it satisfies hypotheses* (i), (iii) *of Proposition 2.4 and*

(ii′) *for every $n \in \mathbb{N}$ the matrix $(b_{mnk})_{m,k}$ maps ω into $\mathfrak{m}$.*

Under these circumstances, $b \in \varphi$ and $\mathcal{C}_{be}\text{-}\lim_{m,n}(Bz)_{mn} = \sum_k b_kz_k$ $(z \in \omega)$.

Proof. Necessity is evident.

Sufficiency. By Proposition 2.4, the limit $\mathcal{C}_e\text{-}\lim_{m,n}(Bz)_{mn}$ exists for every $z \in \omega$. By (ii′), for every $n \in \mathbb{N}$ there exists $N(n) \in \mathbb{N}$ such that $b_{mnk} = 0$ for $k > N(n)$ $(m \in \mathbb{N})$. So

$$\sup_m |(Bz)_{mn}| \leq \max_{k=1,\dots,N(n)} |z_k| \sup_m \sum_{k=1}^{N(n)} |b_{mnk}| \quad (n \in \mathbb{N};\ z \in \omega).$$

Hence $\mathcal{C}_{be}$-$\lim_{m,n}(Bz)_{mn}$ exists for every $z \in \omega$. $\square$

3. Four-dimensional matrix maps

In this section we obtain necessary and sufficient conditions for a matrix map $A: \ X \to Y$, where $X, Y \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$, and the double series $\sum_{k,l} a_{mnkl}x_{kl}$ (cf. (1)) are either $\mathcal{C}_e$- or $\mathcal{C}_{be}$-convergent.

First we verify that $\mathcal{C}_e^{\beta(e)} = \mathcal{C}_e^{\beta(be)} = \Phi$. Let $u \in \mathcal{C}_e^{\beta(e)}$ be fixed. Since $x\mathbf{e^l}$ and $x\mathbf{e_k}$ are in $\mathcal{C}_e$ for every $k, l \in \mathbb{N}$ and $x \in \Omega$, the sequences $(u_{kl_0})_k$, $(u_{k_0l})_l$ are finite ($k_0, l_0 \in \mathbb{N}$). Moreover, there exists $N \in \mathbb{N}$ such that $u = \sum_{l=1}^{N} u\mathbf{e^l}$. If we suppose the contrary that there exist increasing index sequence (l_i) and index sequence (k_i) such that $u_{k_il_i} \neq 0$ ($i \in \mathbb{N}$), then for $x \in \mathcal{C}_0$ with $x_{k_il_i} := 1/u_{k_il_i}$ ($i \in \mathbb{N}$) and $x_{kl} := 0$ for $(k,l) \notin \{(k_i, l_i) |\ i \in \mathbb{N}\}$ we get

$$\sum_{k=1}^{\max\{k_s|s=1,\dots,i\}} \sum_{l=1}^{l_i} u_{kl}x_{kl} = \sum_{s=1}^{i} 1 = i.$$

This proves our statement. Hence $u \in \Phi$, implying $\mathcal{C}_e^{\beta(e)} \subset \Phi$. So $\mathcal{C}_e^{\beta(e)} = \mathcal{C}_e^{\beta(be)} = \Phi$.

Now we verify that

$$\mathcal{C}_{be}^{\beta(e)} = \mathcal{C}_{be}^{\beta(be)} = \varphi(\ell) := \Big\{ u \in \Omega |\ \forall l \in \mathbb{N}: \ (u_{kl})_k \in \ell \text{ and } \exists l_0 \in \mathbb{N}: \ u = \sum_{l=1}^{l_0} u\mathbf{e^l} \Big\}.$$

Let $u \in \mathcal{C}_{be}^{\beta(e)}$ be fixed. Since every $x \in \Omega$ such that $x = x\mathbf{e^{l_0}}$ and $(x_{kl_0})_k \in \mathfrak{m}$ for some $l_0 \in \mathbb{N}$ is in $\mathcal{C}_{be}$, then $(u_{kl})_k \in \ell$ for every $l \in \mathbb{N}$. On the other hand, since $x\mathbf{e_k}$ is in $\mathcal{C}_{be}$ for every $k \in \mathbb{N}$ and $x \in \Omega$, then $(u_{kl})_l \in \varphi$ ($k \in \mathbb{N}$). Moreover, in the same way as for $\mathcal{V} = \mathcal{C}_e$ we can prove that there exists $N \in \mathbb{N}$ such that $u = \sum_{l=1}^{N} u\mathbf{e^l}$. Hence $\mathcal{C}_{be}^{\beta(e)} \subset \varphi(\ell)$. A direct check shows that $\varphi(\ell) \subset \mathcal{C}_{be}^{\beta(be)}$. Hence $\mathcal{C}_{be}^{\beta(e)} = \mathcal{C}_{be}^{\beta(be)} = \varphi(\ell)$.

So, given a 4-dimensional matrix $A = (a_{mnkl})$, we get $(\mathcal{C}_e)_A^{(\mathcal{C}_e)} = (\mathcal{C}_e)_A^{(\mathcal{C}_{be})}$ and $(\mathcal{C}_{be})_A^{(\mathcal{C}_e)} = (\mathcal{C}_{be})_A^{(\mathcal{C}_{be})}$. Therefore it makes no difference, whether we require the $\mathcal{C}_e$- or $\mathcal{C}_{be}$-convergence of double series $\sum_{k,l} a_{mnkl}x_{kl}$ ($x \in E$; $k, l \in \mathbb{N}$) for matrix maps $A: \ E \to F$; $E, F \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$.

Theorem 3.1. *A 4-dimensional matrix $A = (a_{mnkl})$ maps $\mathcal{C}_{be}$ into $\mathcal{C}_e$ if and only if the following conditions hold:*

(i) *for every $l_0 \in \mathbb{N}$ the matrix $(a_{mnkl_0})_{m,n,k}$ maps $\mathfrak{m}$ into $\mathcal{C}_e$,*
(ii) *the limit $v := \mathcal{C}_e$-$\lim_{m,n} \sum_{k,l} a_{mnkl}$ exists,*

(iii) *for every* $m, n \in \mathbb{N}$: $a^{(m,n)} \in \varphi(\ell)$,

(iv) *for every index sequence* (k_l) *and sequence* (ε_l) *of positive numbers there exist* $N_1, P \in \mathbb{N}$ *such that*

$$\forall n > N_1 \; \exists m_n : \; m > m_n \; \Rightarrow \; \sum_{k=1}^{k_l} |a_{mnkl}| < \varepsilon_l \quad (l > P),$$

(v) *there exists* $N' \in \mathbb{N}$ *and a sequence* (m_n) *such that*

$$\sup_{\substack{n > N' \\ m \geq m_n}} \sum_{k,l} |a_{mnkl}| < \infty.$$

Under these hypotheses, $a := (a_{kl}) \in \varphi(\ell)$, *where* $a_{kl} := \mathcal{C}_e\text{-}\lim_{m,n} a_{mnkl}$ $(k, l \in \mathbb{N})$, *and*

$$\mathcal{C}_e\text{-}\lim_{m,n} (Ax)_{m,n} = \sum_{k,l} a_{kl} x_{kl} + \Big(v - \sum_{k,l} a_{kl}\Big) \mathcal{C}_{be}\text{-}\lim_{m,n} x_{m,n} \quad (x \in \mathcal{C}_{be}).$$

Remark 3.2. Statement (iv) implies

$$\exists n_0 \in \mathbb{N} \;\; \forall n \geq n_0 \;\; \exists L(n) \in \mathbb{N}\text{: } a^{(m,n)} = \sum_{l=1}^{L(n)} a^{(m,n)} \mathbf{e}^{\mathbf{l}} \;\; (m \in \mathbb{N}); \tag{2}$$

$$\exists L, N \in \mathbb{N}\text{: } l \geq L, \; n \geq N \; \Rightarrow \; \lim_m a_{mnkl} = 0 \quad (k \in \mathbb{N}). \tag{3}$$

In fact, if (2) fails, then there exist an increasing sequence (n_i) and double sequences (m_{ij}), (k_{ij}), (l_{ij}) such that $m_{i,j+1} > m_{ij}$, $l_{\varphi^{-1}(i+1)} > l_{\varphi^{-1}(i)}$ (cf. Remark 2.3), and $a_{m_{ij} n_i k_{ij} l_{ij}} \neq 0$ $(i, j \in \mathbb{N})$. Put

$$\varepsilon_{l_{\varphi^{-1}(r)}} := |a_{m_{\varphi^{-1}(r)} n_{\lambda_1(r)} k_{\varphi^{-1}(r)} l_{\varphi^{-1}(r)}}| \text{ and } \varepsilon_l := 1 \text{ for } l \notin \{l_{\varphi^{-1}(r)} | \; r \in \mathbb{N}\}.$$

By (iv), there exist $i_0 \in \mathbb{N}$ and a sequence (j_i) such that for $i \geq i_0$, $j \geq j_i$ we have $|a_{m_{ij} n_i k_{ij} l_{ij}}| \leq \sum_{k=1}^{k_{ij}} |a_{m_{ij} n_i k l_{ij}}| < |a_{m_{ij} n_i k_{ij} l_{ij}}|$. This yields a contradiction.

If (3) does not hold, then there exist increasing sequences (l_i), (n_i), an index sequence (k_i) and a double sequence (m_{ij}) with $m_{i,j+1} > m_{ij}$ $(i, j \in \mathbb{N})$ such that $\nu_i := \inf_j |a_{m_{ij} n_i k_i l_i}| > 0$ $(i \in \mathbb{N})$. Putting $\varepsilon_{l_i} := \nu_i$ $(i \in \mathbb{N})$ and $\varepsilon_l := 1$ for $l \notin \{l_i | \; i \in \mathbb{N}\}$, we obtain a contradiction by (iv).

Proof of Theorem 3.1. Necessity of **(i)–(ii)** follows, since $x\mathbf{e}^{\mathbf{l_0}}$ and $\mathbf{e}$ are in $\mathcal{C}_{be}$ for every $l_0 \in \mathbb{N}$ and $x \in \Omega$ with $\sup_k |x_{kl}| < \infty$ $(l \in \mathbb{N})$. **(iii)** follows, since $\mathcal{C}_{be}^{\beta(e)} = \mathcal{C}_{be}^{\beta(be)} = \varphi(\ell)$.

(iv) Let (k_l) and (ε_l) with $\varepsilon_l > 0$ be fixed. We put $b_{mnk} := a_{mnk1}$, $\varepsilon'_k := \varepsilon_1/k_1$ for $k = 1, \dots, k_1$, $b_{mnk} := a_{m,n,k-k_1,2}$, $\varepsilon'_k := \varepsilon_2/k_2$ for $k = k_1+1, \dots, k_1+k_2$ $(m, n \in \mathbb{N})$. Continuing inductively, for $\sum_{j=1}^{i-1} k_j < k \le \sum_{j=1}^{i} k_j =: k'_i$ we put $b_{mnk} := a_{m,n,k-k'_{i-1},i}$, $\varepsilon'_k := \varepsilon_i/k_i$ $(i, m, n \in \mathbb{N})$. The obtained matrix $B := (b_{mnk})$ maps ω into $\mathcal{C}_e$.

So by Proposition 2.4 (iii), there exist $P, N \in \mathbb{N}$ and a sequence (m_n) such that $|b_{mnk}| \le \varepsilon'_k$ for $k > \sum_{j=1}^{P-1} k_j$, $n > N$ and $m > m_n$. Hence for $l > P$, $n > N$ and $m > m_n$ we get

$$\sum_{k=1}^{k_l} |a_{mnkl}| = \sum_{k=k'_{l-1}+1}^{k'_{l-1}+k_l} |b_{mnk}| \le k_l \frac{\varepsilon_l}{k_l} = \varepsilon_l.$$

(v) By (i) and Proposition 2.1 (iii), for every $p \in \mathbb{N}$ we can find $N \in \mathbb{N}$ such that $\sup_{n\ge N} \overline{\lim}_m \sum_{l=1}^{p} \sum_k |a_{mnkl}| < \infty$. Hence if (v) fails, there exist increasing sequences (n_i), (p^i) and a double sequence (m_{ij}) with $m_{i,j+1} > m_{ij}$ $(i, j \in \mathbb{N})$ such that

$$\sum_{l=p^i}^{L_i} \sum_k |a_{m_{ij}n_i kl}| \ge 2i^2 + 2i \sum_{l=1}^{p^i-1} \sum_k |a_{m_{ij}n_i kl}| \quad (i, j \in \mathbb{N}),$$

where $L_i := L(n_i)$. We may assume that $p^{i+1} > L_i$ and

$$\sum_{l=p^i}^{L_i} \sum_k |\Re(a_{m_{ij}n_i kl})| \ge i^2 + i \sum_{l=1}^{p^i-1} \sum_k |a_{m_{ij}n_i kl}| \quad (i, j \in \mathbb{N}).$$

Now for every $i, j \in \mathbb{N}$ we fix $M(i,j) \in \mathbb{N}$ such that

$$\sum_{l=p^i}^{L_i} \sum_{k=M(i,j)+1}^{\infty} |a_{m_{ij}n_i kl}| \le 1 \quad (i, j \in \mathbb{N}).$$

By Remark 3.2, we may suppose that $\sum_{l=p^i}^{L_i} \sum_{k=1}^{M(i,j)} |a_{m_{i,j+1}n_i kl}| \le 1$ $(i, j \in \mathbb{N})$. Hence

$$\sum_{l=p^i}^{L_i} \sum_{k=M(i,j)+1}^{M(i,j+1)} |\Re(a_{m_{i,j+1}n_i kl})| \ge i^2 - 2 + i \sum_{l=1}^{p^i-1} \sum_k |a_{m_{i,j+1}n_i kl}| \ (i, j \in \mathbb{N}).$$

We put $x_{kl} := \operatorname{sgn} \Re(a_{m_{ij}n_ikl})/i$ if $l = p^i, \ldots, L_i$ and $k = M(i, j-1) + 1, \ldots, M(i,j)$ for some $i, j \in \mathbb{N}$, $j > 1$ and $x_{kl} := 0$ otherwise. Therefore $x = (x_{kl}) \in \mathcal{C}_{be}$ and

$$\begin{aligned} \left|(Ax)_{m_{ij}n_i}\right| \geq \frac{1}{i} \sum_{l=p^i}^{L_i} \sum_{k=M(i,j-1)+1}^{M(i,j)} |\Re(a_{m_{ij}n_ikl})| - \sum_{l=1}^{p^i-1} \sum_k |a_{m_{ij}n_ikl}| \\ - \sum_{l=p^i}^{L_i} \sum_{k=1}^{M(i,j-1)} |a_{m_{ij}n_ikl}| - \sum_{l=p^i}^{L_i} \sum_{k=M(i,j)+1}^{\infty} |a_{m_{ij}n_ikl}| \geq i - \frac{2}{i} - 2 \end{aligned}$$

for $i, j \in \mathbb{N}$, $j > 1$, yielding a contradiction.

Sufficiency. First note that (i) implies $(a_{kl_0})_k \in \ell$ for every fixed $l_0 \in \mathbb{N}$. Moreover, in the same way as we proved that $b \in \varphi$ in Proposition 2.4 we verify that there exist $L \in \mathbb{N}$ such that $a = \sum_{l=1}^{L} a\mathbf{e}^{\mathbf{l}}$.

Further, we can suppose that $a_{kl} = 0$ $(k, l \in \mathbb{N})$, because if A maps $\mathcal{C}_{be}$ into $\mathcal{C}_e$, then $A' = (a'_{mnkl})$ with $a'_{mnkl} := a_{mnkl} - a_{kl}$ $(m, n, k, l \in \mathbb{N})$ also does and $\mathcal{C}_e\text{-}\lim_{m,n} a'_{mnkl} = 0$ $(k, l \in \mathbb{N})$. Moreover,

$$\mathcal{C}_e\text{-}\lim_{m,n}(Ax)_{m,n} = \mathcal{C}_e\text{-}\lim_{m,n}(A'x)_{m,n} + \sum_{l=1}^{L} \sum_k a_{kl}x_{kl} \quad (x \in \mathcal{C}_{be}).$$

First we prove the statement for $x \in \mathcal{C}_{be}$ having $\mathcal{C}_{be}\text{-}\lim_{k,l} x_{kl} = 0$. For a fixed $\varepsilon > 0$ we choose $P > K$ and a sequence $(k_l)_{l=P}^{\infty}$ such that $|x_{kl}| < \varepsilon$ $(k \geq k_l,\ l \geq P)$. By (iv) and (v), we can find find $N_1, P_1 \in \mathbb{N}$ with $P_1 \geq P$ and a sequence (m_n) such that

$$n > N_1,\ m > m_n \implies \sum_{k=1}^{k_l} |a_{mnkl}| < \frac{\varepsilon}{2^l \max\{1, |x_{1l}|, \ldots, |x_{k_ll}|\}} \quad (l > P_1),$$

$$M := \sup_{\substack{n \geq N_1 \\ m \geq m_n}} \sum_{k,l} |a_{mnkl}| < \infty.$$

We will estimate

$$\begin{aligned} \left|\sum_{k,l} a_{mnkl}x_{kl}\right| \leq \left|\sum_{l=P_1}^{\infty} \sum_{k=k_l}^{\infty} a_{mnkl}x_{kl}\right| + \sum_{l=P_1}^{\infty} \sum_{k=1}^{k_l-1} |a_{mnkl}x_{kl}| \\ + \left|\sum_{l=1}^{P_1-1} \sum_k a_{mnkl}x_{kl}\right| =: A_{mn} + B_{mn} + C_{mn}. \end{aligned} \tag{4}$$

Keeping in mind that $a_{kl} = 0$ $(k, l \in \mathbb{N})$, by (i) and Proposition 2.1 (iv), we can find $N_2 > N_1$ and a sequence (m'_n) such that $m'_n > m_n$ and

$$n > N_2,\ m > m'_n \quad \Rightarrow \quad \sum_k |a_{mnkl}| \leq \frac{\varepsilon}{P_1 \max_k |x_{kl}|} \quad (l = 1, \ldots, P_1).$$

Hence for $n > N_2$ and $m \geq m'_n$ we get $A_{mn} \leq M\varepsilon$, $C_{mn} \leq \varepsilon$ and

$$B_{mn} \leq \sum_{l=P_1}^{\infty} \max_{k=1,\ldots,k_l} |x_{kl}| \sum_{k=1}^{k_l} |a_{mnkl}| < \sum_{l=P_1}^{\infty} \frac{\varepsilon}{2^l} \leq \varepsilon.$$

So $\mathcal{C}_e\text{-}\lim_{m,n}(Ax)_{m,n} = 0$.

Now, let x be an arbitrary element in $\mathcal{C}_{be}$. We put $\tilde{x} := \mathcal{C}_{be}\text{-}\lim_{m,n} x_{mn}$. Then the element $y := (x_{kl} - \tilde{x}) \in \mathcal{C}_{be}$ meets $\mathcal{C}_{be}\text{-}\lim_{k,l} y_{kl} = 0$. Hence

$$\begin{aligned} \mathcal{C}_e\text{-}\lim_{m,n}(Ax)_{m,n} &= \mathcal{C}_e\text{-}\lim_{m,n}(Ay)_{m,n} + \mathcal{C}_e\text{-}\lim_{m,n} \tilde{x}(A\mathbf{e})_{m,n} \\ &= \sum_{l=1}^{L} \sum_k a_{kl} x_{kl} + \tilde{x}\Big(v - \sum_{l=1}^{L} \sum_k a_{kl}\Big). \end{aligned}$$

□

Note that the conditions (i)–(v) in Theorem 3.1 are independent. In five following examples the considered matrix meets all of the hypotheses of Theorem 3.1 except one with a number corresponding to the number of the example. None of these matrices maps $\mathcal{C}_{be}$ into $\mathcal{C}_e$.

Example 3.3. **1)** The matrix $A = (a_{mnkl})$ with $a_{mnn1} := 1$ and $a_{mnkl} := 0$ for $l \neq 1$ or $k \neq n$ $(m, n, k \in \mathbb{N})$ does not sum the double sequence $\sum_k (-1)^k \mathbf{e^{k1}} \in \mathcal{C}_{be}$.

2) The matrix $A = (a_{mnkl})$ with $a_{mnmn} := (-1)^n$ and $a_{mnkl} := 0$ for $(k, l) \neq (m, n)$ $(m, n \in \mathbb{N})$ does not sum the double sequence $\mathbf{e} \in \mathcal{C}_{be}$.

3) The matrix $A = (a_{mnkl})$ with $a_{11kk} := 1$, $a_{1,1,k+1,k} = -1$, $a_{11kl} := 0$ for $k \neq l$ and $a_{mnkl} := 0$ for $(m, n) \neq (1, 1)$ $(m, n, k, l \in \mathbb{N})$ does not sum the double sequence $\sum_k \mathbf{e^{kk}} \in \mathcal{C}_{be}$.

4) The matrix $A = (a_{mnkl})$ with $a_{mnnn} := 1$ and $a_{mnkl} := 0$ for $(k, l) \neq (n, n)$ $(m, n \in \mathbb{N})$ does not sum the double sequence $\sum_k k\mathbf{e^{kk}} \in \mathcal{C}_{be}$.

5) The matrix $A = (a_{mnkl})$ with $a_{mnmn} := n^2$, $a_{m,n,m+1,n} := -n^2$ and $a_{mnkl} := 0$ for $(k, l) \notin \{(m, n), (m + 1, n)\}$ $(m, n \in \mathbb{N})$ does not sum the double sequence $x \in \mathcal{C}_{be}$ with $x_{kl} := (-1)^k / l$ $(k, l \in \mathbb{N})$.

Theorem 3.4. *A 4-dimensional matrix $A = (a_{mnkl})$ maps $\mathcal{C}_e$ into $\mathcal{C}_e$ if and only if it satisfies hypotheses* (ii), (iv) and (v) *of Theorem 3.1 and*

(i′) *for every $l_0 \in \mathbb{N}$ the matrix $(a_{mnkl_0})_{m,n,k}$ maps ω into $\mathcal{C}_e$,*
(iii′) *for every $m, n \in \mathbb{N}$: $a^{(m,n)} := (a_{mnkl})_{k,l} \in \Phi$,*

Under these circumstances, $a = (a_{kl}) \in \Phi$ and

$$\mathcal{C}_e\text{-}\lim_{m,n}(Ax)_{m,n} = \sum_{k,l} a_{kl}x_{kl} + \Big(v - \sum_{k,l} a_{kl}\Big)\mathcal{C}_e\text{-}\lim_{m,n} x_{m,n} \quad (x \in \mathcal{C}_e).$$

Proof is analogous to that of Theorem 3.1. □

Theorem 3.5. *A 4-dimensional matrix $A = (a_{mnkl})$ maps $\mathcal{C}_{be}$ into $\mathcal{C}_{be}$ if and only if A satisfies* (iii) *of Theorem 3.1,*

(i″) *for every $l_0 \in \mathbb{N}$ the matrix $(a_{mnkl_0})_{m,n,k}$ maps $\mathfrak{m}$ into $\mathcal{C}_{be}$,*
(ii′) *the limit $v := \mathcal{C}_{be}\text{-}\lim_{m,n} \sum_{k,l} a_{mnkl}$ exists,*
(iv′) *for every index sequence (k_l) and sequence (ε_l) of positive numbers there exists $P \in \mathbb{N}$ such that*

$$\forall n \in \mathbb{N}\ \exists m_n :\ m > m_n\ \Rightarrow\ \sum_{k=1}^{k_l} |a_{mnkl}| < \varepsilon_l \quad (l > P),$$

(v′) $\sup_n \overline{\lim}_m \sum_{k,l} |a_{mnkl}| < \infty$.

Under these circumstances, $a = (a_{kl}) \in \varphi(\ell)$ and

$$\mathcal{C}_{be}\text{-}\lim_{m,n}(Ax)_{m,n} = \sum_l \sum_k a_{kl}x_{kl} + \Big(v - \sum_l \sum_k a_{kl}\Big)\mathcal{C}_{be}\text{-}\lim_{m,n} x_{m,n} \quad (x \in \mathcal{C}_{be}).$$

Proof. Necessity of **(iii)** follows from Theorem 3.1. **(i″)** and **(ii′)** can be obtained analogously with Theorem 3.1. **(iv′)** follows similarly to Theorem 3.1 (iv) with the help of Proposition 2.5.

(v′) From (iv′) we get (cf. Remark 3.2) that for every $n \in \mathbb{N}$ there exists $L(n) \in \mathbb{N}$ such that $a^{(m,n)} = \sum_{l=1}^{L(n)} a^{(m,n)}\mathbf{e}^l$ $(m \in \mathbb{N})$. Hence by (i″) and Proposition 2.2 (ii′), we get

$$\sup_m \sum_{k,l} |a_{mnkl}| \le \sum_{l=1}^{L(n)} \sup_m \sum_k |a_{mnkl}| < \infty.$$

Now (v′) follows from Theorem 3.1 (v).

Sufficiency. By Theorem 3.1, the limit $\mathcal{C}_e\text{-}\lim_{m,n}(Ax)_{m,n}$ exists for every $x \in \mathcal{C}_{be}$. So we should just prove that $\sup_m \big|(Ax)_{mn}\big| < \infty$ for a fixed $x \in \mathcal{C}_{be}$ and $n \in \mathbb{N}$.

Let $P_1 \in \mathbb{N}$ and a sequence (k_l) be such that $M := \sup\{|x_{kl}| \,:\, l \geq P_1,\ k \geq k_l\} < \infty$. Using the notation of (4), by (v′), we get $\sup_m A_{mn} \leq \sup_m \sum_{k,l} |a_{mnkl}| \cdot M < \infty$. By (iv′), for every $n \in \mathbb{N}$ there exists $L(n) \in \mathbb{N}$ such that $a^{(m,n)} = \sum_{l=1}^{L(n)} a^{(m,n)} \mathbf{e}^{\mathbf{l}}$ $(m \in \mathbb{N})$ (cf. Remark 3.2). Therefore

$$\sum_{l} \sum_{k=1}^{k_l - 1} a^{(m,n)} \mathbf{e}^{\mathbf{kl}} = \sum_{l=1}^{L(n)} \sum_{k=1}^{k_l - 1} a^{(m,n)} \mathbf{e}^{\mathbf{kl}} \quad (m \in \mathbb{N}).$$

Hence $\sup_m B_{mn} < \infty$. Finally, (i″) implies

$$\sup_m C_{mn} \leq \sum_{l=1}^{P_1} \sup_m \sum_k |a_{mnkl}| \sup_{\substack{l=1,\dots,P_1 \\ k \in \mathbb{N}}} |x_{kl}| < \infty.$$

Hence $\sup_m \left|(Ax)_{mn}\right| < \infty$. □

Theorem 3.6. *A 4-dimensional matrix $A = (a_{mnkl})$ maps $\mathcal{C}_e$ into $\mathcal{C}_{be}$ if and only if it satisfies hypotheses* (iii′) *of Theorem 3.4,* (ii′), (iv′), *and* (v′) *of Theorem 3.5 and*

(i‴) *for every $l_0 \in \mathbb{N}$ the matrix $(a_{mnkl_0})_{m,n,k}$ maps ω into $\mathcal{C}_{be}$.*

Under these circumstances, $a = (a_{kl}) \in \Phi$ and

$$\mathcal{C}_{be}\text{-}\lim_{m,n} (Ax)_{m,n} = \sum_{k,l} a_{kl} x_{kl} + \Big(v - \sum_{k,l} a_{kl}\Big) \mathcal{C}_e\text{-}\lim_{m,n} x_{m,n} \quad (x \in \mathcal{C}_e).$$

Proof. Necessity. of (i‴) is evident. Other statements follow by Theorems 3.4 and 3.5.

Sufficiency. By Theorem 3.4, the limit $\mathcal{C}_e\text{-}\lim_{m,n} (Ax)_{m,n}$ exists for every $x \in \mathcal{C}_e$. Fix $x \in \mathcal{C}_e$ and $n \in \mathbb{N}$. Using the notation of (4), in the same way as in Theorem 3.5 we get that $\sup_m A_{mn} < \infty$ and $\sup_m B_{mn} < \infty$. By (i‴) and Proposition 2.5 (ii′), for every $n \in \mathbb{N}$ we can find $K(n) \in \mathbb{N}$ such that $\sum_{l=1}^{P_1 - 1} \sum_k a_{mnkl} \mathbf{e}^{\mathbf{kl}} = \sum_{l=1}^{P_1 - 1} \sum_{k=1}^{K(n)} a_{mnkl} \mathbf{e}^{\mathbf{kl}}$ $(m \in \mathbb{N})$. Hence $\sup_m C_{mn} < \infty$. So $\sup_m \left|(Ax)_{mn}\right| < \infty$. □

In addition to the double sequences considered above, we now also treat the space $\mathcal{C}_p$ of double sequences convergent in Pringsheim's sense, and its subspace

$$\mathcal{C}_{bp} := \Big\{ x \in \mathcal{C}_p \big|\ \sup_{k,l} |x_{kl}| < \infty \Big\}.$$

At the end of our note we will verify that a 4-dimensional matrix A, meeting

$$\mathcal{V} \subset (\mathcal{V})_A^{(\mathcal{V})} \qquad (5)$$

for some space $\mathcal{V} \in \{\mathcal{C}_e, \mathcal{C}_{be}, \mathcal{C}_p, \mathcal{C}_{bp}\}$, should not meet (5) for any other space from the same set.

Example 3.7. **1)** Put $(Ax)_{mn} := 0$ for $m \leq n$ and $(Ax)_{mn} := x_{m-n,n}$ for $m > n$ $(n \in \mathbb{N})$. Then A satisfies (5) for $\mathcal{V} \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$, but not for $\mathcal{V} \in \{\mathcal{C}_p, \mathcal{C}_{bp}\}$.

2) Let $A : \Omega \to \Omega$ be a matrix map, defined by $(Ax)_{m1} := mx_{m1}$, $(Ax)_{mn} := x_{mn}$ $(m, n \in \mathbb{N},\ n > 1)$. Then A satisfies (5) for $\mathcal{V} \in \{\mathcal{C}_e, \mathcal{C}_p\}$, but not for $\mathcal{V} \in \{\mathcal{C}_{be}, \mathcal{C}_{bp}\}$.

3) Put $(Ax)_{mn} := x_{nn}$ $(x \in \Omega;\ m, n \in \mathbb{N})$. Then A satisfies (5) for $\mathcal{V} \in \{\mathcal{C}_p, \mathcal{C}_{bp}\}$, but not for $\mathcal{V} \in \{\mathcal{C}_e, \mathcal{C}_{be}\}$.

4) Consider the Cesáro 4-dimensional matrix map $A : \Omega \to \Omega$, defined by $(Ax)_{mn} := \left(\sum_{k=1}^{m} \sum_{l=1}^{n} x_{kl}\right)/mn$ $(x \in \Omega;\ m, n \in \mathbb{N})$. The matrix A satisfies (5) for $\mathcal{V} \in \{\mathcal{C}_{be}, \mathcal{C}_{bp}\}$, but not for $\mathcal{V} \in \{\mathcal{C}_e, \mathcal{C}_p\}$.

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