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Empty-cored sequences in Banach spaces

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ABSTRACT. The purpose of the present paper is to describe sequences in Banach spaces in terms of Knopp core. For matrix summability methods, the conditions for core-shrinkingness are also given. In investigations of core inclusions and several other summability problems in Banach spaces it is vital to be aware of elements with empty cores. Empty-cored sequences that remain empty-cored under transformation by an arbitrary regular matrix method are described.

1. Preliminaries

Let X be a Banach space and let X^* be its topological dual. Let $\omega(X)$ denote the set of all sequences $x = (\xi_n)$ with $\xi_n \in X$, $n = 1, 2, \ldots$.

Let $E_n(x) = \{\xi_n, \xi_{n+1}, \ldots\}$, where $x = (\xi_n) \in \omega(X)$, and let $R_n(x)$ be the closure of the convex hull of $E_n(x)$ in E, i.e.

$$R_n(x) = \operatorname{cl}\operatorname{conv} E_n(x).$$

The intersection

$$K(x) = \bigcap_{n=1}^{\infty} R_n(x)$$

is called *Knopp core* of the sequence x (see [2]; cf. also [1], Chpt. VI). If $K(x) = \emptyset$, then we say that x is *empty-cored.*

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In the sequel we consider in particular the following subsets of $\omega(X)$:

$$m(X) = \{x = (\xi_n) \mid \sup\{\|\xi_n\| : n \in \mathbb{N}\} < \infty\},$$
$$m^\sharp(X) = \{x = (\xi_n) \mid \sup\{\|\xi\| : \xi \in K(x)\} < \infty,\ K(x) \neq \emptyset\},$$
$$\tilde{c}_w(X) = \{x = (\xi_n) \mid x \text{ is a weakly Cauchy sequence in } X\},$$
$$c_w(X) = \{x = (\xi_n) \mid x \text{ is a weakly convergent sequence in } X\},$$
$$c(X) = \{x = (\xi_n) \mid x \text{ is a convergent sequence in } X\}.$$
$$c^\sharp(X) = \{x = (\xi_n) \mid K(x) \text{ is a singleton}\}.$$

The sets $m(X)$, $\tilde{c}_w(X)$, $c_w(X)$ and $c(X)$ are linear subspaces of $\omega(X)$. In the general case the sets $m^\sharp(x)$ and $c^\sharp(X)$ are not linear subspaces of $\omega(X)$.

It is shown in [5] that for any Banach space X

$$c_w(X) \subset c^\sharp(X) \tag{1}$$

and if X is reflexive, then

$$m(X) \subset m^\sharp(X). \tag{2}$$

Proposition 1. *Let X be a complex Banach space and let $x = (\xi_n) \subset X$, then*

$$K(x) = \{\xi \in X \mid \operatorname{Re} f(\xi) \leq \limsup_n \operatorname{Re} f(\xi_n) \quad \forall f \in X^*\}$$
$$= \{\xi \in X \mid \liminf_n \operatorname{Re} f(\xi_n) \leq \operatorname{Re} f(\xi) \leq \limsup_n \operatorname{Re} f(\xi_n) \quad \forall f \in X^*\}.$$

Proof. This proposition was proved in [5] for the real Banach spaces.

Let us consider now X as a real Banach space and let X_R^* be its topological dual. Knopp core of the sequence x does not change if we change the field of scalars from $\mathbb{C}$ to $\mathbb{R}$. Let

$$f_R(x) = \operatorname{Re} f(x) \qquad \forall x \in X.$$

Since, for every $g \in X_R^*$ there is an $f \in X^*$ such that $g = f_R$, the complex case follows from the real one. □

Let X and Y be Banach spaces. Recall, that two sequences $(\xi_n) \subset X$ and $(\eta_n) \subset Y$ are *equivalent* if there exists an isomorphism T from $\operatorname{cl\,span}\{\xi_n : n \in \mathbb{N}\}$ onto $\operatorname{cl\,span}\{\eta_n : n \in \mathbb{N}\}$ such that $T\xi_n = \eta_n$ for all $n \in \mathbb{N}$.

By Proposition 1 it is easy to see that every sequence which is equivalent to an empty-cored sequence has empty Knopp core itself.

Proposition 2. *If $x = (\xi_n)$ is a weakly Cauchy sequence in X with no weak limit or a sequence that is equivalent to the unit vector basis (e_n) of ℓ_1, then it is empty-cored.*

Proof. Suppose that $\xi \in K(x)$. If x is weakly Cauchy, then by Proposition 1

$$f(\xi) = \lim_n f(\xi_n) \qquad \forall f \in X^*.$$

As x has no weak limit, this is a contradiction. Suppose now that $x = (e_n)$. Let $f_k = (0, ..., 0, 1, 0, ...)$, where 1 stands on the k-th position. As $f_k \in \ell_1^*$ for all $k \in \mathbb{N}$, using Proposition 1 one can see that $\xi = 0$. But for the element $f_e = (1, 1, ...) \in \ell_1^*$, we have that $f_e(\xi) = 1$, which is a contradiction, and therefore $K((e_n)) = \emptyset$. Since equivalent sequences have empty cores simultaneously, the proposition is proved. □

2. Empty-cored subsequences

The property that $x = (\xi_n) \subset \mathbb{R}$ is convergent if and only if its Knopp core is a singleton is important for the investigation of Knopp core in $\omega(\mathbb{R})$. Using Rosenthal's ℓ_1-theorem we shall describe in terms of Knopp cores the sequences in a Banach space X. This description provides us with an account of the set $c^\sharp(X) \setminus c_w(X)$.

Theorem 3. *Let $x = (\xi_n)$ be a sequence in a Banach space X. Then x has a subsequence (ξ_{n_k}) satisfying one of the two mutually exclusive conditions:*

(i) (ξ_{n_k}) *is empty-cored,*

(ii) (ξ_{n_k}) *is weakly convergent.*

Proof. If $x = (\xi_n)$ is not bounded, then there exists $f \in X^*$ and a subsequence (ξ_{n_k}) such that

$$\lim_k \operatorname{Re} f(\xi_{n_k}) = -\infty,$$

by Proposition 1 we get that $K((\xi_{n_k})) = \emptyset$. If the sequence $x = (\xi_n)$ is bounded, then by Rosenthal's ℓ_1-theorem (see e.g. [4], p. 43) one can extract a subsequence that is weakly-Cauchy or a subsequence that is equivalent to the unit vector basis (e_k) of ℓ_1. Thus by Proposition 2, if this subsequence is not weakly convergent, then its Knopp core is empty. □

Corollary 4. *Let $x \in c^\sharp(X)$. If x has no empty-cored subsequences, then it is weakly convergent.*

Proof. Let $K(x) = \{\xi\}$. By Theorem 3 every subsequence of x has a subsequence that is weakly convergent. As Knopp core contains all weak cluster points of x, the sequence x is weakly convergent to ξ. □

Theorem 5. *Let $x = (\xi_n)$ and $y = (\eta_n)$ be sequences in a Banach space X.*

(i) *If*

$$\limsup_n \operatorname{Re} f(\eta_n) \le \limsup_n \operatorname{Re} f(\xi_n) \qquad \forall f \in X^*, \tag{3}$$

then

$$K(y) \subset K(x). \tag{4}$$

(ii) *If the sequence y has no empty-cored subsequences, then the conditions (3) and (4) are equivalent.*

Proof. (i) is a direct consequence of Proposition 1.

(ii) Let us assume that (4) is true but there exists $f_0 \in X^*$ such that

$$\alpha = \limsup_n \operatorname{Re} f_0(\eta_n) > \limsup_n \operatorname{Re} f_0(\xi_n). \tag{5}$$

Consequently there exists a subsequence (η_{n_k}) such that

$$\lim_k \operatorname{Re} f_0(\eta_{n_k}) = \alpha.$$

This subsequence possesses no subsequence with empty Knopp core. By Theorem 3 we can extract a weakly convergent subsequence $(\eta'_n) \subset (\eta_{n_k})$. This means that there exists $\eta \in X$ such that

$$\lim_n f(\eta'_n) = f(\eta) \qquad \forall f \in X^*.$$

Knopp core $K(y)$ contains all weak cluster points of y, therefore $\eta \in K(y)$, and by (4) and Proposition 1

$$\operatorname{Re} f(\eta) \le \limsup_n \operatorname{Re} f(\xi_n) \qquad \forall f \in X^*.$$

As $\operatorname{Re} f_0(\eta) = \alpha$, this is in contradiction with (5). □

3. Empty-cored sequences on the unit sphere

The property that every bounded sequence in $\omega(\mathbb{R})$ has the nonempty Knopp core is yet another important property in the investigation of Knopp cores. In this section we shall show that this property is true in $\omega(X)$ if and only if X is reflexive. On the whole, this fact is well known (see e.g. [3], p. 58). We shall provide here a direct proof, based on the concept of Knopp core and the James condition for the reflexivity.

Theorem 6. *A Banach space X is reflexive if and only if every sequence on the unit sphere S_X has nonempty Knopp core.*

Proof. Necessity. Assume that X is reflexive. If $(\xi_n) \subset S_X$ then $(\xi_n) \in m(X)$ and by (2) its Knopp core is nonempty.

Sufficiency. We need the following version of the James condition for reflexivity (see e.g. [4]; for a short proof see [6]): A Banach space X is reflexive if and only if there is an $\varepsilon \in (0,1)$ such that if $(\xi_n) \subset S_X$, with $\|u\| > \varepsilon$ for all $u \in \operatorname{conv}\{\xi_1, \xi_2, ...\}$, then there are $n_0 \in \mathbb{N}$, $u \in \operatorname{conv}\{\xi_1, \xi_2, ..., \xi_{n_0}\}$ and $v \in \operatorname{conv}\{\xi_{n_0+1}, \xi_{n_0+2}, ...\}$ such that $\|u - v\| \leq \varepsilon$.

Let $x = (\xi_n) \subset S_X$ be an arbitrary sequence and let $\xi \in K(x)$. This means that there exists a sequence (σ_n) of convex combinations

$$\sigma_n = \sum_{k=m_n+1}^{m_{n+1}} \lambda_k \xi_k$$

such that $\sigma_n \to \xi$. Note that

$$\sigma_n \in \operatorname{conv}\{\xi_1, \xi_2, ..., \ \xi_{m_{n+1}}\}$$

and

$$\sigma_{n+1} \in \operatorname{conv}\{\xi_{m_{n+1}+1}, \ \xi_{m_{n+1}+2}, ...\}.$$

Since (σ_n) is a Cauchy sequence, for arbitrary $\varepsilon \in (0,1)$ there exists n_ε such that

$$\|\sigma_{n_\varepsilon} - \sigma_{n_\varepsilon+1}\| < \varepsilon.$$

We may choose for the James condition $n_0 = m_{n_\varepsilon+1}$, $u = \sigma_{n_\varepsilon}$ and $v = \sigma_{n_\varepsilon+1}$, and this finishes the proof of the theorem. □

Corollary 7. *A Banach space X is reflexive if and only if*

$$m(X) \subset m^\sharp(X).$$

Corollary 8. *If X is a reflexive Banach space, then*

$$c_w(X) = c^\sharp(X) \cap m(X).$$

Proof follows from Corollary 7 by using Corollary 4 and inclusion (1).

4. Regular matrix methods of summability and empty-cored sequences

Recall that a real or complex matrix $A = (a_{nk})$ is called a *regular* method of summability if, given a sequence of scalars (ξ_n) converging to ξ, the sequence (η_n), where

$$\eta_n = \sum_k a_{nk}\xi_k, \tag{6}$$

also converges to ξ. It is well known (see e.g. [1]) that A is a regular method if and only if

$$\lim_n a_{nk} = 0 \qquad \forall k \in \mathbb{N}, \tag{7}$$

$$\lim_n \sum_k a_{nk} = 1, \tag{8}$$

$$\sup_n \sum_k |a_{nk}| < \infty \tag{9}$$

Let A be a matrix method of summability and let $x = (\xi_k)$ be the sequence in a Banach space X. Let $y = Ax$ be the sequence (η_n) in X which is given by the formula (6). If $Ax \in c(X)$ (resp. $Ax \in c_w(X)$), then we say that x is *summable* (resp. *w-summable*) by A.

The set

$$\omega_A(X) = \{x = (\xi_k) \in \omega(X) |\ \text{series (6) converge in } X \text{ for all } n \in \mathbb{N}\}$$

is called a *domain* of matrix method $A = (a_{nk})$ in $\omega(X)$. It follows from (9) that if A is a regular matrix method, then $m(X) \subset \omega_A(X)$.

Let $Z \subset \omega_A(X)$. A matrix method A is called *core-shrinking* on Z if

$$K(Ax) \subset K(x) \qquad \forall x \in Z.$$

Note that a core-shrinking method on Z preserves empty cores in Z.

Let

$$(Z; X^*) = \{(\text{Re } f(\xi_n)) \in \omega(\mathbb{R}) |\ (\xi_n) \in Z,\ \ f \in X^*\}.$$

Observe that if $Z \subset \omega_A(X)$, then $(Z; X^*) \subset \omega_A(\mathbb{R})$. Obviously,

$$(m(X); X^*) = m(\mathbb{R}) \tag{10}$$

and

$$(c(X); X^*) = (c_w(X); X^*) = (\tilde{c}_w(X); X^*) = c(\mathbb{R}). \tag{11}$$

Theorem 9. *Let X be a Banach space and let $A = (a_{nk})$ be a matrix method such that $Z \subset \omega_A(X)$. If A is a core-shrinking matrix method on $(Z; X^*)$, then it is a core-shrinking matrix method on Z.*

Proof. Let $x = (\xi_n) \in Z$ and let $f \in X^*$ be an arbitrary functional. Therefore, $(\text{Re} f(\xi_n)) \in (Z; X^*)$. Since method A is core-shrinking on $(Z; X^*)$, due to the concept of Knopp core in $\mathbb{R}$, we have

$$\limsup_n \sum_k a_{nk} \text{Re} f(\xi_k) \le \limsup_n \text{Re} f(\xi_n),$$

meaning that

$$\limsup_n \text{Re} f(\sum_k a_{nk}\xi_k) \le \limsup_n \text{Re} f(\xi_n).$$

It is clear from part (i) of Theorem 5 that A is core-shrinking on Z. □

Corollary 10. *Every regular matrix method $A = (a_{nk})$ is core-shrinking on $\tilde{c}_w(X)$.*

Proof. Since $\tilde{c}_w(X) \subset \omega_A(X)$, the proof follows from the observations that a regular method is core-shrinking on $c(\mathbb{R})$ and (11) holds. □

Corollary 11. *If $A = (a_{nk})$ is a regular matrix method and if*

$$\lim_n \sum_k |a_{nk}| = 1, \tag{12}$$

then A is core-shrinking on $m(X)$.

Proof. It is well known that a regular metod $A = (a_{nk})$ is core-shrinking on $m(\mathbb{R})$ if and only if (12) holds (see e.g. [1]). The assertion follows now immediately from Theorem 9 and the inclusion (10). □

Corollary 12. *If $A = (a_{nk})$ is a positive and regular matrix method on $\omega(\mathbb{R})$, then it is core-shrinking on $\omega(X)$.*

Proof. A matrix method that has $\omega(\mathbb{R})$ for its domain is row-finite. It is obvious that then $\omega_A(X) = \omega(X)$. As a positive and regular method is core-shrinking on $\omega(\mathbb{R})$ (see e.g. [1]), the assertion follows from Theorem 9. □

Proposition 13. *Let x be a weakly Cauchy sequence in X with no weak limit or a sequence that is equivalent to the unit vector basis (e_k) of ℓ_1.*

Then for every regular matrix method A, Knopp core $K(Ax)$ is empty. Consequently, there exists no regular matrix method A such that x is w-summable by A.

Proof. If x is weakly Cauchy with no weak limit then the statement follows from Corollary 10.

It can be easily verified that if A is regular and if x is equivalent to $\hat{x} = (e_k)$ then Ax is equivalent to $A\hat{x}$. As equivalent sequences have empty cores simultaneously, we have to prove that the statement holds for $\hat{x} = (e_k)$.

Assume that there exist regular A and $\xi \in \ell_1$ such that $\xi \in K(A\hat{x})$. Now we can construct a contradiction in the same way as in the proof of Proposition 2. Indeed, by using the fact that (7) holds, we get from Proposition 1 for $f_k \in \ell_1^*$ that

$$f_k(\xi) = \lim_n f_k(\sum_j a_{nj} e_j) = \lim_n a_{nk} = 0, \qquad \forall k \in \mathbb{N},$$

i.e. $\xi = 0$. But for $f_e \in \ell_1^*$ we get from (8) that

$$f_e(\xi) = \lim_n f_e(\sum_j a_{nj} e_j) = \lim_n \sum_j a_{nj} = 1,$$

which is a contradiction, therefore $K(A\hat{x}) = \emptyset$. □

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