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Riesz summability with speed of orthogonal series

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ABSTRACT. Sufficient conditions for summability with speed of orthogonal series are found.

1. Main result

Let $\varphi = \{\varphi_k\}$ be a system of orthogonal functions on $[a,b]$, and let $\lambda = (\lambda_k)$ be a sequence with $0 < \lambda_k \nearrow \infty$. We shall consider the series of the form

$$\sum \xi_k \varphi_k(t),$$

where $x = (\xi_k) \in \ell^2_\lambda$, i.e. $\sum \xi_k^2 \lambda_k^2 < \infty$.

We shall use the following definitions from [1].

Let $A = (a_{nk})$ be a triangular summability method and let $z = (\zeta_k) \in c$ with $\lim \zeta_k = \zeta$.

The sequence z is said to be *convergent with speed* λ or λ*-convergent*, if the limit

$$\lim_n \lambda_n(\zeta_n - \zeta)$$

exists. The set of all λ-convergent sequences is denoted by c^λ.

The sequence z is said to be *A-summable with speed* λ or A^λ*-summable*, if $y = (\eta_n) \in c^\lambda$, where

$$\eta_n = \sum_{k=0}^{n} a_{nk}\zeta_k.$$

The summability method A is said to be λ*-convergence preserving* if every element of the set c^λ is A^λ-summable.

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The series $\sum \xi_k \varphi_k(t)$ is said to be *A^λ-summable almost everywhere (a.e.) on* $[a,b]$ if it is A-summable a.e. on $[a,b]$, i.e. the limit

$$\lim_n \sum_{k=0}^{n} \alpha_{nk} \xi_k \varphi_k(t) = f_x(t) \tag{1}$$

exists a.e. on $[a,b]$, and the limit

$$\lim_n \beta_n(A,x,t) \tag{2}$$

exists a.e. on $[a,b]$, where

$$\beta_n(A,x,t) = \lambda_n \left(\sum_{k=0}^{n} \alpha_{nk} \xi_k \varphi_k(t) - f_x(t) \right)$$

and

$$\alpha_{nk} = \sum_{\nu=k}^{n} a_{n\nu}.$$

The series $\sum \xi_k \varphi_k(t)$ is said to be *maximally A^λ-summable* if the limits (1) and (2) exist and

$$\int_a^b \sup_n |\beta_n(A,x,t)| dt < \infty.$$

The starting point of this paper is the following theorem.

Theorem 1 (see [7]). *Let A be λ^2-convergence preserving and let*

$$\lim_n \alpha_{nk} = 1 \quad \textit{for all} \quad k \in \mathbb{N}.$$

The series $\sum \xi_k \varphi_k(t)$ is A^λ-summable a.e. on $[a,b]$ for all $x \in \ell^2_\lambda$ if and only if the following conditions hold:

1° *$\sum \xi_k \varphi_k(t)$ is A-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$;*

2° *For each $\varepsilon > 0$ there exist a measurable subset $T_\varepsilon \subset [a,b]$ satisfying* mes$T_\varepsilon > b - a - \varepsilon$ *and a constant $M_\varepsilon > 0$ such that, for all mesurable decompositions*

$$\mathfrak{M}_m = \{\mathfrak{M}_{mn}\colon n=0,1,\dots,m; \mathfrak{M}_{mk} \bigcap \mathfrak{M}_{mn} = \emptyset \text{ if } k \neq n; \bigcup_{n=0}^{m} \mathfrak{M}_{mn} = [a,b]\}, \tag{3}$$

one has

$$A_m(\varepsilon)=\left|\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\sum_{\nu=0}^{m}\varphi_\nu(t)\varphi_\nu(\tau)D_{np\nu}^{m}\,dt\,d\tau\right|\le M_\varepsilon,$$

where $\chi_{mn}=\chi_{\mathfrak{M}_{mn}}$ *and*

$$D_{np\nu}^{m}=\begin{cases}(\alpha_{m\nu}-\alpha_{n\nu})(\alpha_{m\nu}-\alpha_{p\nu})\dfrac{\lambda_n\lambda_p}{\lambda_\nu^2}, & \text{if } 0\le v\le n<p<m,\\[2mm] \alpha_{m\nu}(\alpha_{m\nu}-\alpha_{p\nu})\dfrac{\lambda_n\lambda_p}{\lambda_\nu^2}, & \text{if } n<\nu\le p<m,\\[2mm] \alpha_{m\nu}^2\dfrac{\lambda_n\lambda_p}{\lambda_\nu^2}, & \text{if } n<p<\nu\le m.\end{cases}$$

In the present paper we shall mainly consider the case, when A is the Riesz summability method P, i.e.

$$a_{nk}=\begin{cases}\dfrac{p_k}{P_n}, & k\le n,\\ 0, & k>n,\end{cases}$$

where $p_k>0$ and $P_n=\sum_{k=0}^{n}p_k\nearrow\infty$.

Note that the Riesz summability method P is λ-convergence preserving if and only if (see [2])

$$\frac{\lambda_n}{P_n}\sum_{k=0}^{n}\frac{p_k}{\lambda_k}=O(1).$$

If P is λ-convergence preserving, then clearly

$$\frac{\lambda_n}{P_n}=O(1)\frac{\lambda_k}{P_k}\quad\text{for } k\le n\ ,\quad k,n\in\mathbb{N}. \tag{4}$$

Hence, if the method P is λ^2-convergence preserving, i.e.

$$\frac{\lambda_n^2}{P_n}\sum_{k=0}^{n}\frac{p_k}{\lambda_k^2}=O(1), \tag{5}$$

then

$$\frac{\lambda_n^2}{P_n}=O(1)\frac{\lambda_k^2}{P_k}\quad\text{for } k\le n\ ,\quad k,n\in\mathbb{N}. \tag{6}$$

Since by the Cauchy inequality

$$\frac{\lambda_n}{P_n}\sum_{k=0}^{n}\frac{p_k}{\lambda_k}\le\Big(\frac{\lambda_n^2}{P_n^2}\sum_{k=0}^{n}\frac{p_k}{\lambda_k^2}\sum_{k=0}^{n}p_k\Big)^{1/2}=\Big(\frac{\lambda_n^2}{P_n}\sum_{k=0}^{n}\frac{p_k}{\lambda_k^2}\Big)^{1/2},$$

we have that *if P is λ^2-convergence preserving, then P is also λ-convergence preserving.*

The main objective of this paper is to prove the following theorem.

Theorem 2. *Let condition* (5) *hold, and let*

$$\frac{\lambda_n^2}{P_{n-1}} \searrow 0\ , \quad p_n = O(P_{n-1}), \tag{7}$$

$$\frac{1}{p_n}\Delta\frac{1}{\lambda_n^2} \searrow 0, \tag{8}$$

where

$$\Delta\frac{1}{\lambda_n^2} = \frac{1}{\lambda_n^2} - \frac{1}{\lambda_{n+1}^2}.$$

If

$$\int_a^b \sup_k L_k(P,t)dt < \infty, \tag{9}$$

where

$$L_k(P,t) = \int_a^b \left|\sum_{\nu=0}^{k}\Big(1 - \frac{P_{\nu-1}}{P_k}\Big)\varphi_\nu(t)\varphi_\nu(\tau)\right| d\tau,$$

with $P_{-1} = 0$, *are the Lebesgue functions of the method* P, *then the series* $\sum \xi_k\varphi_k(t)$ *is maximally* P^λ*-summable a.e. on* $[a,b]$ *for every* $x \in \ell_\lambda^2$.

Let us remark that, in 1969, G. Kangro proved the following result.

Theorem 3 (cf. [2])**.** *If* $(1/\lambda_k)$ *is a sequence of summability factors of type* (A, A^λ), *i.e. the series*

$$\sum \frac{1}{\lambda_k}\zeta_k$$

is A^λ*-summable for every* A*-summable series* $\sum \zeta_k$, *then the* A*-summability a.e. on* $[a,b]$ *of the series* $\sum \xi_k^0\varphi_k(t)$, *where* $x_0 \in \ell^2$, *implies the* A^λ*-summability of the series* $\sum \frac{\xi_k^0}{\lambda_k}\varphi_k(t)$ *a.e. on* $[a,b]$.

If conditions (5), (7) and (8) are fulfilled, then from Theorem 29.3 of [1], it follows that $(1/\lambda_k)$ is a sequence of summability factors of type (P, P^λ). Therefore we have that if conditions (5), (7) and (8) hold, then the P-summability a.e. of the series $\sum \xi_k\varphi_k(t)$ for every $x \in \ell^2$ implies the P^λ-summability of the series $\sum \xi_k\varphi_k(t)$ for every $x \in \ell_\lambda^2$. Note that the above argument does not imply the maximal P^λ-summability of the series $\sum \xi_k\varphi_k(t)$ for every $x \in \ell_\lambda^2$.

2. Main Lemma

The proof of Theorem 2 is based on the following lemma.

Lemma 4. *If conditions* (5), (7) *and* (8) *hold, then for each* $\varepsilon > 0$ *there exists a mesurable subset* $T_\varepsilon \subset [a,b]$ *satisfying* $\mathrm{mes} T_\varepsilon > b - a - \varepsilon$ *such that for all decompositions* (3) *one has*

$$A_m(\varepsilon) = O(1) \int_{T_\varepsilon} \sup_{k \le m} L_k(P,t)dt. \tag{10}$$

Proof. Denote

$$R_j(t,\tau) = \sum_{\nu=0}^{j} \alpha_{j\nu} \varphi_\nu(t) \varphi_\nu(\tau),$$

where

$$\alpha_{j\nu} = 1 - \frac{P_{\nu-1}}{P_j}.$$

Then

$$\varphi_\nu(t)\varphi_\nu(\tau) = \sum_{k=0}^{\nu} \eta_{\nu k} R_k(t,\tau),$$

where $(\eta_{nk}) = P^{-1}$ is the inverse matrix of P.

From [1] (see p. 193) it follows that

$$\sum_{\nu=k}^{m} \eta_{\nu k} D_{np\nu}^{m} = P_k \Delta \frac{\Delta D_{npk}^{m}}{p_k},$$

and therefore

$$\begin{aligned} A_m(\varepsilon) &= \left| \int_{T_\varepsilon} \int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \sum_{\nu=0}^{m} \sum_{k=0}^{\nu} \eta_{\nu k} R_k(t,\tau) D_{np\nu}^{m} dt d\tau \right| \\ &= \left| \int_{T_\varepsilon} \int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \sum_{k=0}^{m} R_k(t,\tau) P_k \Delta \frac{\Delta D_{np\nu}^{m}}{p_k} dt d\tau \right| \\ &= \left| \int_{T_\varepsilon} \int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t) \sum_{p=n+1}^{m-1} \chi_{mp}(\tau) \lambda_n \lambda_p \left[\sum_{k=0}^{m} R_k(t,\tau) \Big(\Delta_k^1(n,p,m) \right. \\ &\quad \left. + \Delta_k^2(p,m) + \Delta_k^3(p,m) \Big) \right] dt d\tau \Bigg|, \end{aligned}$$

where

$$\Delta_k^1(n,p,m) = \begin{cases} P_k \Delta \dfrac{\Delta \left[(\alpha_{mk} - \alpha_{nk})(\alpha_{mk} - \alpha_{pk}) \frac{1}{\lambda_k^2} \right]}{p_k} & \text{if } 0 \le k < n, \\ \dfrac{P_n \alpha_{nn} (\alpha_{mn} - \alpha_{pn})}{\lambda_n^2 p_n} & \text{if } k = n, \\ 0 & \text{if } k > n, \end{cases}$$

$$\Delta_k^2(p,m)=\begin{cases} P_k\Delta\dfrac{\Delta\left[\alpha_{mk}(\alpha_{mk}-\alpha_{pk})\frac{1}{\lambda_k^2}\right]}{p_k} & \text{if } n\le k<p,\\ \dfrac{P_p\alpha_{mp}\alpha_{pp}}{\lambda_p^2p_p} & \text{if } k=p,\\ 0 & \text{if } k<n \text{ or } k>p,\end{cases}$$

and

$$\Delta_k^3(p,m)=\begin{cases} P_k\Delta\dfrac{\Delta\left[\alpha_{mk}^2\frac{1}{\lambda_k^2}\right]}{p_k} & \text{if } p\le k<m-1,\\ P_{m-1}\left(\dfrac{\alpha_{m,m-1}^2}{\lambda_{m-1}^2p_{m-1}}-\dfrac{\alpha_{mm}^2}{\lambda_m^2p_{m-1}}-\dfrac{\alpha_{mm}^2}{\lambda_m^2p_m}\right) & \text{if } k=m-1,\\ P_m\dfrac{\alpha_{mm}^2}{\lambda_m^2p_m} & \text{if } k=m,\\ 0 & \text{if } k<p.\end{cases}$$

Observe, that

$$\left|\Delta_k^1(n,p,m)\right|\le\frac{P_k}{P_nP_p}\left|\Delta\frac{\Delta\frac{P_{k-1}^2}{\lambda_k^2}}{p_k}\right| \quad \text{for } 0<k<n,$$

by (4)

$$\Delta_n^1(n,p,m)=O(1)\frac{1}{\lambda_n^2P_p}=O(1)\frac{1}{\lambda_n\lambda_pP_n},$$

and

$$\Delta_p^2(p,m)=O(1)\frac{1}{\lambda_p^2}.$$

Denote

$$M_n=\frac{P_n}{p_n}\lambda_n^2\Delta\frac{1}{\lambda_n^2}.$$

From (7) it follows that $M_n\le 1$ for all $n\in\mathbb{N}$. Therefore

$$\Delta_{m-1}^3(p,m)=\frac{P_{m-1}}{p_{m-1}}\Delta\frac{\alpha_{m,m-1}^2}{\lambda_{m-1}^2}-\frac{p_m}{\lambda_m^2P_m^2}=O(1)M_{m-1}\frac{1}{\lambda_{m-1}^2}+O(1)\frac{1}{\lambda_m^2},$$

and

$$\Delta_m^3(p,m)=O(1)\frac{1}{\lambda_m^2}.$$

Thus

$$A_m(\varepsilon) \le \int_{T_\varepsilon}\int_{T_\varepsilon} \sum_{n=0}^{m-2} \chi_{mn}(t)\frac{\lambda_n^2}{P_nP_{n-1}} \sum_{k=0}^{n-1}\left| P_k\Delta\frac{\Delta\frac{P_{k-1}^2}{\lambda_k^2}}{p_k}\right| |R_k(t,\tau)|\, dtd\tau$$

$$+\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{p=1}^{m-1}\chi_{mp}(\tau)\lambda_p^2\left[\sum_{k=0}^{p-1}\left|\Delta_k^2(p,m)\right||R_k(t,\tau)|\right.$$

$$\left.+\sum_{k=p}^{m-2}\left|\Delta_k^3(p,m)\right||R_k(t,\tau)|\right]dtd\tau$$

$$+O(1)\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{n=0}^{m-2}\chi_{mn}(t)\sum_{p=n+1}^{m-1}\chi_{mp}(\tau)\Big[|R_n(t,\tau)|+|R_p(t,\tau)|$$

$$+|R_{m-1}(t,\tau)|+|R_m(t,\tau)|\Big]dtd\tau.$$

Now we have

$$A_m(\varepsilon) \le \int_{T_\varepsilon}\sup_{k<m} L_k(P,t)dt \sup_{n<m}\frac{\lambda_n^2}{P_nP_{n-1}}\sum_{k=0}^{n-1}\left|P_k\Delta\frac{\Delta\frac{P_{k-1}^2}{\lambda_k^2}}{p_k}\right|$$

$$+\int_{T_\varepsilon}\sup_{k<m} L_k(P,\tau)d\tau \sup_{p<m}\lambda_p^2\sum_{k=0}^{p-1}\left|\Delta_k^2(p,m)\right|$$

$$+\int_{T_\varepsilon}\sup_{k<m} L_k(P,\tau)d\tau \sup_{p}\lambda_p^2\sum_{k=p}^{m-2}\left|\Delta_k^3(p,m)\right|$$

$$+O(1)\int_{T_\varepsilon}\sup_{k<m} L_k(P,t)dt.$$

Therefore, in order to prove (10), it is sufficient to show that

$$V_{npm}^i = O(1), \quad i=1,2,3,$$

where

$$V_{npm}^1 = \frac{\lambda_n^2}{P_nP_{n-1}}\sum_{k=0}^{n-1}P_k\left|\Delta\frac{\Delta\frac{P_{k-1}^2}{\lambda_k^2}}{p_k}\right|,$$

$$V_{npm}^2=\lambda_p^2\sum_{k=0}^{p-1}\left|\Delta_k^2(p,m)\right|,\ V_{npm}^3=\lambda_p^2\sum_{k=p}^{m-2}\left|\Delta_k^3(p,m)\right|.$$

By [4] (see p. 220) we have that, for any sequence $(a_n) \subset \mathbb{R}$,

$$
\begin{aligned}
P_k \Delta \frac{\Delta \frac{a_k}{\lambda_k^2}}{p_k} = \Big(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2}\Big) P_k \frac{\Delta a_k}{p_k} + \frac{1}{\lambda_{k+2}^2} P_k \Delta \frac{\Delta a_k}{p_k} \\
+ P_k a_{k+1} \Delta \Big(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2}\Big).
\end{aligned}
\tag{11}
$$

Consider the case when $i = 1$; then, by (11), we have

$$
\begin{aligned}
V_{npm}^1 \le \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \Big| \Big(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2}\Big) \frac{\Delta P_{k-1}^2}{p_k} \\
+ \frac{1}{\lambda_{k+2}^2} \Delta \frac{\Delta P_{k-1}^2}{p_k} + P_k^2 \Delta \Big(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2}\Big) \Big|.
\end{aligned}
$$

Since by (7)

$$
\sum_{k=0}^{n-1} \Big| P_k^2 \Delta \Big(\frac{1}{p_k} \Delta \frac{1}{\lambda_k^2}\Big) \Big| = P_0^2 \frac{1}{p_0} \Delta \frac{1}{\lambda_0^2} - \sum_{k=0}^{n-1} \Delta P_k^2 \frac{1}{p_{k+1}} \Delta \frac{1}{\lambda_{k+1}^2} - \frac{P_{n-1}^2}{p_n} \Delta \frac{1}{\lambda_n^2},
$$

we have

$$
\begin{aligned}
V_{npm}^1 \le \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \Big(\Big(\Delta \frac{1}{\lambda_k^2} + \Delta \frac{1}{\lambda_{k+1}^2}\Big)(P_{k-1} + P_k) + \frac{1}{\lambda_{k+2}^2}(p_k + p_{k+1}) \Big) \\
+ \frac{\lambda_n^2}{P_n} O(1) + \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \Big| \Delta P_k^2 \frac{1}{p_{k+1}} \Delta \frac{1}{\lambda_{k+1}^2} \Big| + \frac{\lambda_n^2}{P_n^2} \frac{P_{n-1}^2}{p_n} \Delta \frac{1}{\lambda_n^2}.
\end{aligned}
$$

Now, by (5) and (6), we have

$$
\begin{aligned}
V_{n,p,m}^1 &\le 2 \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \Big(\frac{p_k}{\lambda_k^2} M_k + \frac{p_{k+1}}{\lambda_{k+1}^2} M_{k+1} + p_{k+1} \Delta \frac{1}{\lambda_{k+1}^2} \Big) \\
&\quad + \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \Big(\frac{p_k}{\lambda_k^2} + \frac{p_{k+1}}{\lambda_{k+1}^2} \Big) + O(1) \\
&\quad + 2 \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \frac{p_{k+1}}{\lambda_{k+1}^2} M_{k+1} + \frac{\lambda_n^2}{P_n^2} \frac{P_n}{\lambda_n^2} M_n + O(1) \\
&= O(1) \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n} \frac{p_k}{\lambda_k^2} + O(1) \frac{\lambda_n^2}{P_n} \sum_{k=0}^{n-1} \frac{p_{k+1}}{\lambda_{k+1}^2} + O(1) \\
&= O(1).
\end{aligned}
$$

Analogously we have

$$V^2_{n,p,m} \le \frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left| P_k \Delta \frac{\Delta \frac{\alpha_{mk}P_{k-1}}{\lambda_k^2}}{p_k}\right|$$

$$\le \frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left|\Big(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\Big)P_k\frac{\Delta(\alpha_{mk}P_{k-1})}{p_k}+\frac{1}{\lambda_{k+2}^2}P_k\Delta\frac{\Delta(\alpha_{mk}P_{k-1})}{p_k}\right|$$

$$+\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left|\alpha_{mk}P_k^2\Delta\Big(\frac{1}{p_k}\Delta\frac{1}{\lambda_k^2}\Big)\right|$$

$$= O(1)\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\left[\Big(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\Big)P_k+\frac{1}{\lambda_{k+2}^2}P_k\left|\Delta\Big(\frac{P_k}{P_m}-(1-\frac{P_{k-1}}{P_m})\Big)\right|\right]$$

$$+\frac{\lambda_p^2}{P_p}O(1)+\frac{\lambda_p^2}{P_p}\sum_{k=0}^{p-1}\Delta(\alpha_{mk}P_k^2)\frac{1}{p_{k+1}}\Delta\frac{1}{\lambda_{k+1}^2}+\alpha_{m,p-1}\frac{\lambda_p^2}{P_p}\frac{P_p}{\lambda_p^2}M_p = O(1).$$

Finally, for $i=3$, we have

$$V^3_{n,p,m}$$

$$\le \lambda_p^2\sum_{k=p}^{m-2}\left|\Big(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\Big)P_k\frac{\Delta\alpha_{mk}^2}{p_k}+\frac{1}{\lambda_{k+2}^2}P_k\Delta\frac{\Delta\alpha_{mk}^2}{p_k}+P_k\alpha^2_{mk+1}\Delta\Big(\frac{1}{p_k}\Delta\frac{1}{\lambda_k^2}\Big)\right|$$

$$= O(1)\lambda_p^2\sum_{k=p}^{m-2}\Big(\Delta\frac{1}{\lambda_k^2}+\Delta\frac{1}{\lambda_{k+1}^2}\Big)+\frac{\lambda_p^2}{P_m}\sum_{k=p}^{m-2}\left|\frac{P_k}{\lambda_{k+2}^2}\Delta(\alpha_{mk}+\alpha_{m,k+1})\right|$$

$$+\alpha_{mp+1}\lambda_p^2\frac{P_p}{p_p}\Delta\frac{1}{\lambda_p^2}+O(1)\lambda_p^2\sum_{k=p}^{m-2}\Delta\frac{1}{\lambda_{k+1}^2}=O(1).$$

The proof is complete.

3. Proof of Theorem 2

In the proof of Theorem 2, we shall make use of the following

Lemma 5 (see [5], pp. 142-144). *Let (f_n) be a sequence of integrable functions on $[a,b]$. Then for each mesurable subset $T\subset[a,b]$ and for each $m\in\mathbb{N}$ one has*

$$\int_T \sup_{n\le m}|f_n(t)|dt \le 2\sup_{\mathfrak{M}_m}\left|\int_T\sum_{n=0}^{m}\chi_{mn}(t)f_n(t)dt\right|,$$

where $\mathfrak{M}_m$ ranges over all decompositions defined by (3).

Proof of Theorem 2. By [6] (see p. 201) the condition

$$L_n(P,t) = O_t(1) \text{ a.e. on } [a,b]$$

implies that the series $\sum \xi_k \varphi_k(t)$ is P-summable a.e. on $[a,b]$ for every $x \in \ell^2$.

From Theorem 1 and Lemma 4 it follows that the series $\sum \xi_k \varphi_k(t)$ is P^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$.

To show the maximal P^λ-summability we prove that

$$\int_{T_\varepsilon} \sup_{n\le m} |\beta_n(A,x,t)|dt = O(\|x\|_{l^2_\lambda}) + \sup_{\mathfrak{M}_m}\{A_m(\varepsilon)\}^{1/2}, \tag{12}$$

where $T_\varepsilon \subset [a,b]$ is a measurable subset with mes$T_\varepsilon > b-a-\varepsilon$ and $\mathfrak{M}_m$ ranges over all decompositions defined by (3).

If condition (12) holds, then from (9) and (10) it follows that the series $\sum \xi_k \varphi_k(t)$ is maximally P^λ-summable a.e. on $[a,b]$ for every $x \in \ell^2_\lambda$.

We now prove (12). By Lemma 5

$$\int_{T_\varepsilon} \sup_{n\le m} |\beta_n(A,x,t)|dt = O(1)\sup_{\mathfrak{M}_m}\left|\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)\beta_n(A,x,t)dt\right|.$$

Denote

$$\bar{\alpha}_{pk} = \alpha_{pk} - \alpha_{p-1,k},$$

then

$$\beta_n(A,x,t) = \lambda_n \sum_{p=n+1}^{\infty}\sum_{k=0}^{p} \bar{\alpha}_{pk}\xi_k\varphi_k(t) = B_{mn}(x,t) + C_{mn}(x,t),$$

where

$$B_{mn}(x,t) = \lambda_n \sum_{p=n+1}^{m}\sum_{k=0}^{p} \bar{\alpha}_{pk}\xi_k\varphi_k(t),$$

$$C_{mn}(x,t) = \lambda_n \sum_{p=m+1}^{\infty}\sum_{k=0}^{p} \bar{\alpha}_{pk}\xi_k\varphi_k(t).$$

Therefore

$$\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)\beta_n(A,x,t)dt$$

$$= O(1)\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)B_{mn}(x,t)dt + O(1)\int_{T_\varepsilon} \sum_{n=0}^{m} \chi_{mn}(t)C_{mn}(x,t)dt.$$

By orthogonality of φ we have

$$\left|\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)C_{mn}(x,t)dt\right|$$
$$\leq \int_a^b \sum_{n=0}^{m}\chi_{mn}(t)\lambda_n\left|\sum_{k=0}^{m}\alpha_{mk}\xi_k\varphi_k(t)-f_x(t)\right|dt$$
$$\leq \sqrt{b-a}\Big[\sup_{k\leq m}\frac{\lambda_m|\alpha_{mk}-1|}{\lambda_k}\Big(\sum_{k=0}^{m}\xi_k^2\lambda_k^2\Big)^{1/2}+\Big(\sum_{k=m+1}^{\infty}\xi_k^2\lambda_k^2\Big)^{1/2}\Big].$$

If A is λ-convergence preserving, then by [3] (see Lemma 3)

$$\lambda_m|\alpha_{mk}-1| = O(\lambda_k)\quad (k\leq m) \tag{13}$$

and therefore

$$\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)C_{mn}(x,t)dt = O(\|x\|_{l^2_\lambda}).$$

Denoting

$$A_p^m(t) = \sum_{n=0}^{p-1}\chi_{mn}(t)\lambda_n,$$

we have

$$\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)B_{mn}(x,t)dt$$
$$=\sum_{k=0}^{m}\xi_k\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt+\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)\lambda_n(\alpha_{m0}-\alpha_{n0})\xi_0\varphi_0(t)dt.$$

Now by (13)

$$\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)B_{mn}(x,t)dt = \sum_{k=0}^{m}\xi_k\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt + O(\|x\|_{\ell^2_\lambda}).$$

Using the principle of uniform boundedness we get

$$\sum_{k=0}^{m}\xi_\nu\int_{T_\varepsilon}\sum_{p=k}^{m}\bar{\alpha}_{pk}\varphi_k(t)A_p^m(t)dt$$
$$=O(1)\Big(\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{k=0}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2}\sum_{p=k}^{m}\bar{\alpha}_{pk}A_p^m(t)\sum_{\nu=k}^{m}\bar{\alpha}_{\nu k}A_\nu^m(\tau)dtd\tau\Big)^{1/2}\|x\|_{\ell^2_\lambda}.$$

Finally

$$\int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{k=0}^{m}\frac{\varphi_k(t)\varphi_k(\tau)}{\lambda_k^2}\sum_{p=k}^{m}\bar{\alpha}_{pk}A_p^m(t)\sum_{\nu=k}^{m}\bar{\alpha}_{\nu k}A_\nu^m(\tau)dtd\tau = A_m(\varepsilon)+E_m,$$

where

$$E_m = \int_{T_\varepsilon}\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)\chi_{mn}(\tau)\sum_{k=0}^{m}\varphi_k(t)\varphi_k(\tau)\Big[\frac{\lambda_n}{\lambda_k}(\alpha_{mk}-\alpha_{nk})\Big]^2 dtd\tau.$$

By (13) and Bessel's inequality we get

$$\begin{aligned}
E_m &= \sum_{n=0}^{m-1}\sum_{k=0}^{m}\Big[\frac{\lambda_n}{\lambda_k}(\alpha_{mk}-\alpha_{nk})\Big]^2\Big(\int_{T_\varepsilon}\varphi_k(t)\chi_{mn}(t)dt\Big)^2 = \\
&O(1)\sum_{n=0}^{m}\sum_{\nu=0}^{\infty}\Big(\int_{T_\varepsilon}\varphi_\nu(t)\chi_{mn}(t)dt\Big)^2 = \\
&O(1)\sum_{n=0}^{m}\int_{T_\varepsilon}\chi_{mn}^2(t)dt = \\
&O(1)\int_{T_\varepsilon}\sum_{n=0}^{m}\chi_{mn}(t)dt = O(1).
\end{aligned}$$

Therefore condition (12) holds. □

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