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A pseudo-metric structure on interpolation functors

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ABSTRACT. We consider topological properties of sets of interpolation functors. In particular we construct a distance function for describing nearness of interpolation functors. The idea of this paper is inspired by the concept of Banach-Mazur distance between Banach spaces.

1. Introduction

It is easy to see that for a given Banach couple the set of all intermediate norms is a compact set (it is closed for the topology of point-wise convergence in a compact set of functions) and that the set of all interpolation norms is a closed subset. If $\mathcal{C}$ is a category of Banach couples, then the set of all interpolation functors can be represented as a projective limit of compact spaces and may hence in itself be considered as a compact (Hausdorff) space. In this paper we shall consider a related problem, namely the problem of defining a metric structure on the set of interpolation functors. We do this by considering *bounded couples*, i. e. couples in which the spaces X_0 and X_1 (and therefore also all interpolation spaces) are isomorphic. Since e. g. the lower and the upper complex methods cannot be separated by bounded couples, this means that the metrics we obtain are only pseudo-metrics on the set of all interpolation functors. Since there are also other sets of functors in functional analysis where a metric structure could be useful, we hope and believe that our method will be of interest, perhaps not only to specialists in interpolation theory. In the main part of the paper we shall consider what

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we call a *distance function*, while we shall see at the end how this can be used to define a *(pseudo-)metric*.

In the sequel we mostly adopt notation and terminology already used either by [B-K] or in some cases by [B-L].

2. Distance function

Let $\Re$ denote the set of real numbers. We shall write $T : A \to B$ to denote a bounded linear map T from a Banach space A into a Banach space B. If, in addition, T is an *injection* satisfying $\|T\|_{\mathcal{L}(A,B)} \leq 1$, then we write $T : A \hookrightarrow B$.

To introduce our distance function we need the following definition.

Definition 2.1. For a fixed $s \geq 1$, a Banach couple $\vec{A}$ is said to be *s-bounded* if the canonical embedding $j : \Delta(\vec{A}) \to \Sigma(\vec{A})$ is invertible and $\|j^{-}\|_{\mathcal{L}(\Sigma(\vec{A}),\Delta(\vec{A}))} \leq s$.

We denote the category consisting of all Banach spaces with morphisms all bounded linear maps between Banach spaces and the category consisting of all Banach couples with morphisms all bounded linear *couple maps* (i. e. all bounded linear maps between Banach couples) by $\mathcal{B}$ and $\vec{\mathcal{B}}$, respectively. We also denote the full subcategory of all s-bounded Banach couples of $\vec{\mathcal{B}}$ by $\vec{\mathcal{B}}(s)$.

Remark 1. Indeed, if $\vec{A}$ is s-bounded, then

$$\begin{aligned}\|a\|_{\Sigma(\vec{A})} &\leq \min(\|a\|_{A_0}, \|a\|_{A_1}) \leq \max(\|a\|_{A_0}, \|a\|_{A_1}) = \|a\|_{\Delta(\vec{A})} \\ &\leq s\|a\|_{\Sigma(\vec{A})}.\end{aligned}$$

Definition 2.2. For each fixed $s \geq 1$, we define the *lower s-bounded couple* $\vec{A}_s$ of a given Banach couple $\vec{A}$ to be the couple $\vec{A}_s = (A_{s0}, A_{s1})$, where as vector spaces $A_{s0} = \Delta(\vec{A}) = A_{s1}$ with norms given by $\|a\|_{A_{si}} = s^{-i}J(s^{(2i-1)}, a)$, as i rangs over $\{0,1\}$. Similarly we define the *upper s-bounded couple* $\vec{A}^s$ of $\vec{A}$ to be the couple $\vec{A}^s = (A_0^s, A_1^s)$, where as vector spaces $A_0^s = \Sigma(\vec{A}) = A_1^s$ endowed with norms $\|a\|_{A_i^s} = s^iK(s^{(1-2i)}, a)$, as i rangs over $\{0,1\}$. Here J and K are the Peetre functionals on the spaces $\Delta(\vec{A})$ and $\Sigma(\vec{A})$, respectively.

Remark 2. For a fixed $s \geq 1$, if $\vec{A}$ is s-bounded and $a \in \Sigma(\vec{A})$, then the K-functional $K(\cdot, a)$ on the intervals $(0, \frac{1}{s}]$ and $[s, \infty)$ is given by

$$K(t,a) = \begin{cases} t\|a\|_{A_1}, & \text{if } 0 < t \leq \frac{1}{s}, \\ \|a\|_{A_0}, & \text{if } s \leq t. \end{cases}$$

Furthermore, for an arbitrary couple $\vec{A}$ and $a \in \Sigma(\vec{A})$, we have

$$K(t, a; \vec{A}^s) = \begin{cases} tsK(\frac{1}{s}, a; \vec{A}) & \text{for } 0 < t \leq \frac{1}{s}, \\ K(t, a; \vec{A}) & \text{for } \frac{1}{s} \leq t \leq s, \\ K(s, a; \vec{A}) & \text{for } s \leq t, \end{cases}$$

whereas

$$J(t, a; \vec{A}_s) = \begin{cases} J(\frac{1}{s}, a; \vec{A}) & \text{for } 0 < t \leq \frac{1}{s}, \\ J(t, a; \vec{A}) & \text{for } \frac{1}{s} \leq t \leq s, \\ \frac{t}{s} J(s, a; \vec{A}) & \text{for } s \leq t. \end{cases}$$

Proposition 2.3. *Given $s \geq 1$ and a Banach couple $\vec{A}$, we have the following:*

(i) *The Banach couples $\vec{A}_s$ and $\vec{A}^s$ are s-bounded.*
(ii) *There are inclusions $i_{ls} : \vec{A}_s \hookrightarrow \vec{A}$ and $i_{us} : \vec{A} \hookrightarrow \vec{A}^s$.*
(iii) $\lim_{r\to\infty} \|a\|_{A_{r0}} = \|a\|_{A_0}$ *and* $\lim_{r\to\infty} \|a\|_{A_{r1}} = \|a\|_{A_1}$ *for every $a \in \Delta(\vec{A})$.*
(iv) $\lim_{r\to\infty} \|a\|_{A_0^r} = \|a\|_{A_0}$ *and* $\lim_{r\to\infty} \|b\|_{A_1^r} = \|b\|_{A_1}$ *whenever $a \in A_0$ and $b \in A_1$.*

Proof. All statements follow from Definition 2.2. □

Proposition 2.4. *For each fixed $s \geq 1$, the inclusion $i : \vec{\mathcal{B}}(s) \hookrightarrow \vec{\mathcal{B}}$ has a left adjoint $l : \vec{\mathcal{B}} \hookrightarrow \vec{\mathcal{B}}(s)$ given by $l(\vec{A}) = \vec{A}^s$ and a right adjoint $r : \vec{\mathcal{B}} \hookrightarrow \vec{\mathcal{B}}(s)$ given by $r(\vec{A}) = \vec{A}_s$.*

Proof. It is enough to show the following two statements:

(i) There is a natural isomorphism between $\mathcal{L}(\vec{X}, \vec{A})$ and $\mathcal{L}(\vec{X}^s, \vec{A})$ whenever $\vec{X} \in \vec{\mathcal{B}}$ and $\vec{A} = \vec{A}^s \in \vec{\mathcal{B}}(s)$;
(ii) There is a natural isomorphism between $\mathcal{L}(\vec{A}, \vec{X})$ and $\mathcal{L}(\vec{A}, \vec{X}_s)$ whenever $\vec{X} \in \vec{\mathcal{B}}$ and $\vec{A} = \vec{A}_s \in \vec{\mathcal{B}}(s)$.

But it follows from Proposition 2.3 (ii) that there are natural inclusions $i_{us} : \mathcal{L}(\vec{X}^s, \vec{A}) \hookrightarrow \mathcal{L}(\vec{X}, \vec{A})$ and $i_{ls} : \mathcal{L}(\vec{A}, \vec{X}_s) \hookrightarrow \mathcal{L}(\vec{A}, \vec{X})$ with norms less than one. So to complete the proof it is enough to show

$$i_{us}^- : \mathcal{L}(\vec{X}, \vec{A}) \hookrightarrow \mathcal{L}(\vec{X}^s, \vec{A}) \quad \text{whenever } \vec{X} \in \vec{\mathcal{B}} \text{ and } \vec{A} = \vec{A}^s \in \vec{\mathcal{B}}(s) \tag{2.1}$$

for part (i) and

$$i_{ls}^- : \mathcal{L}(\vec{A}, \vec{X}) \hookrightarrow \mathcal{L}(\vec{A}, \vec{X}_s) \quad \text{whenever } \vec{X} \in \vec{\mathcal{B}} \text{ and } \vec{A} = \vec{A}_s \tag{2.2}$$

for part (ii), respectively. To show the inclusion in (2.1) assume that $\vec{A} = \vec{A}^s$ and $S \in \mathcal{L}(\vec{X}, \vec{A})$ such that $\|S\|_{\mathcal{L}(\vec{X}, \vec{A})} \leq 1$. Suppose $\|x\|_{X_0^s} < 1$.

Then we can find $x_0 \in X_0$ and $x_1 \in X_1$ such that $x = x_0 + x_1$ with $\|x_0\|_{X_0} + s\|x_1\|_{X_1} < 1$. Since $Sx = Sx_0 + Sx_1$ we have

$$\begin{aligned}\|Sx\|_{A_0} &\le \|Sx_0\|_{A_0} + \|Sx_1\|_{A_0} \le \|Sx_0\|_{A_0} + s\|Sx_1\|_{A_1} \\ &\le \|S\|_{\mathcal{L}(X_0,A_0)}\|x_0\|_{X_0} + \|S\|_{\mathcal{L}(X_1,A_1)}\, s\,\|x_1\|_{X_1} \\ &\le \max(\|S\|_{\mathcal{L}(X_0,A_0)}, \|S\|_{\mathcal{L}(X_1,A_1)})\cdot(\|x_0\|_{X_0} + s\,\|x_1\|_{X_1}) < 1.\end{aligned}$$

This shows that $\|S\|_{\mathcal{L}(X_0^s,A_0)} \le 1$. By the same argument we have $\|S\|_{\mathcal{L}(X_1^s,A_1)} \le 1$. It follows that $S \in \mathcal{L}(\vec{X}, \vec{A})$ implies $S \in \mathcal{L}(\vec{X}^s, \vec{A})$, thus completing the proof of (i). By a similar argument (2.2) follows, and we are done. □

Definition 2.5. An exact interpolation functor $F : \vec{\mathcal{B}} \to \mathcal{B}$ is called a *normalized interpolation functor* if both the inclusions $\delta : \Delta(\vec{A}) \hookrightarrow F(\vec{A})$ and $\sigma : F(\vec{A}) \hookrightarrow \Sigma(\vec{A})$ have norms $\|\delta\| \le 1$ and $\|\sigma\| \le 1$.

For two normalized interpolation functors F and G and a Banach couple $\vec{A}$, let $\Delta(F,G)(\vec{A})$ and $\Sigma(F,G)(\vec{A})$ be the intersection and sum spaces, respectively, of the Banach spaces $F(\vec{A})$ and $G(\vec{A})$ endowed with the norms

$$\|a\|_{\Delta(F,G)(\vec{A})} = \max(\|a\|_{F(\vec{A})}, \|a\|_{G(\vec{A})})$$

and

$$\|a\|_{\Sigma(F,G)(\vec{A})} = \inf\{\|a_0\|_{F(\vec{A})} + \|a_1\|_{G(\vec{A})} : (a = a_0 + a_1) \wedge (a_0 \in F(\vec{A})) \wedge (a_1 \in G(\vec{A}))\}.$$

If, for some $s \ge 1$, $\vec{A}$ is an element of $\vec{\mathcal{B}}(s)$, then we see that the canonical embedding $j : \Delta(F,G)(\vec{A}) \to \Sigma(F,G)(\vec{A})$ also has an inverse j^- with $\|j^-\|_{\mathcal{L}(\Sigma(F,G)(\vec{A}),\Delta(F,G)(\vec{A}))} \le s$.

Definition 2.6. The *distance function* $d(F,G)(s)$ between two normalized interpolation functors F and G is defined on the half line $s \ge 1$ by the value

$$d(F,G)(s) = \sup\{\|j^-\|_{\mathcal{L}(\Sigma(F,G)(\vec{A}),\Delta(F,G)(\vec{A}))} : \vec{A} \in \vec{\mathcal{B}}(s)\}.$$

Proposition 2.7. *Let F and G be two normalized interpolation functors. Then for each fixed $s \ge 1$, we have*

$$d(F,G)(s) = \sup\left\{\max\left(\frac{\|a\|_{F(\vec{A})}}{\|a\|_{G(\vec{A})}}, \frac{\|a\|_{G(\vec{A})}}{\|a\|_{F(\vec{A})}}\right) : (a \in \Delta(F,G)(\vec{A}) \setminus \{0\}) \wedge (\vec{A} \in \vec{\mathcal{B}}(s))\right\}.$$

Proof. By symmetry it is enough to prove the norm inequality $\|a\|_{F(\vec{A})} \leq \frac{\|a\|_{F(\vec{A})}}{\|a\|_{G(\vec{A})}} \cdot \|a\|_{\Sigma(F,G)(\vec{A})}$. Clearly $\|a\|_{\Sigma(F,G)(\vec{A})} < 1$ implies that there exist $a_F \in F(\vec{A})$ and $a_G \in G(\vec{A})$ such that $a = a_F + a_G$ and $\|a_F\|_{F(\vec{A})} + \|a_G\|_{G(\vec{A})} \leq 1$. But then

$$\|a_F + a_G\|_{F(\vec{A})} \leq \|a_F\|_{F(\vec{A})} + \|a_G\|_{F(\vec{A})} = \|a_F\|_{F(\vec{A})} + \|a_G\|_{G(\vec{A})} \cdot \frac{\|a_G\|_{F(\vec{A})}}{\|a_G\|_{G(\vec{A})}}$$

$$\leq \|a_F\|_{F(\vec{A})} + \|a_G\|_{G(\vec{A})} \cdot \max\left(\frac{\|a_G\|_{F(\vec{A})}}{\|a_G\|_{G(\vec{A})}},\, 1\right).$$

By symmetry we also have

$$\|a\|_{G(\vec{A})} \leq \max\left(1,\, \frac{\|a_F\|_{G(\vec{A})}}{\|a_F\|_{F(\vec{A})}}\right)$$

and this completes the proof. □

Before going into details, we shall motivate the reader by the following two simple examples of distance functions.

Example 2.1. *For the normalized interpolation functors $\vec{A} \mapsto F(\vec{A}) = \Delta(\vec{A})$ and $\vec{A} \mapsto G(\vec{A}) = \Sigma(\vec{A})$ and for any $s \geq 1$, we obtain*

$$d(F, G)(\vec{A})(s) = \sup\left\{ \|j^-\|_{\mathcal{L}(\Sigma,\Delta)(\vec{A})} \;:\; \vec{A} \in \vec{\mathcal{B}}(s) \right\} = s.$$

Example 2.2. *Likewise, if the normalized interpolation functors F and G are such that $\vec{A} \mapsto F(\vec{A}) = A_0$ and $\vec{A} \mapsto G(\vec{A}) = A_1$, then for any $s \geq 1$, an easy calculation gives that*

$$d(F, G)(\vec{A})(s) = s.$$

We shall state and prove the following basic properties of the distance function.

Theorem 2.8. *For fixed normalized interpolation functors F, G and H on the category $\vec{\mathcal{B}}$ and for any s and t in the open interval $(1, \infty)$, the distance function d has the following properties:*

(i) $d(F, G)(s) = d(G, F)(s) \leq s$;
(ii) $d(F, G)(s) \leq \max(1, \frac{s}{t}) \cdot d(F, G)(t)$;
(iii) $d(F, H)(s) \leq d(F, G)(s) \cdot d(G, H)(s)$.

For the proof we need the following

Lemma 2.9. *Let $1 \leq t < s$. Then, for any s-bounded Banach couple $\vec{A}$, the contractible isomorphism $j^t : \vec{A} \hookrightarrow \vec{A}^t$ satisfies $\|(j^t)^-\|_{\mathcal{L}(\vec{A}^t, \vec{A})} \leq \frac{s}{t}$.*

Proof. Suppose $a \in A_0^t$ with $\|a\|_{A_0^t} < 1$. Then we can find $a_0 \in A_0$ and $a_1 \in A_1$ such that $a = a_0 + a_1$ and $\|a_0\|_{A_0} + t\|a_1\|_{A_1} < 1$. Since

$$\begin{aligned}\|a\|_{A_0} \leq \|a_0\|_{A_0} \dot{+} \|a_1\|_{A_0} &\leq \|a_0\|_{A_0} + s\|a_1\|_{A_1} \\ &\leq \frac{s}{t} \cdot \left(\|a_0\|_{A_0} + t\|a_1\|_{A_0}\right) < \frac{s}{t},\end{aligned}$$

it follows that $\|(j^t)^{-}\|_{\mathcal{L}(\vec{A}^t, \vec{A})} \leq \frac{s}{t}$ as required. □

Proof of Theorem 2.8. Part (i) follows from the definition of $d(F,G)(s)$ and the obvious inclusion maps $\Delta(\vec{A}) \hookrightarrow \Delta(F,G)(\vec{A}) \hookrightarrow \Sigma(F,G)(\vec{A}) \hookrightarrow \Sigma(\vec{A})$.

To prove (ii) it is enough to show that

$$\|\gamma_a\|_{\Delta(F,G)(\vec{A})} \leq \max(1, \frac{s}{t}) \, \|\gamma_b\|_{\Delta(F,G)(\vec{B})},$$

where $\vec{A} \in \vec{\mathcal{B}}(s)$, $\vec{B} \in \vec{\mathcal{B}}(t)$, $1 < \min(s,t)$, and γ_a and γ_b are the inverses of the canonical inclusions of $\Delta(F,G)(\vec{A})$ into $\Sigma(F,G)(\vec{A})$ and $\Delta(F,G)(\vec{A}^t)$ into $\Sigma(F,G)(\vec{A}^t)$, respectively. Recall that $\Sigma(F,G)(\cdot) = \Sigma(F(\cdot),G(\cdot))$ and $\Delta(F,G)(\cdot) = \Delta(F(\cdot), G(\cdot))$. For $s \leq t$, the conclusion is trivial. Assume that $t < s$. Let $\vec{A} \in \vec{\mathcal{B}}(s)$ and $I : \vec{A} \hookrightarrow \vec{A}^t$ be the inclusion map. Clearly

$$\|Ia\|_{\Sigma(F,G)(\vec{A}^t)} \leq \|a\|_{\Sigma(F,G)(\vec{A})}. \tag{2.3}$$

Applying Lemma 2.9 to $\Delta(F,G)(\vec{A})$ and $\Delta(F,G)(\vec{A}^t)$, we get

$$\|a\|_{\Delta(F,G)(\vec{A})} \leq \frac{s}{t}\|Ia\|_{\Delta(F,G)(\vec{A}^t)}. \tag{2.4}$$

Now from (2.3) and (2.4), it easily follows that

$$\frac{t}{s}\frac{\|a\|_{\Delta(F,G)(\vec{A})}}{\|a\|_{\Sigma(F,G)(\vec{A})}} \leq \frac{\|Ia\|_{\Delta(F,G)(\vec{A}^t)}}{\|Ia\|_{\Sigma(F,G)(\vec{A}^t)}}.$$

Now since

$$\|\gamma_a\|_{\Delta(F,G)(\vec{A})} = \sup\left\{\frac{\|a\|_{\Delta(F,G)(\vec{A})}}{\|a\|_{\Sigma(F,G)(\vec{A})}} : a \in \Sigma(F,G)(\vec{A})\right\}$$

and

$$\|\gamma_b\|_{\Delta(F,G)(\vec{A}^t)} = \sup\left\{\frac{\|Ia\|_{\Delta(F,G)(\vec{A}^t)}}{\|Ia\|_{\Sigma(F,G)(\vec{A}^t)}} : Ia \in \Sigma(F,G)(\vec{A}^t)\right\},$$

we get

$$\|\gamma_a\|_{\Delta(F,G)(\vec{A})} \leq \frac{s}{t}\|\gamma_b\|_{\Delta(F,G)(\vec{A}^t)} \leq \max(1, \frac{s}{t})\|\gamma_b\|_{\Delta(F,G)(\vec{A}^t)}.$$

This settles (ii).

To prove (iii) we need to show that

$$\frac{\|a\|_{\Delta(F,H)(\vec{A})}}{\|a\|_{\Sigma(F,H)(\vec{A})}} \leq d(F,G)(s) \cdot d(G,H)(s)$$

for any s-bounded couple $\vec{A}$ and $a \in \Delta(\vec{A})$. To that end, we shall include the following notation which slightly 'generalizes' their counterparts mentioned earlier in this paper. Let $\{F_i\}_{i=0}^{n}$ be interpolation functors. Let $\Delta(F_0, \cdots, F_n)(\vec{A})$ and $\Sigma(F_0, \cdots, F_n)(\vec{A})$ be the intersection space and the algebraic sum space of the spaces $F_i(\vec{A})$ as i varies over the set $\{0, \cdots, n\}$, with the respective norms

$$\|a\|_{\Delta(F_0,\cdots,F_n)(\vec{A})} = \max\{\, \|a\|_{F_i(\vec{A})} \,:\, i \in \{0, \cdots, n\}\,\}$$

and

$$\|a\|_{\Sigma(F_0,\cdots,F_n)(\vec{A})} = \inf\{\, \sum_{i=0}^{n} \|a_i\|_{F_i(\vec{A})} \,:\, (a = \sum_{i=0}^{n} a_i) \wedge (a_0 \in F_0(\vec{A})) \wedge \cdots \wedge (a_n \in F_n(\vec{A}))\}.$$

Continuing the proof, we assume that $\|a\|_{\Sigma(F,G,H)(\vec{A})} < 1$ and let $a = a_F + a_G + a_H$ be a representation of a as a sum where $a_F \in F(\vec{A})$, $a_G \in G(\vec{A})$ and $a_H \in H(\vec{A})$ in such a way that $\|a\|_{\Sigma(F,G,H)(\vec{A})} \leq \|a_F\|_{F(\vec{A})} + \|a_G\|_{G(\vec{A})} + \|a_H\|_{H(\vec{A})} < 1$. Putting $z_{FG} = a_F + a_G$ and $z_{GH} = a_G + a_H$, we know that

$$\|z_{FG}\|_{\Delta(F,G)(\vec{A})} \leq d(F,G)(s) \cdot \big(\|a_F\|_{F(\vec{A})} + \|a_G\|_{G(\vec{A})}\big)$$

and

$$\|z_{GH}\|_{\Delta(G,H)(\vec{A})} \leq d(G,H)(s) \cdot \big(\|a_G\|_{G(\vec{A})} + \|a_H\|_{H(\vec{A})}\big).$$

Thus we have

$$\begin{aligned}
\|a\|_{\Delta(F,G)(\vec{A})} &\leq d(F,G)(s) \cdot \|a\|_{\Sigma(F,G)(\vec{A})} \\
&\leq d(F,G)(s) \cdot \big(\|a_F\|_{F(\vec{A})} + \|z_{GH}\|_{G(\vec{A})}\big) \\
&\leq d(F,G)(s) \cdot \big(\|a_F\|_{F(\vec{A})} + \|z_{GH}\|_{\Delta(G,H)(\vec{A})}\big) \\
&\leq d(F,G)(s) \cdot \Big(\|a_F\|_{F(\vec{A})} + d(G,H)(s) \cdot \big(\|a_G\|_{G(\vec{A})} \\
&\qquad + \|a_H\|_{H(\vec{A})}\big)\Big) \\
&\leq d(F,G)(s) \cdot d(G,H)(s) \cdot \|a\|_{\Sigma(F,G,H)(\vec{A})}.
\end{aligned}$$

Similarly $\|a\|_{\Delta(G,H)(\vec{A})} \leq d(F,G)(s)\cdot d(G,H)(s)\cdot\|a\|_{\Sigma(F,G,H)(\vec{A})}$ and it follows that

$$\begin{aligned}
\|a\|_{\Delta(F,H)(\vec{A})} &= \max\Big(\|a\|_{F(\vec{A})},\ \|a\|_{H(\vec{A})}\Big)\\
&\leq \max\Big(\|a\|_{F(\vec{A})},\ \|a\|_{G(\vec{A})},\ \|a\|_{H(\vec{A})}\Big)\\
&\leq \max\Big(\|a\|_{\Delta(F,G)(\vec{A})},\ \|a\|_{\Delta(G,H)(\vec{A})}\Big)\\
&\leq d(F,G)(s)\cdot d(G,H)(s)\cdot\|a\|_{\Sigma(F,G,H)(\vec{A})}\\
&\leq d(F,G)(s)\cdot d(G,H)(s)\cdot\|a\|_{\Sigma(F,H)(\vec{A})}.
\end{aligned}$$

The last inequality follows from the property of the norm in the sum space. Thus we obtain the desired result. □

We have the following

Theorem 2.10. *Let E, F, G and H be fixed normalized interpolation functors such that $E(\vec{A}) \hookrightarrow F(\vec{A}) \hookrightarrow H(\vec{A})$ and $E(\vec{A}) \hookrightarrow G(\vec{A}) \hookrightarrow H(\vec{A})$ are sequences of embeddings for each Banach couple $\vec{A}$. Then for any $s \geq 1$, we have the following relations:*

(i) $d(F,G)(s) \leq d(E,H)(s)$;
(ii) $\max(d(E,G)(s),\ d(E,F)(s),\ d(F,H)(s),\ d(G,H)(s)) \leq d(E,H)(s)$.

Proof. The proof of (i) is based on the observations that the inclusion map $E(\vec{A}) \hookrightarrow \Delta(F,G)(\vec{A})$ together with the inclusion map $\Sigma(F,G)(\vec{A}) \hookrightarrow H(\vec{A})$ and $E(\vec{A}) = \Delta(E,H)(\vec{A}) = \Delta(E,F)(\vec{A}) = \Delta(E,G)(\vec{A})$ and $H(\vec{A}) = \Sigma(E,H)(\vec{A})$. For $\epsilon > 0$, let $a \in \Sigma(\vec{A})$ satisfy the inequality $\|a\|_{\Delta(F,G)(\vec{A})} > (1-\epsilon)\cdot d(F,G)(s)\cdot\|a\|_{\Sigma(F,G)(\vec{A})}$. Then since $\|a\|_E \geq \|a\|_{\Delta(F,G)(\vec{A})}$ and $\|a\|_H \leq \|a\|_{\Sigma(F,G)(\vec{A})}$, it follows that

$$(1-\epsilon)\cdot d(F,G)(s) < \frac{\|a\|_{\Delta(F,G)(\vec{A})}}{\|a\|_{\Sigma(F,G)(\vec{A})}} \leq \frac{\|a\|_E}{\|a\|_{\Sigma(F,G)(\vec{A})}} \leq \frac{\|a\|_E}{\|a\|_H} \leq d(E,H)(s).$$

Thus we have proved part (i).

For the proof of (ii), it is enough to show that $d(E,F)(s) \leq d(E,H)(s)$ and $d(F,H)(s) \leq d(E,H)(s)$. Suppose that the inequality $\|a\|_{\Delta(E,F)(\vec{A})} > (1-\epsilon)\cdot d(E,F)(s)\cdot\|a\|_{\Sigma(E,F)(\vec{A})}$ is satisfied for some $a \in \Sigma(\vec{A})$. Again since $\|a\|_E \geq \|a\|_H$, we obtain the inequality $\|a\|_E > (1-\epsilon)\cdot d(E,F)(s)\cdot\|a\|_H$ which implies $d(E,F)(s) \leq d(E,H)(s)$. On the other hand, if we let $x \in \Sigma(\vec{A})$ satisfy the inequality $\|x\|_{\Delta(F,H)(\vec{A})} > (1-\epsilon)\cdot d(F,H)(s)\cdot\|x\|_{\Sigma(F,H)(\vec{A})}$, then since $\|x\|_{\Delta(E,H)(\vec{A})} > \|x\|_{\Delta(F,H)(\vec{A})}$ and $\|x\|_{\Sigma(F,H)(\vec{A})} \geq \|x\|_{\Sigma(E,H)(\vec{A})}$, we have $\|x\|_{\Delta(E,H)(\vec{A})} > (1-\epsilon)\cdot d(F,H)(s)\cdot\|x\|_{\Sigma(E,H)(\vec{A})}$. This gives $(1-\epsilon)\cdot d(F,H)(s) \leq d(E,H)(s)$. Thus (ii) follows and we are done. □

We may somewhat generalize our distance function to normalized regular interpolation functors from a category of Banach families into the category $\mathcal{B}$ as follows.

Let X be a Banach space and let $P(\Omega)$ denote a probability space consisting of a probability measure P on the Borel subsets of a given topological set Ω. Suppose that a measurable function $n : \Omega \times X \to \Re_+ \cup \{0\}$ given by $n(\omega, \cdot) = \|\cdot\|_\omega$ generates a family of complete normed spaces (viz X_ω) on X for almost every $\omega \in \Omega$. We call such a function n an interpolation generating function (IG for short).

Let $\widetilde{X}(n,I) = \left\{ X_\omega \ : \ \omega \in \Omega \ \text{ and } \ X_\omega \ \text{ is a Banach space} \right\}$ be the family of spaces generated by an IG-function n and a dense range contractive map I of the space

$$\Delta(\widetilde{X}) = \left\{ x \in X \ : \ \|x\|_{\Delta(\tilde{X})} = \operatorname*{ess\,sup}_{\omega\in\Omega} n(\omega, x) < \infty \right\}$$

into the space $\Sigma(\widetilde{X})$ endowed with the norm

$$\|Ix\|_{\Sigma(\tilde{X})} = \inf\Big\{ \int_\Omega n(\omega, f(\omega))dP(\omega) \ : \ f \in S(\Omega, \Delta(\widetilde{X})) \\ \text{and} \ \ x = \int_\Omega f(\omega)\, dP(\omega) \ \Big\}.$$

Here $S(\Omega, \Delta(\widetilde{X}))$ is the set of all simple $\Delta(\widetilde{X})$-valued functions on Ω.

Definition 2.11. For a given $P(\Omega)$, let n and I be as above, then the family $\widetilde{X}(n,I)$ is called a *regular interpolation family*. Such a family will be called, for $s \geq 1$, a *regular s-bounded family*, provided that the inequality

$$\|a\|_{\Delta(\tilde{X})} \leq s \cdot \inf_{\omega\in\Omega} \|a\|_\omega$$

holds for all a in $\Delta(\widetilde{X})$.

For a fixed Borel subset Θ of Ω, we define the corresponding K- and J-*functionals* on a regular Banach family $\widetilde{X}(n,I)$ by

$$K(t,x;\tilde{X}(n,I)) = \inf\Big\{ \int_\Omega t^{\chi_\Theta(\omega)} \cdot n(\omega,\ f(\omega))\, dP(\omega) \ : \ f \in S(\Omega, \Delta(\widetilde{X})) \\ \text{and} \quad x = \int_\Omega f(\omega)\, dP(\omega) \ \Big\}$$

and

$$J(t,x;\tilde{X}(n,I)) = \operatorname{ess\,sup}\{t^{\chi_\Theta(\omega)}\cdot n(\omega,x) : \omega\in\Omega \text{ and } x\in\Delta(\tilde{X})\},$$

respectively. Here χ_Θ is the characteristic function for the set Θ.

Remark 3. The K- and J-functionals for a family have been defined somewhat differently by different authors, but we prefer this one because it preserves the quasi-concavity as a function of t, similarly to the K-functional on Banach couples.

Definition 2.12. For a fixed $s\geq 1$ and a fixed Borel subset Θ of Ω, we define the *lower s-bounded* Banach family $\widetilde{X_s}(n,I)$ of a given regular Banach family $\tilde{X}(n,I)$ by $\widetilde{X_s}(n,I) := \{X_{s\omega} : \omega\in\Omega\}$, where as vector spaces $X_{s\omega} = \Delta(\tilde{X})$ for each $\omega\in\Omega$ with norms given by $\|x\|_{X_{s\omega}} = s^{-\chi_{(\Omega\setminus\Theta)}(\omega)}J(s^{(2\chi_{(\Omega\setminus\Theta)}(\omega)-1)},x;\tilde{X}(n,I))$. Similarly we define the *upper s-bounded* Banach family $\widetilde{X^s}(n,I)$ of a given regular Banach family $\tilde{X}(n,I)$ by $\widetilde{X^s}(n,I) := \{X^s_\omega : \omega\in\Omega \text{ and } X^s_\omega \text{ is a Banach space}\}$, where as vector spaces $X^s_\omega = \Sigma(\tilde{X})$ for every $\omega\in\Omega$ endowed with norms $\|x\|_{X^s_\omega} = s^{\chi_{(\Omega\setminus\Theta)}(\omega)}K(s^{(1-2\chi_{(\Omega\setminus\Theta)}(\omega))},x;\tilde{X}(n,I))$.

To help us define distance between regular interpolation families we need to characterize the category of regular interpolation families $\tilde{\mathcal{B}}$ whose objects are all interpolation families subject to a preassigned probability space $P(\Omega)$, and whose morphisms are bounded maps $T : \tilde{X}(n,I)\to\tilde{Y}(m,J)$ such that $T|_{X_\omega} : X_\omega\to Y_\omega$ with

$$\|T\|_{\mathcal{L}(\tilde{X},\tilde{Y})} = \operatorname{ess\,sup}\left\{\|T\|_{\mathcal{L}(X_\omega,Y_\omega)} : X_\omega\in\tilde{X}(n,I) \text{ and } Y_\omega\in\tilde{Y}(m,J)\right\}$$

is finite. We denote the subcategory of all regular s-bounded interpolation families by $\tilde{\mathcal{B}}(s)$.

Definition 2.13. The *distance function* $d_f(F,G)(s)$ between two interpolation functors F and G from the category $\tilde{\mathcal{B}}$ of *regular interpolation families* to the category $\mathcal{B}$ of all Banach spaces is defined on the half line $s\geq 1$ by the value

$$d_f(F,G)(s) = \sup\{\|I^{-1}\|_{\mathcal{L}(\Sigma(F,G)(\tilde{X}),\,\Delta(F,G)(\tilde{X}))} : \tilde{X}\in\tilde{\mathcal{B}}(s)\}.$$

Remark 4. Results similar to Proposition 2.7 and Theorem 2.8 are now easy to prove for the distance function d_f between two normalized interpolation functors on $\tilde{\mathcal{B}}(s)$ with the obvious modifications.

3. Maximal and minimal methods

Suppose that $\vec{\mathcal{C}}$ is a full subcategory of the category $\vec{\mathcal{B}}$ and let H be an interpolation functor from $\vec{\mathcal{C}}$ to $\mathcal{B}$. Then we recall that an interpolation functor F from $\vec{\mathcal{B}}$ to $\mathcal{B}$ is called a *minimal extension* (or *left Kan extension* or at times *left adjoint*, see [K-P, pp. 93–98]) for H denoted by $\mathrm{Lan}_{\vec{\mathcal{C}}H}$ if $F(\vec{X}) \hookrightarrow G(\vec{X})$ for every $\vec{X} \in \vec{\mathcal{B}}$ and each interpolation functor G from $\vec{\mathcal{B}}$ to $\mathcal{B}$ satisfying $H(\vec{A}) = F(\vec{A}) = G(\vec{A})$ for all $\vec{A} \in \vec{\mathcal{C}}$. The interpolation functor F is called a *maximal extension* (or *right Kan extension* or at times *right adjoint*) for H denoted by $\mathrm{Ran}_{\vec{\mathcal{C}}H}$ if the inclusion above is reversed. (Equivalently F from $\vec{\mathcal{B}}$ to $\mathcal{B}$ is called a *minimal extension* for H if for every G from $\vec{\mathcal{B}}$ to $\mathcal{B}$ and for each $\vec{A} \in \vec{\mathcal{C}}$ such that $H(\vec{A}) = F(\vec{A}) = G(\vec{A})$ for all $\vec{A} \in \vec{\mathcal{C}}$ and $G(\vec{X}) \hookrightarrow F(\vec{X})$ for all $\vec{X} \in \vec{\mathcal{B}}$ imply that $G = F$. There is an analogous formulation for *maximal extensions*.) Furthermore, when $\vec{\mathcal{C}}$ consists of only one object, the couple $\vec{A}$ with morphisms which are automorphisms of couple maps, the minimal extensions are called *minimal interpolation functors*, while the maximal extensions are called *maximal interpolation functors*. We use the notations $F_{\vec{A}}$ and $F^{\vec{A}}$ to denote respectively the minimal and maximal extensions to the whole category $\vec{\mathcal{B}}$ of the interpolation functor F previously defined on the full subcategory with only one object, the couple $\vec{A}$.

In this case the minimal extension functor $F_{\vec{A}}$ on the Banach couple $\vec{X}$ is the Banach space consisting of all $x \in \Sigma(\vec{X})$ for which

$$\|x\|_{F_{\vec{A}}(\vec{X})} = \inf\left\{ \sum_{n\in\mathbf{N}} \|S_n\|_{\mathcal{L}(\vec{A};\vec{X})} \|a_n\|_A \ : \ x = \sum_{n\in\mathbf{N}} S_n a_n \ \text{ and } \ a_n \in F(\vec{A}) \right\}$$

is finite. Similarly the maximal extension is the Banach space consisting of all $x \in \Sigma(\vec{X})$ such that

$$\|x\|_{F^{\vec{A}}(\vec{X})} = \sup\Big\{ \|S(x)\|_{F(\vec{A})} \ : \ (S \in \mathcal{L}(\vec{X},\vec{A})) \wedge (\|S\|_{\mathcal{L}(\vec{X},\vec{A})} = 1) \wedge (S(x) \in \Phi(\vec{A})) \Big\}$$

is finite.

Suppose that $\vec{\mathcal{F}}$ denotes a family of interpolation functors from $\vec{\mathcal{C}}$ to $\mathcal{B}$. We also recall that $\vec{\mathcal{F}}$ is said to be an *interpolation method* (or merely *method* if $\vec{\mathcal{C}} = \vec{\mathcal{B}}$) on $\vec{\mathcal{C}}$ if for any functors F, G and H in $\vec{\mathcal{F}}$ and every $\vec{A} \in \vec{\mathcal{C}}$, the condition that $\Delta(G,H)(\vec{A}) \hookrightarrow F(G(\vec{A}), H(\vec{A})) \hookrightarrow \Sigma(G,H)(\vec{A})$ implies the existence of an interpolation functor R in $\vec{\mathcal{F}}$ such that $F(G(\vec{A}), H(\vec{A})) =$

$R(\vec{A})$. Accordingly, *the maximal* and *the minimal methods* are the families of maximal interpolation functors and minimal interpolation functors, respectively, each one of them satisfying the property of being a method.

We have the following description of distance between two minimal interpolation functors and two maximal interpolation functors, respectively.

Theorem 3.1. *Suppose that F, G, Φ and Ψ are normalized interpolation functors defined on a full subcategory consisting of a single object, the s-bounded couple $\vec{A}$, where $s \geq 1$.*

(i) *If $A = F(\vec{A}) = F_{\vec{A}}(\vec{A})$ and $B = G(\vec{A}) = G_{\vec{A}}(\vec{A})$, then we have*

$$d(F_{\vec{A}}, G_{\vec{A}})(s) = \sup\left\{ \max\left(\frac{\|a\|_A}{\|a\|_B}, \frac{\|a\|_B}{\|a\|_A} \right) : 0 \neq a \in A \cap B \right\}.$$

(ii) *Again if $A = \Phi(\vec{A}) = \Phi^{\vec{A}}(\vec{A})$ and $B = \Psi(\vec{A}) = \Psi^{\vec{A}}(\vec{A})$ instead, then*

$$d(\Phi^{\vec{A}}, \Psi^{\vec{A}})(s) = \sup\left\{ \max\left(\frac{\|a\|_A}{\|a\|_B}, \frac{\|a\|_B}{\|a\|_A} \right) : 0 \neq a \in A \cap B \right\}.$$

Proof. (i) For a given $\epsilon > 0$, let $\vec{X}$ be an s-bounded couple and $x \in \Sigma(\vec{X})$ such that

$$d(F_{\vec{A}}, G_{\vec{A}})(s) - \epsilon \leq \max\left(\frac{\|x\|_{F_{\vec{A}}(\vec{X})}}{\|x\|_{G_{\vec{A}}(\vec{X})}}, \frac{\|x\|_{G_{\vec{A}}(\vec{X})}}{\|x\|_{F_{\vec{A}}(\vec{X})}} \right).$$

Without loss of generality assume

$$\max\left(\frac{\|x\|_{F_{\vec{A}}(\vec{X})}}{\|x\|_{G_{\vec{A}}(\vec{X})}}, \frac{\|x\|_{G_{\vec{A}}(\vec{X})}}{\|x\|_{F_{\vec{A}}(\vec{X})}} \right) = \frac{\|x\|_{F_{\vec{A}}(\vec{X})}}{\|x\|_{G_{\vec{A}}(\vec{X})}}.$$

Since

$$\begin{aligned}
\|x\|_{F_{\vec{A}}(\vec{X})} &= \inf\left\{ \sum_{n\in\mathbf{N}} \|S_n\|_{\mathcal{L}(\vec{A},\vec{X})} \|a_n\|_A \ : x = \sum_{n\in\mathbf{N}} S_n a_n \text{ and } a_n \in F(\vec{A}) \right\} \\
&= \inf\left\{ \sum_{n\in\mathbf{N}} \|S_n\|_{\mathcal{L}(\vec{A},\vec{X})} \|a_n\|_B \cdot \frac{\|a_n\|_A}{\|a_n\|_B} \ : x = \sum_{n\in\mathbf{N}} S_n a_n \right. \\
&\qquad \left. \text{and } a_n \in F(\vec{A}) \right\} \\
&\leq \sup\left\{ \frac{\|a_n\|_A}{\|a_n\|_B} : a_n \in A \cap B \right\} \cdot \|x\|_{G_{\vec{A}}(\vec{X})} \\
&\leq \sup\left\{ \max\left(\frac{\|a\|_A}{\|a\|_B}, \frac{\|a\|_B}{\|a\|_A} \right) : 0 \neq a \in A \cap B \right\} \cdot \|x\|_{G_{\vec{A}}(\vec{X})},
\end{aligned}$$

we settle the proof of the first part of the theorem.

To prove (ii) let $s > 1$ be fixed and let $\epsilon > 0$. Choose $\vec{X}$ in $\vec{\mathcal{B}}(s)$ and let $x \in \Sigma(\vec{X})$ satisfy $\frac{\|x\|_{\Phi^{\vec{A}}(\vec{X})}}{\|x\|_{\Psi^{\vec{A}}(\vec{X})}} > (1-\epsilon)\cdot d(\Psi^{\vec{A}}, \Phi^{\vec{A}})(s)$. Since by Aronszajn-Gagliardo theorem every maximal interpolation functor is a Coorbit functor, see [B-K, Theorem 2.3.17, pp. 153], so are both $\Phi^{\vec{A}}$ and $\Psi^{\vec{A}}$. Thus we have

$$\begin{aligned}\|x\|_{A^{\vec{\Phi}}(\vec{X})} &= \sup\Big\{ \|T(x)\|_{\Phi(\vec{A})} \ : (T \in \mathcal{L}(\vec{X}, \vec{A})) \wedge (\|T\|_{\mathcal{L}(\vec{X},\vec{A})} = 1) \\ &\qquad \wedge\ (T(x) \in \Phi(\vec{A})) \Big\} \\ &\le \sup\Big\{ \sup\Big\{ \frac{\|T(y)\|_{\Phi(\vec{A})}}{\|T(y)\|_{\Psi(\vec{A})}} \ : 0 \neq y \in \Sigma(\vec{X}) \Big\} \cdot \|T(x)\|_{\Psi(\vec{A})} \ : \\ &\qquad (\|T\|_{\mathcal{L}(\vec{X},\vec{A})} = 1) \ \wedge\ (x \in \Sigma(\vec{X}))\Big\} \\ &\le \ \sup\Big\{ \frac{\|T(y)\|_{A}}{\|T(y)\|_{B}} \ : 0 \neq T(y) \in A \cap B \ \Big\} \cdot \|x\|_{\Psi^{\vec{A}}(\vec{X})}.\end{aligned}$$

Likewise, should we assume $\frac{\|x\|_{\Psi^{\vec{A}}(\vec{X})}}{\|x\|_{\Phi^{\vec{A}}(\vec{X})}} > (1-\epsilon)\cdot d(\Psi^{\vec{A}}, \Phi^{\vec{A}})(s)$, then we can similarly show that

$$\|x\|_{\Psi^{\vec{A}}(\vec{X})} \ \le \ \sup\Big(\frac{\|T(y)\|_{B}}{\|T(y)\|_{A}} \ : 0 \neq T(y) \in A \cap B \ \Big) \cdot \|x\|_{\Phi^{\vec{A}}(\vec{X})}.$$

Thus part (ii) and hence Theorem 3.1 is proved. □

We have the following description of distance between two minimal extension functors and two maximal extension functors, respectively.

Theorem 3.2. *Let $\vec{\mathcal{C}}$ be a full subcategory of $\vec{\mathcal{B}}$ whose objects are all regular couples. Let $F : \vec{\mathcal{C}} \to \mathcal{B}$ and $G : \vec{\mathcal{C}} \to \mathcal{B}$ be uniformly regular (i. e. $\Delta(\vec{A})$ is dense in both $F(\vec{A})$ and $G(\vec{A})$ for every $\vec{A} \in \vec{\mathcal{C}}$), normalized functors on $\vec{\mathcal{C}}$. Suppose that $(F(\vec{A}), G(\vec{A}))$ is a non-trival and s-bounded couple for each $\vec{A} \in \vec{\mathcal{C}}$. Then for each $s \ge 1$, we have*

(i) $d(Lan_{\vec{\mathcal{C}}F}, Lan_{\vec{\mathcal{C}}G})(s) = \sup\Big\{ d(F_{\vec{W}}, G_{\vec{W}})(s) : \vec{W} \in \vec{\mathcal{C}} \cap \vec{\mathcal{B}(s)} \Big\}$;

(ii) $d(Ran_{\vec{\mathcal{C}}F}, Ran_{\vec{\mathcal{C}}G})(s) = \sup\Big\{ d(F^{\vec{W}}, G^{\vec{W}})(s) : \vec{W} \in \vec{\mathcal{C}} \cap \vec{\mathcal{B}(s)} \Big\}$.

Proof. The proof follows from Theorem 3.1. □

Given an intermediate space A of a Banach couple $\vec{A} = (A_0, A_1)$, we define its dual A' (in the sense of interpolation theory unlike the duality A^* in the sense of Banach), to be $A' = \{a^* \in \Delta(\vec{A})^* \ : \|a^*\|_{A'} = \sup\{|\langle a^*, a\rangle| : a \in \Delta(\vec{A})$ and $\|a\|_A \leq 1\} < \infty\}$ (i.e. the subspace of $\Delta(\vec{A})^*$ containing only those functionals that are bounded for the norm of A). One can easily see that A' is a Banach space. Regarding both A_0 and A_1 as intermediate spaces of $\vec{A}$, we also define the *dual Banach couple* to be the couple $\vec{A'} = (A'_0, A'_1)$. Moreover for an operator $T \in \mathcal{L}(\vec{A}, \vec{X})$, if we put its *adjoint operator* to be $T' = \left(T|_{\Delta(\vec{A})}\right)^*$, then it is clear that $T' \in \mathcal{L}(\vec{X'}, \vec{A'})$.

Definition 3.3. i) Let A be an interpolation space of the Banach couple $\vec{A}$. Then $\vec{A}$ is called a *weak compliant Banach couple*, if A' is an interpolation space of the couple $\vec{A'}$.

ii) Let $\vec{A}$ be a weak compliant Banach couple. Then for any interpolation functor $F : \vec{A} \mapsto \mathcal{B}$ we define its *dual interpolation functor* $\mathcal{D}F$ to be the maximal interpolation functor $\mathcal{D}F^{\vec{A'}}$ such that $\mathcal{D}F(\vec{A'}) = \mathcal{D}F^{\vec{A'}}(\vec{A'}) = F(\vec{A})'$.

We have the following corollary to Theorem 3.1.

Corollary 3.4. *Let a weak compliant Banach couple $\vec{A}$ be s-bounded and let two normalized dual interpolation functors $\mathcal{D}F$ and $\mathcal{D}G$ of the regular interpolation functors $F : \vec{A} \mapsto \mathcal{B}$ and $G : \vec{A} \mapsto \mathcal{B}$, respectively, be given. Putting $\mathcal{D}F(\vec{A'}) = F(\vec{A})' = A'$ and $\mathcal{D}G(\vec{A'}) = G(\vec{A})' = B'$, for each $s \geq 1$, we have*

$$d(\mathcal{D}F, \mathcal{D}G)(s) = \sup\left\{\max\left(\frac{\|a\|_A}{\|a\|_B}, \frac{\|a\|_B}{\|a\|_A}\right) : 0 \neq a \in A \cap B\right\}.$$

Proof. The proof is similar to the proof of Theorem 3.1 and is omitted. □

4. Relative distance

In this section we restrict the definition of our distance to some subcategories $\vec{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ of the categories $\vec{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$, respectively. But first we introduce the following terms:

Definition 4.1. For each fixed $s \geq 1$, the *lower s-closure* $\mathrm{lc}(\vec{\mathcal{C}})$, the *upper s-closure* $\mathrm{uc}(\vec{\mathcal{C}})$ and the *s-closure* $\mathrm{sc}(\vec{\mathcal{C}})$ of a subcategory $\vec{\mathcal{C}}$, and the *lower* *s-closure* $\mathrm{lc}(\widetilde{\mathcal{C}})$, the *upper s-closure* $\mathrm{uc}(\widetilde{\mathcal{C}})$) and the *s-closure* $\mathrm{sc}(\widetilde{\mathcal{C}})$ of a subcategory $\widetilde{\mathcal{C}}$ are defined by the identities

$$\begin{aligned}
\mathrm{lc}(\vec{\mathcal{C}}) &= \{\vec{X}_s \in \vec{\mathcal{B}} \ : \ \vec{X} \in \vec{\mathcal{C}}\} \cup \vec{\mathcal{C}}, \\
\mathrm{uc}(\vec{\mathcal{C}}) &= \{\vec{X}^s \in \vec{\mathcal{B}} \ : \ \vec{X} \in \vec{\mathcal{C}}\} \cup \vec{\mathcal{C}}, \\
\mathrm{sc}(\vec{\mathcal{C}}) &= \mathrm{uc}(\mathrm{lc}(\vec{\mathcal{C}})) \cup \mathrm{lc}(\mathrm{uc}(\vec{\mathcal{C}})), \\
\mathrm{lc}(\widetilde{\mathcal{C}}) &= \{\widetilde{X_s} \in \widetilde{\mathcal{B}} \ : \ \widetilde{X} \in \widetilde{\mathcal{C}}\} \cup \widetilde{\mathcal{C}}, \\
\mathrm{uc}(\widetilde{\mathcal{C}}) &= \{\widetilde{X^s} \in \widetilde{\mathcal{B}} \ : \ \widetilde{X} \in \widetilde{\mathcal{C}}\} \cup \widetilde{\mathcal{C}}, \\
\mathrm{sc}(\widetilde{\mathcal{C}}) &= \mathrm{uc}(\mathrm{lc}(\widetilde{\mathcal{C}})) \cup \mathrm{lc}(\mathrm{uc}(\widetilde{\mathcal{C}})),
\end{aligned}$$

respectively. A subcategory $\vec{\mathcal{C}}$ is said to be *s-closed* if $\vec{\mathcal{C}} = \mathrm{sc}(\vec{\mathcal{C}})$, i. e. if $\vec{X} \in \vec{\mathcal{C}}$ imply that both $\vec{X}^s$ and $\vec{X}_s$ belong to $\vec{\mathcal{C}}$. Similarly $\widetilde{\mathcal{C}}$ is said to be *s-closed* if $\widetilde{\mathcal{C}} = \mathrm{sc}(\widetilde{\mathcal{C}})$.

We can now make the following definitions.

Definition 4.2. The *lower relative distance function*, the *upper relative distance function* and the *relative distance function* between two normalized interpolation functors F and G, denoted accordingly by $d_{lc(\vec{\mathcal{C}})}(F, G)(s)$, $d_{uc(\vec{\mathcal{C}})}(F, G)(s)$ and $d_{sc(\vec{\mathcal{C}})}(F, G)(s)$, on the half line $s \geq 1$ are defined by the values:

$$d_{\vec{\mathcal{N}}}(F, G)(s) = \sup\{\|\gamma\|_{\mathcal{L}(\Sigma(F,G)(\vec{A}),\, \Delta(F,G)(\vec{A}))} \ : \ \vec{A} \in \vec{\mathcal{N}} \cap \vec{\mathcal{B}}(s)\ \},$$

where $\vec{\mathcal{N}}$ varies over the set $\{\mathrm{lc}(\vec{\mathcal{C}}),\ \mathrm{uc}(\vec{\mathcal{C}}),\ \mathrm{sc}(\vec{\mathcal{C}})\}$.

Proposition 4.3. *Using the above notation, for each $s \geq 1$, we have the following inequalities:*

$$\max\left\{d_{lc(\vec{\mathcal{C}})}(F, G)(s),\ d_{uc(\vec{\mathcal{C}})}(F, G)(s)\right\} \ \leq d_{sc(\vec{\mathcal{C}})}(F, G)(s) \ \leq \ d(F, G)(s).$$

Proof. The easy proof is omitted. □

Proposition 4.4. *Let F and G be two normalized interpolation functors. Then, for each $s \geq 1$,*

$$d_{\vec{\mathcal{N}}}(F, G)(s) = \sup\left\{\max\left(\frac{\|a\|_{F(\vec{A})}}{\|a\|_{G(\vec{A})}},\ \frac{\|a\|_{G(\vec{A})}}{\|a\|_{F(\vec{A})}}\right) \ : \ a \in \Delta(F,G)(\vec{A}) \setminus \{0\} \right.$$
$$\left. \text{and} \quad \vec{A} \in \vec{\mathcal{N}} \cap \vec{\mathcal{B}}(s) \ \right\},$$

where $\vec{\mathcal{N}}$ is varying over the set $\{\,\vec{\mathcal{B}},\ lc(\vec{\mathcal{C}}),\ uc(\vec{\mathcal{C}}),\ sc(\vec{\mathcal{C}})\,\}$ and $\vec{\mathcal{C}}$ is a full subcategory of $\vec{\mathcal{B}}$.

Proof. The result follows by repeating the same line of argument as in the proof of Proposition 2.7 and is omitted. □

Remark 5. Once again, results similar to Proposition 2.7 and Theorem 2.8 are now easy to prove for the relative distance functions between two normalized interpolation functors on $\mathrm{lc}(\vec{C})$, $\mathrm{uc}(\vec{C})$ and $\mathrm{sc}(\vec{C})$, respectively, with the obvious modifications.

5. Relatively easy examples of a distance function between two functors in the classical $K_{(\theta,p)}$-method

In this section we compute the exact distance functions for some classical functors of the real $K_{(\theta,p)}$-method by giving both upper and lower estimates. For simplicity in presenting the computation we choose to proceed as follows.

Given $s \geq 1$, let K_0 denote the Banach space consisting of all continuous functions defined on the closed interval $[\frac{1}{s}, s]$ and endowed with the supremum norm. We let K_1 be the same linear space but equipped with the norm $\|f\|_{K_1} = \|\frac{f(t)}{t}\|_{K_0}$ and consider the Banach couple $\vec{K} = (K_0, K_1)$. For $f \in \Delta(\vec{K})$, we have

$$\|f\|_{\Delta(\vec{K})} = \max(\|f\|_{K_0}, \|f\|_{K_1}) = \max\Big(\sup_{\frac{1}{s}\leq t\leq s} |f(t)|,\ \sup_{\frac{1}{s}\leq t\leq s} \frac{|f(t)|}{t}\Big).$$

From the inequalities

$$\begin{aligned}
\sup_{\frac{1}{s}\leq t\leq s} |f(t)| &\leq \inf\Big\{ \sup_{\frac{1}{s}\leq t\leq s} f_0(t) + \sup_{\frac{1}{s}\leq t\leq s} (|f(t)| - f_0(t)) \ :\ |f| = f_0 + f_1 \\
&\qquad\qquad \text{and} \quad \min(f_0(t), f_1(t)) \geq 0 \Big\} \\
&\leq \inf\Big\{ \sup_{\frac{1}{s}\leq t\leq s} f_0(t) + s \sup_{\frac{1}{s}\leq t\leq s} \frac{|f(t)| - f_0(t)}{t} \ :\ |f| = f_0 + f_1 \\
&\qquad\qquad \text{and} \quad \min(f_0(t), f_1(t)) \geq 0 \Big\} \\
&\leq s\|f\|_{\Sigma(\vec{K})}
\end{aligned}$$

and

$$\sup_{\frac{1}{s}\le t\le s}\frac{|f(t)|}{t}\le \inf\left\{\sup_{\frac{1}{s}\le t\le s}\frac{f_0(t)}{t}+\sup_{\frac{1}{s}\le t\le s}\frac{|f(t)|-f_0(t)}{t}\ :\ |f|=f_0+f_1 \right.$$
$$\left.\text{and}\quad \min(f_0(t),f_1(t))\ge 0\right\}$$
$$\le \inf\left\{s\sup_{\frac{1}{s}\le t\le s} f_0(t)+\sup_{\frac{1}{s}\le t\le s}\frac{|f(t)|-f_0(t)}{t}\ :\ |f|=f_0+f_1 \right.$$
$$\left.\text{and}\quad \min(f_0(t),f_1(t))\ge 0\right\}$$
$$\le s\|f\|_{\Sigma(\vec{K})},$$

we conclude that $\vec{K}$ is an s-bounded Banach couple.

We compute the following distances.

Example 5.1. *The distance function between $K_{(\theta,\infty)}$ and $K_{(\beta,\infty)}$ satisfies the following equality:*

$$d(K_{(\theta,\infty)}\,,\,K_{(\beta,\infty)})(s)\ =\ s^{|\theta-\beta|}.$$

It should be noted that we can find non-decreasing concave functions $f\in\Sigma(\vec{K})$ which do not coincide with the restictions of their K-functionals on the interval $[\frac{1}{s},\,s]$.

Verification. By the definition of the $K_{(\theta,\infty)}$-norm (see [B-L] or [B-K]), for any $f\in K_{(\theta,\infty)}(\vec{K})$, since

$$\|f\|_{K_{(\theta,\infty)}}=\sup_{\frac{1}{s}\le t\le s} t^{-\theta}K(t,f;\vec{K}),$$

we have

$$\|f\|_{K_{(\theta,\infty)}}\le s^{|\theta-\beta|}\sup_{\frac{1}{s}\le t\le s} t^{-\beta}K(t,f;\vec{K})=s^{|\theta-\beta|}\|f\|_{K_{(\beta,\infty)}}.$$

Similarly we easily see that

$$\|f\|_{K_{(\beta,\infty)}}\le s^{|\theta-\beta|}\|f\|_{K_{(\theta,\infty)}}$$

and by Proposition 2.7 and Theorem 3.1(ii) we conclude that

$$d(K_{(\theta,\infty)}\,,\,K_{(\beta,\infty)})(s)\ \le\ s^{|\theta-\beta|},$$

giving the upper bound. To finish the verification we must be able to show that $s^{|\theta-\beta|}$ is also a lower bound. To that end we consider the function $m(t)\,=\,\min(1,\,st)$ defined on the interval $[\frac{1}{s},\,s]$ which is clearly in $\Sigma(\vec{K})$. We see then that

$$\max\left(\frac{\|m\|_{K_{(\beta,\infty)}}}{\|m\|_{K_{(\theta,\infty)}}},\ \frac{\|m\|_{K_{(\theta,\infty)}}}{\|m\|_{K_{(\beta,\infty)}}}\right)$$

gives a lower bound, and since $\|m\|_{K_{(\beta,\infty)}} = s^{\beta}$ and $\|m\|_{K_{(\theta,\infty)}} = s^{\theta}$, we conclude that

$$\max\left(\frac{\|m\|_{K_{(\beta,\infty)}}}{\|m\|_{K_{(\theta,\infty)}}}, \frac{\|m\|_{K_{(\theta,\infty)}}}{\|m\|_{K_{(\beta,\infty)}}}\right) = \max\left(s^{\beta-\theta}, s^{\theta-\beta}\right) = s^{|\theta-\beta|},$$

thus verifying that the lower bound and the upper bound coincide. Hence we have

$$d(K_{(\theta,\infty)}\,,\,K_{(\beta,\infty)})(s) \;=\; s^{|\theta-\beta|}.$$

□

Example 5.2. *The distance function between $K_{(\theta,1)}$ and $K_{(\theta,\infty)}$ satisfies the following equality:*

$$d(K_{(\theta,1)}\,,\,K_{(\theta,\infty)})(s) \;=\; 1 \;+\; 2\theta(1-\theta)\log s.$$

Verification. Let $f \in K_{(\theta,1)}(\vec{K}) \cap K_{(\theta,\infty)}(\vec{K})$. Using Remark 2 and the definition of a K-functional we can easily obtain the following three inequalities

$$\int_0^{\frac{1}{s}} t^{-\theta}K(t,f)\,\frac{dt}{t} \;=\; \frac{s^{\theta}K(\frac{1}{s},f)}{(1-\theta)},$$

$$\int_{\frac{1}{s}}^{s} t^{-\theta}K(t,f)\,\frac{dt}{t} \;\le\; (2\log s)\sup_{\frac{1}{s}\le t\le s} t^{-\theta}K(t,f;\vec{K}),$$

$$\int_s^{\infty} t^{-\theta}K(t,f)\,\frac{dt}{t} \;=\; \frac{s^{-\theta}K(s,f)}{\theta}.$$

Consequently, since the normalized $K_{(\theta,1)}$-norm by definition is given by

$$\|f\|_{K_{(\theta,1)}} = \theta(1-\theta)\int_0^{\infty} r^{-\theta}K(r,f)\,\frac{dr}{r},$$

it follows that

$$\begin{aligned}
\|f\|_{K_{(\theta,1)}} &= \theta(1-\theta)\int_0^{\infty} t^{-\theta}K(t,f)\,\frac{dt}{t}\\
&= \theta(1-\theta)\Big(\int_0^{\frac{1}{s}} t^{-\theta}K(t,f)\,\frac{dt}{t} \;+\; \int_{\frac{1}{s}}^{s} t^{-\theta}K(t,f)\,\frac{dt}{t}\\
&\qquad +\; \int_s^{\infty} t^{-\theta}K(t,f)\,\frac{dt}{t}\Big)\\
&\le \Big(\theta \;+\; 2\theta(1-\theta)\log s \;+\; (1-\theta)\Big)\sup_{\frac{1}{s}\le t\le s} t^{-\theta}K(t,f;\vec{K})\\
&\le \Big(1 \;+\; 2\theta(1-\theta)\log s\Big)\|f\|_{K_{(\theta,\infty)}}.
\end{aligned}$$

On the other hand, for each fixed $t > 0$, we have

$$\begin{aligned} t^{-\theta}K(t, f; \vec{K}) &= \theta(1-\theta)\int_0^\infty r^{-\theta}\min(1, \frac{r}{t})K(t, f)\,\frac{dr}{r} \\ &\le \theta(1-\theta)\int_0^\infty r^{-\theta}K(r, f)\,\frac{dr}{r}. \end{aligned}$$

The last inequality follows from the fact that $\min(1, \frac{r}{t})K(t, f) \le K(r, f)$ for all quasi concave functions (in particular, for the K-functional see [B-L, p. 38] or [B-K, p. 290]). It follows that

$$\begin{aligned} \|f\|_{K_{(\theta,\infty)}} &= \sup_{\frac{1}{s}\le t\le s} t^{-\theta}K(t, f; \vec{K}) \\ &= \sup_{\frac{1}{s}\le t\le s}\left(\theta(1-\theta)\int_0^\infty r^{-\theta}\min(1, \frac{r}{t})K(t, f)\,\frac{dr}{r}\right) \\ &\le \theta(1-\theta)\int_0^\infty r^{-\theta}K(r, f)\,\frac{dr}{r} = \|f\|_{K_{(\theta,1)}} \end{aligned}$$

and thus

$$\|f\|_{K_{(\theta,\infty)}} \le \|f\|_{K_{(\theta,1)}}.$$

Hence by Proposition 2.7 and Theorem 3.1(ii), we conclude that

$$d(K_{(\theta,1)}, K_{(\theta,\infty)})(s) \le 1 + 2\theta(1-\theta)\log s,$$

giving the required upper bound. To show that $1 + 2\theta(1-\theta)\log s$ is also a lower bound we consider the function $g(t) = t^\theta$ defined on the interval $[\frac{1}{s}, s]$ which is clearly in $\Sigma(\vec{K})$. Since by definition

$$\max\left(\frac{\|g\|_{K_{(\theta,\infty)}}}{\|g\|_{K_{(\theta,1)}}}, \frac{\|g\|_{K_{(\theta,1)}}}{\|g\|_{K_{(\theta,\infty)}}}\right)$$

gives a lower bound, we see that $\|g\|_{K_{(\theta,1)}} = 1 + 2\theta(1-\theta)\log s$ and $\|g\|_{K_{(\theta,\infty)}} = 1$, and we conclude that

$$\max\left(\frac{\|g\|_{K_{(\theta,1)}}}{\|g\|_{K_{(\theta,\infty)}}}, \frac{\|g\|_{K_{(\theta,\infty)}}}{\|g\|_{K_{(\theta,1)}}}\right) = 1 + 2\theta(1-\theta)\log s,$$

thus verifying that the lower bound and the upper bound coincide. Hence

$$d(K_{(\theta,1)}, K_{(\theta,\infty)})(s) = 1 + 2\theta(1-\theta)\log s.$$

□

6. Pseudo-metric

We shall conclude by indicating how our distance function can be used to define a (pseudo-) metric on the set of interpolation functors.

Proposition 6.1. *The set $\mathcal{F}$ of all interpolation functors becomes a pseudo-metric space if we define (for interpolation functors F, G)*

$$\delta(F,G) = \int_1^\infty \frac{\log d(F,G)(s)}{s^2}\,ds.$$

Furthermore we have $\delta(F,G) \leq 1$ for all F and G in $\mathcal{F}$.

Proof. Since we obviously have $\delta(F,F) = 0$ and $\delta(F,G) = \delta(G,F)$ we shall only prove the triangle inequality. We observe therefore that it follows from Theorem 2.8 that for every $s \geq 1$ we have $\log d(F,H)(s) \leq \log d(F,G)(s) + \log d(G,H)(s)$ and integrating this the triangle inequality for δ follows. It also follows from Theorem 2.8 that $d(F,G)(s) \leq s$ and therefore

$$\delta(F,G) \leq \int_1^\infty \frac{\log s}{s^2}\,ds = 1.$$

□

Problem. It would be interesting to know if the topology on $\mathcal{F}$ induced by this metric is the compact topology referred to in the Introduction.

References

[B-L] J. Bergh and L. Löfström, *Interpolation Spaces,* Springer-Verlag, Berlin, 1976.
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