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Some dual Tauberian embeddings

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Abstract. We show that if the famous construction of Davis, Figiel, Johnson and Pełczyński [1] is worked out on a weak-star compact set in a dual Banach space, then the resulting Banach space is a dual space. Next, we apply this result to show that either a set is weak-star thick or it is contained in the operator range of a weak-star continuous Tauberian embedding. This result improves and, in some sense, completes the theory of thin sets and surjectivity described in [8].

1. Introduction

It was a longstanding problem whether every weakly compact operator $T : X \to Y$ from a Banach space X into a Banach space Y factors through a reflexive Banach space. More precisely, the problem was to find a reflexive space Z and (weakly compact) operators $S_1 : X \to Z$, $S_2 : Z \to Y$ such that $T = S_2 S_1$.

Suppose K is a closed, bounded and symmetric subset of a Banach space. The Banach disc on K, $BD(K)$, is the normed linear space $\operatorname{span} K$ with K as the unit ball. It is well-known and easy to show that $BD(K)$ is complete. A natural way of solving the factorization problem is to build a reflexive Banach space Z "inside" Y and put $T = j \circ \hat{T}$, where $\hat{T}$ is T viewed as an operator into Z and j is the embedding of Z into Y.

Note that the Banach disc with the unit ball $\overline{TB_X}$ does not solve the factorization problem. This is easily seen by taking X non-reflexive and T weakly compact and 1-1. Then, by construction and by a classical lemma due to Banach, T is onto the Banach disc $BD(\overline{TB_X})$. Hence, by the Open Mapping Theorem, $BD(\overline{TB_X})$ is isomorphic to the non-reflexive space X.

In the important paper [1], Davis, Figiel, Johnson and Pełczyński solved the factorization problem by introducing a general technique (below called

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the DFJP-technique) to transform a given relatively weakly compact, convex and symmetric set K in a Banach space X into a weakly compact, convex and symmetric set $\tilde{K} \subset X$ such that $BD(\tilde{K})$ is reflexive. Many additional properties come along for free: the most basic of them are listed below. Recall that an operator $T : Z \to X$ is called Tauberian if $T^{**-1}(X) \subset Z$, that is, a Tauberian operator is "opposite" to a weakly compact operator. The theory of Tauberian operators is summarized very clearly in [3], while the use of Tauberian operators in the DFJP-technique was investigated in [6].

Theorem 1.1. *Let K be a convex, symmetric and bounded subset of a Banach space X. Let $\tilde{K}$ be the result after using the DFJP-technique on K. Let $Z = BD(\tilde{K})$ and let $j : Z \to X$ be the natural embedding. Then*

(a) *$\tilde{K}$ is closed in X, thus $BD(\tilde{K})$ is a complete normed space.*
(b) *$j, j^{**}, \dots$ are all Tauberian embeddings, so $j^{*}, j^{***}, \dots$ all have norm-dense ranges.*
(c) *$\tilde{K}$ is weakly compact in X exactly when K is relatively weakly compact in X.*
(d) *The weak topologies in X and Z coincide on $\tilde{K}$, hence by* (a), *Z is reflexive exactly when K is relatively weakly compact.*

The solution of the factorization problem now easily follows by taking $K = TB_X$. It is an important observation that the DFJP-technique uses no results beyond a first year graduate course in functional analysis. The DFJP-technique was re-examined in [5]; the main conclusions can be summarized as follows:

Theorem 1.2. *Let the assumptions be as in the theorem above but now assume that $K \subset B_X$.*

(a) *The DFJP-technique can be adjusted so that $K \subset \tilde{K} \subset B_X$.*
(b) *$j|_K$ is a norm-norm homeomorphism.*
(c) *If K is respectively contained in a finite dimensional subspace, is compact or separable, then j is respectively finite rank, compact or separably valued.*

The factorization of an operator $T : X \to Y$ can now be expressed as $T = j \circ \hat{T}$, where $\|j\| = 1$ and $\|\hat{T}\| = \|T\|$. Moreover j and $\hat{T}$ are finite rank, compact, weakly compact or separably valued exactly when T has the same property. In [5] it was also shown that, for weakly compact operators, there is a factorization with the 1-1 operator first, that is $T = \tilde{T} \circ i$. This is shown by doing the adjusted DFJP-technique on $T^{*}B_{Y^{*}}$.

We now address the following problem:

Suppose K is a weak-star compact, convex and symmetric set in a dual Banach space. Which conclusions can be drawn in addition to the general conclusions described above?

In [1, Lemma 1 (vi)] it is shown that $\tilde{K}$ is weak-star compact in X^* whenever K is. We will see that $BD(\tilde{K})$ is a dual space.

2. DFJP-technique on a weak-star compact set

We start by repeating the main steps in the construction of $\tilde{K}$ from K. Let $K \subset B_X$. First, for every $k \in \mathbb{N}$ and $a > 1$, let

$$B_k = a^{\frac{k}{2}} K + a^{-\frac{k}{2}} B_X$$

and note that the gauge (Minkowski functional) on B_k defines an equivalent norm on X, say $\|\cdot\|_k$. Let $\tilde{K}$ be the set

$$\tilde{K} = \left\{ x \in X \ : |||x||| \stackrel{def}{=} \left(\sum_{k=1}^{\infty} \|x\|_k^2 \right)^{\frac{1}{2}} \leq 1 \right\}.$$

The set $\tilde{K}$ is by definition a countable intersection of X-closed sets and is thus closed in X. Therefore $Z = BD(\tilde{K})$ is a Banach space. This is the DFJP-construction except that Davis, Figiel, Johnson and Pełczyński used $a = 4$. A simple calculation given in [5] shows that

$$\frac{1}{f(a)} K \subset \tilde{K} \subset B_X \quad \text{where} \quad f(a) = \sqrt{\sum_{n=1}^{\infty} \frac{a^n}{(a^n+1)^2}}.$$

There is a unique a such that $f(a) = 1$. We choose this a for the future, and thus obtain $K \subset \tilde{K} \subset B_X$. In [8] this relation between sets was further investigated and a convexity argument was given to show that $\tilde{K}$ is not norming unless K is norming. Recall that a set $A \subset X$ is called (δ-)norming (for X^*) if, for some $\delta > 0$, $\overline{\text{conv}}(\pm A) \supset \delta B_X$.

It is immediate from the construction of $\tilde{K}$ that $Z = BD(\tilde{K})$ is isometrically isomorphic to the "diagonal" subspace $D = \{\underline{d} = (x, x, x, ...) \ : x \in X\}$ in the Banach space

$$S = \left(\sum_{k=1}^{\infty} BD(B_k) \right)_{\ell_2}.$$

The coincidence of the relative weak topologies on $\tilde{K}$ from Z and X can be shown by considering $\tilde{K}$ in D. We will now show that if K is a weak-star compact subset of a dual space X^*, then S is a dual space and D is a weak-star closed subspace. Thus we will obtain the following result.

Proposition 2.1. *Suppose K is a weak-star compact, convex and symmetric subset of a space Y which is the dual of a Banach space X. Then $Z = BD(\tilde{K})$ is the dual of a Banach space and the natural embedding operator $j : Z \to Y$ is weak-star continuous, that is, j is a dual operator.*

Proof. Referring to the discussion above, we first show that S is a dual space. Since B_k is weak-star closed in X^*, by a well-known result (see e.g. [2, Fact 5.4]), the spaces $BD(B_k)$ are all dual spaces, say X_k^*. Let X_k be a pre-dual of X_k^*. Then a standard argument shows that

$$S = \left(\sum_{k=1}^{\infty} BD(B_k)\right)_{\ell_2} = \left(\sum_{k=1}^{\infty} X_k^*\right)_{\ell_2} = \left(\sum_{k=1}^{\infty} X_k\right)_{\ell_2}^*,$$

which shows that S is the dual of a Banach space, and we know the form of a pre-dual of S.

We now prove that D is weak-star closed. To this end, let $(\underline{d}_\alpha) \subset D$ be a weak-star convergent net and suppose $\underline{s}$ is the limit. We know that $\underline{s} \in S$, so we need to show that $\underline{s} \in D$. Write

$$\underline{d}_\alpha = (x_\alpha^*, x_\alpha^*, ...) \quad \text{and} \quad \underline{s} = (x_1^*, x_2^*, ...).$$

Every element in the pre-dual of S can be written $(x_1, x_2, ...)$. The weak-star convergence $\underline{d}_\alpha \to \underline{s}$ can thus be written

$$\sum_{i=1}^{\infty} x_\alpha^*(x_i) \to \sum_{k=1}^{\infty} x_i^*(x_i)$$

for all $(x_1, x_2, ...) \in (\sum_{k=1}^{\infty} X_k)_{\ell_2}$. In particular, the convergence must hold for $(x, 0, 0, ...)$, $(0, x, 0, ...)$,... which shows at once that

$$x_\alpha^*(x) \to x_i^*(x)$$

for all $x \in X$ and every $i = 1, 2, 3, ...$ Thus $x_1^* = x_2^* = ...$ and so $\underline{s} \in D$.

It remains to verify that j is an adjoint operator. Let $\phi : Z \to D$ be the isometry which identifies Z and D. We have just shown that ϕ is weak-star continuous. Let π be the projection onto the first coordinate in S. Since weak-star convergence in S implies coordinatewise convergence, π is weak-star continuous. Finally, let i be the isomorphism $X_1^* \to X^*$. Then also i is weak-star continuous. Since $j = i \circ \pi \circ \phi$, j is weak-star continuous. □

3. An application to the theory of thin and thick sets

The theory of thick and thin sets in Banach spaces was introduced by M.I. Kadets and V.P. Fonf in [4]. They studied conditions to be put on a *bounded* set A in a Banach space X such that any operator range containing A is X. Say that such a set A has the surjectivity property. Here "operator range" means the range of some linear, continuous operator from some Banach space Y. The difficulty is of course that operators may very well have dense ranges without being onto. Easy examples show that we may even have $\overline{\text{conv}}(A) = B_X$ and still A does not have the surjectivity property (take, for instance, $A = (\pm e_i) \subset \ell_1$ and let T be some compact operator onto A).

The main result in [4] is that A has the surjectivity property if and only if A can not be written as an increasing, countable union of non-norming sets. Such a set is called thick. In [7] is was shown that (still on bounded sets) thickness is equivalent to the following property: every family in X^* which is pointwise bounded on A is uniformly bounded. This property is called the boundedness property.

In [8] the assumption that A is bounded was removed. Thus it was shown that a set is thick if and only if it has the surjectivity property if and only if it has the boundedness property. By classical results of Banach it follows that every set of the second Baire-category is thick.

It was also sought for a similar result if A is a subset of the dual X^* of a Banach space X and it was shown that the corresponding thickness condition is weak-star thickness. That is, the set is weak-star thick if and only if it is not a countable increasing union of sets which are non-norming for the pre-dual X.

In [8] is was further investigated how restrictive conditions one can put on the operator whose range is containing A, when A is thin. Using the DFJP-technique it was shown that if A is thin, then there is a Tauberian embedding onto A but not onto X. We were not able to show the corresponding result for weak-star thin sets, since we did not have Proposition 2.1. Now we can obtain a result to complete the theory in a satisfactory way. In the following theorem, note that interesting cases are when the set A is norming. We give a complete proof even though part (a) of the theorem was already proved in [8].

Theorem 3.1. *Let X be a real Banach space.*

(a) *Suppose A is a thin subset of X. Then there exist a Banach space Y and a Tauberian embedding $T : Y \to X$ such that $TY \supset A$, but T is not onto.*

(b) *Suppose A is a weak-star thin subset of X^*. Then there exist a dual Banach space Y^* and a dual Tauberian embedding $T^* : Y^* \to X^*$ such that $T^*Y^* \supset A$, but T^* is not onto.*

Proof. Assume without loss of generality that A is convex and symmetric.
First assume in (a) that A is non-norming and bounded. Then, since $\tilde{A}$ is not norming unless A is, the operator range formed by $BD(\tilde{A})$ and the natural embedding $j : BD(\tilde{A}) \to X$ does the trick. Next, suppose A is norming, but thin. We will find a set $\hat{A}$ such that $\hat{A}$ is bounded and not norming but has the same linear span as A, and (a) will be proved.

Let (A_i) be an increasing family of non-norming subsets of A such that $A = \cup_{i=1}^{\infty} A_i$. Since $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} A_i \cap i \cdot B_X$ and $A_i \cap i \cdot B_X$ is non-norming, we may assume each A_i to be contained in $i \cdot B_X$. Put $C_1 = A_1$ and $C_i = A_i \setminus A_{i-1}$. Define

$$\hat{A} = \overline{\text{conv}}(\pm \bigcup_{i=1}^{\infty} \frac{C_i}{i^2}).$$

Then $\hat{A}$ is closed, convex, symmetric and obviously has the same span as A. We now show that $\hat{A}$ is non-norming for X^*. Since $\hat{A}$ is closed, convex and symmetric, we only have to show that $\hat{A}$ contains no ball at the origin. We thus show that for every $\varepsilon > 0$, there is an $f \in S_{X^*}$ such that $\sup f(\hat{A}) < \varepsilon$. To do this, let $\varepsilon > 0$ and take j such that $1/j < \varepsilon$. Since A_j is not a norming set, there is a functional $f \in S_{X^*}$ such that $\sup_{x \in A_j} |f(x)| < \varepsilon$. By the definition of $\hat{A}$,

$$\sup_{x \in \hat{A}} |f(x)| = \sup_i \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\}. \tag{1}$$

We devide the process of finding this supremum into two parts: $i \le j$ or $i > j$. If $i \le j$, then

$$\sup_i \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\} = \max_{1 \le i \le j} \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\} \le \sup_{x \in A_j} |f(x)| < \varepsilon.$$

If $i > j$, then, since $C_i \subset i \cdot B_X$,

$$\sup_i \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\} = \sup_{i>j} \left\{ \frac{1}{i^2} \sup_{x \in C_i} |f(x)| \right\} \le \sup_{i>j} \left\{ \frac{1}{i^2} \cdot i \right\} \le \frac{1}{j} < \varepsilon.$$

By (1), we are done.

We now prove (b). If A is not weak-star norming, A is of course not norming and, by Proposition 2.1, the proof is done by taking $j : BD(\tilde{A}) \to X^*$. The problem is when A is weak-star norming. Hovever, the argument used in (a) to reduce a thin, norming set to a non-norming set also works to reduce a weak-star thin, weak-star norming set to a non-weak-star norming one. The proof is complete. □

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