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Graph operations and categorical constructions

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ABSTRACT. Most of the usual binary graph operations from disjoint union up to the complete product are interpreted categorically, using the categories **Gra**, **CGra** and **EGra**. This way it is proved that these categories have coproducts, products and tensor products. As a consequence it turns out that the respective categories with strong morphisms **SGra** and **SEGra** do not admit any of these categorial constructions. It is shown that the functors derived from the respective tensor products and products in **Gra**, **CGra** and **EGra** have right adjoints.

Here we revisit the topic of graph operations (cf., for example, [5], [9], [13]) and their interpretations in graph categories (cf. [2], [9] and others). In considering three different types of morphisms for graphs we obtain three different categories which admit interpretations of the most common graph operations as coproducts, tensor products and products in the respective categories. For the latter two we also consider the associated functors and their right adjoints. Part of the results is folklore or can be deduced from [10]. But we will be very elementary and give all constructions and prototypes of the proofs explicitly.

By **Set** we denote the category of sets with mappings and their composition as morphisms.

1. Categories of graphs

We consider here finite undirected graphs G without multiple edges and without loops. The *vertex set* of G will be denoted by $V(G)$ or just G, the *edge set* by $E(G)$. If $x, y \in G$ are adjacent, we denote the *edge connecting x and y* by $\{x, y\}$ and write $\{x, y\} \in E(G)$.

Let G and H be graphs, $x, y \in G$. A mapping $f : V(G) \longrightarrow V(H)$ is called a *graph homomorphism* if $\{x, y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$.

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A bijective graph homomorphism f such that f^{-1} is also a graph homomorphism is called a *graph isomorphism.* A graph homomorphism f is called a *strong homomorphism* if $\{f(x), f(y)\} \in E(H)$ implies $\{x,y\} \in E(G)$. A mapping $f : V(G) \longrightarrow V(H)$ is called a *comorphism* if $\{f(x), f(y)\} \in E(H)$ implies $\{x,y\} \in E(G)$. It is clear that comomorphisms which are graph homomorphisms are exactly the strong homomorphisms.

From theoretical computer science we take another concept of morphisms which seems especially useful for data compression (compare, for example, A. Buldas' dissertation [1]). Namely, we call a mapping $f : V(G) \longrightarrow V(H)$ a *(strong) egamorphism* if for $x, y \in G$ one has $f(x) = f(y)$ or $\{x,y\} \in E(G)$ implies $\{f(x), f(y)\} \in E(H)$ (and $\{f(x), f(y)\} \in E(H)$ implies $\{x,y\} \in E(G)$).

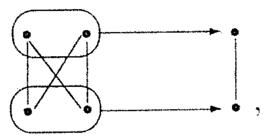
As a common term for these different classes of structure preserving and/or reflecting mappings we use morphism. It is easy to check that the classes of all 5 types of morphisms are closed with respect to composition. This gives the following 5 categories of graphs:

Gra	graphs with graph homomorphisms,
SGra	graphs with strong homomorphisms [3],
CGra	graphs with comorphisms [6],
EGra	graphs with egamorphisms [7],
SEGra	graphs with strong egamorphisms.

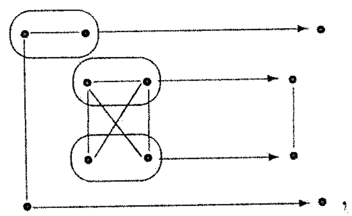
If G and H are two graphs, the set of all morphisms from G into H in the respective category is denoted by **Gra**(G, H), **SGra**(G, H) etc.

We give some examples of the different types of morphisms, encircled vertices are mapped onto the same image, arrows describe the mapping:

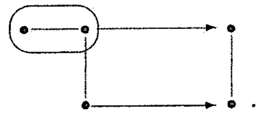
– strong homomorphism



– comorphism, not a graph homomorphism



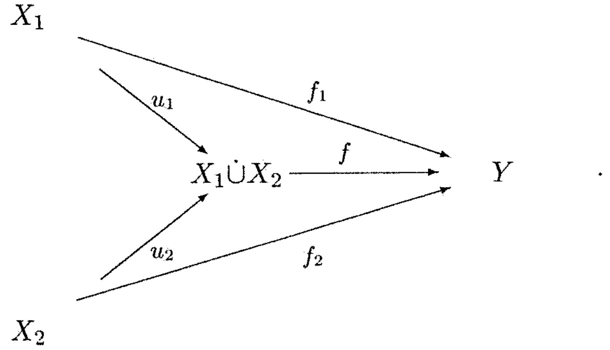
– (not strong) egamorphism, not a graph homomorphism



2. Coproducts, products and tensor products

Coproducts. Recall that the *coproduct* of objects A_1 and A_2 in a category $\mathbf{C}$ is a pair $((u_1, u_2), A_1 \coprod A_2)$ where $A_1 \coprod A_2$ is an object of $\mathbf{C}$ and $u_i : A_i \longrightarrow A_1 \coprod A_2$, $i \in \{1, 2\}$, are morphisms of $\mathbf{C}$ such that for any object B in $\mathbf{C}$ and any morphisms $f_i : A_i \longrightarrow B$, $i = 1, 2$, in $\mathbf{C}$ there exists exactly one morphism $f : A_1 \coprod A_2 \longrightarrow B$ in $\mathbf{C}$ so that $fu_i = f_i$, $i \in \{1, 2\}$.

If X_1 and X_2 are objects in **Set** then the disjoint union$X_1 \dot\cup X_2$ together with the embeddings $u_i : X_i \longrightarrow X_1 \dot\cup X_2$, $i \in \{1, 2\}$, is the coproduct of X_1 and X_2 in **Set**. This can be illustrated by the following commutative diagram



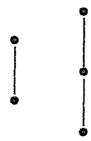
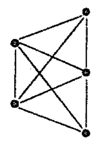
Recall now two compositions of graphs G_1 and G_2 such that the resulting graphs have the union $V(G_1) \cup V(G_2)$ as vertex sets.

The *disjoint union* $G_1 \dot\cup G_2$ of G_1 and G_2: $E(G_1 \dot\cup G_2) = E(G_1) \cup E(G_2)$. Other names: union, sum.

The *join* $G_1 + G_2$ of G_1 and G_2:

$$E(G_1 + G_2) = E(G_1) \dot\cup E(G_2) \cup \{\{x_1, x_2\} \mid x_1 \in V(G_1),\, x_2 \in V(G_1)\}.$$

If $G_1 =$ and $G_2 =$ then

$G_1 \dot{\cup} G_2$:  $G_1 + G_2$:  .

Proposition 2.1. (a) *The disjoint union* $G_1 \dot{\cup} G_2$ *together with the embeddings* $u_i : G_i \longrightarrow G_1 \dot{\cup} G_2$, $i \in \{1,2\}$, *is the coproduct of* G_1 *and* G_2 *in* **Gra**.

(b) *The disjoint union* $G_1 \dot{\cup} G_2$ *together with the embeddings* $u_i : G_i \longrightarrow G_1 \dot{\cup} G_2$, $i \in \{1,2\}$, *is the coproduct of* G_1 *and* G_2 *in* **EGra**.

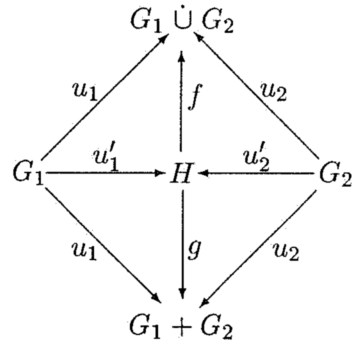
(c) *The join* $G_1 + G_2$ *together with the embeddings* $u_i : G_i \longrightarrow G_1 + G_2$, $i \in \{1,2\}$, *is the coproduct of* G_1 *and* G_2 *in* **CGra**.

Proof. Define f required in the definition of the coproduct by $f(x) = f_i(x)$ if $x \in V(G_i)$, $i \in \{1,2\}$, and check that f and u_i, $i \in \{1,2\}$, belong to the corresponding category. □

Corollary 2.2 ([4]). *In* **SGra** *and in* **SEGra** *there do not exist coproducts.*

Proof. We present the proof for **SGra**.

Suppose that $((u'_1, u'_2), H)$ is a coproduct of G_1 and G_2 in **SGra**. Since the embeddings $u_i : G_i \longrightarrow G_1 \dot{\cup} G_2$ and $u_i : G_i \longrightarrow G_1 + G_2$, $i \in \{1,2\}$, are strong homomorphisms, there exist strong homomorphisms $f : H \longrightarrow G_1 \dot{\cup} G_2$ and $g : H \longrightarrow G_1 + G_2$ such that the following diagram is commutative



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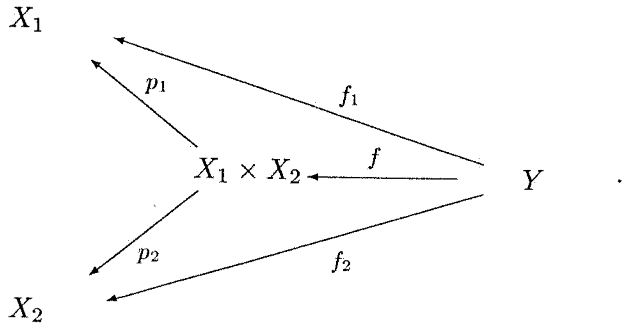
Take $x_1 \in G_1$ and $x_2 \in G_2$. Then, since $\{u_1(x_1), u_2(x_2)\} \in E(G_1 + G_2)$ and g is a comorphism, one has that $\{u'_1(x_1), u'_2(x_2)\} \in E(H)$. Since f is a graph homomorphism, the latter implies $\{u_1(x_1), u_2(x_2)\} \in E(G_1 \dot{\cup} G_2)$, contradicting the definition of the disjoint union. □

Note that the graphtheoretical edge sum can be described as an amalgamated coproduct, i.e., as a pushout.

Products. Recall that the *product* of objects A_1 and A_2 in a category $\mathbf{C}$ is a pair $(A_1 \prod A_2, (\pi_1, \pi_2))$ where $A_1 \prod A_2$ is an object of $\mathbf{C}$ and $\pi_i : A_1 \prod A_2 \longrightarrow A_i$, $i \in \{1, 2\}$, are morphisms of $\mathbf{C}$ such that for any object B in $\mathbf{C}$ and any morphisms $f_i : B \longrightarrow A_i$, $i \in \{1, 2\}$, in $\mathbf{C}$ there exists exactly one morphism $f : B \longrightarrow A_1 \prod A_2$ in $\mathbf{C}$ so that $\pi_i f = f_i$, $i \in \{1, 2\}$.

If X_1 and X_2 are objects in **Set** then the cartesian product $X_1 \times X_2$ together with the projections $p_i : X_1 \times X_2 \longrightarrow X_i$, $i \in \{1, 2\}$, is the product of X_1 and X_2 in **Set**. This can be illustrated by the following commutative diagram



Now we recall some compositions of graphs G_1 and G_2 such that the resulting graphs have the cartesian product $V(G_1) \times V(G_2)$ as vertex sets.

The *cross product* $G_1 \times G_2$ of G_1 and G_2:
$\{(x, i), (x', i')\} \in E(G_1 \times G_2)$ if and only if $\{x, x'\} \in E(G_1)$ and $\{i, i'\} \in E(G_2)$.

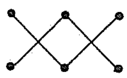
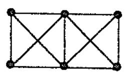
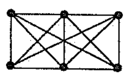
Other names: categorical product, conjunction, tensor product (a misleading name in view of Proposition 2.3).

The *box-cross product* $G_1 \boxtimes G_2$ of G_1 and G_2:

$$\{(x,i),(x',i')\} \in E(G_1 \boxtimes G_2) \text{ if } \begin{array}{l} x = x' \text{ and } \{i,i'\} \in E(G_2), \\ \text{or } \{x,x'\} \in E(G_1) \text{ and } i = i', \\ \text{or } \{x,x'\} \in E(G_1) \text{ and } \{i,i'\} \in E(G_2). \end{array}$$

The *disjunction* $G_1 \vee G_2$ of G_1 and G_2:
$\{(x, i), (x', i')\} \in E(G_1 \vee G_2)$ if and only if $\{x, x'\} \in E(G_1)$ or $\{i, i'\} \in E(G_2)$.

If $G_1 =$ and $G_2 =$ then

$G_1 \times G_2$:  $G_1 \boxtimes G_2$:  $G_1 \vee G_2$:  .

Proposition 2.3. (a) *The cross product* $G_1 \times G_2$ *together with the projections* $p_i : V(G_1) \times V(G_2) \longrightarrow V(G_i)$, $i \in \{1,2\}$, *is the product of* G_1 *and* G_2 *in* **Gra**.

(b) *The box-cross product* $G_1 \boxtimes G_2$ *together with the projections* $p_i : G_1 \boxtimes G_2 \longrightarrow G_i$, $i \in \{1,2\}$, *is the product of* G_1 *and* G_2 *in* **EGra**.

(c) *The disjunction* $G_1 \vee G_2$ *together with the projections* $p_i : G_1 \vee G_2 \longrightarrow G_i$, $i \in \{1,2\}$, *is the product of* G_1 *and* G_2 *in* **CGra**.

Proof. In all three cases we define f required in the definition of the product using the morphisms $f_i : H \longrightarrow G_i$ by $f(x) = (f_1(x), f_2(x))$ for $x \in V(H)$ and check that f and p_i, $i \in \{1,2\}$, belong to the corresponding category. We demonstrate this in the case of **CGra**.

If $\{p_1((x_1,x_2)), p_1((x_1',x_2'))\} \in E(G_1)$, i.e., $\{x_1, x_1'\} \in E(G_1)$, then $\{(x_1,x_2),(x_1',x_2')\} \in E(G_1 \vee G_2)$ by the definition of $E(G_1 \vee G_2)$. Hence p_1 belongs to **CGra**. Similarly, p_2 belongs to **CGra**.

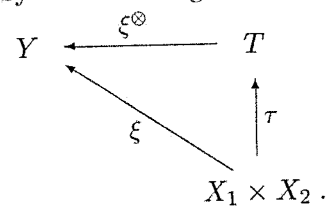
If for $f : H \longrightarrow G_1 \vee G_2$ corresponding to morphisms $f_i : H \longrightarrow G_i$, $i = 1,2$, one has $\{f(y), f(y')\} \in E(G_1 \vee G_2)$ for $y, y' \in H$, i.e., $\{(f_1(y), f_2(y)), (f_1(y'), f_2(y'))\} \in E(G_1 \vee G_2)$, then either $\{f_1(y), f_1(y')\} \in E(G_1)$ or $\{f_2(y), f_2(y')\} \in E(G_2)$, which both imply $\{y, y'\} \in E(H)$ since f_1, f_2 belong to **CGra**. Thus, f belongs to **CGra**. □

By an argument similar to the proof of Corollary 2.2 one obtains the following

Corollary 2.4. *In* **SGra** *and* **SEGgra** *products do not exist in general.* □

Tensor products. Let X_1, X_2 and Y be objects of a concrete category **C**. A mapping $\xi : X_1 \times X_2 \longrightarrow Y$ in **Set** such that $\xi(x_1, \ ) : X_2 \longrightarrow Y$ and $\xi(\ , x_2) : X_1 \longrightarrow Y$ for every $x_1 \in X_1$, $x_2 \in X_2$, belong to **C** is called a *tensorial mapping.* An object $T \in \mathbf{C}$ together with a tensorial mapping $\tau : X_1 \times X_2 \longrightarrow T$ is called the *tensor product of* X_1 *and* X_2 *in* **C** if for every $Y \in \mathbf{C}$ and every tensorial mapping $\xi : X_1 \times X_2 \longrightarrow Y$ there exists a unique *tensor induced morphism* $\xi^{\otimes} : T \longrightarrow Y$ in **C** such that $\xi^{\otimes}\tau = \xi$.

This can be illustrated by the following commutative diagram



Note that in this diagram only the upper line is in **C**, the rest belongs to **Set**.

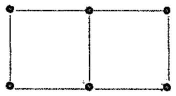
Now we consider two more compositions of graphs G_1 and G_2 such that the resulting graphs have the cartesian product $V(G_1) \times V(G_2)$ as vertex sets.

The *box product* $G_1 \,\Box\, G_2$ of G_1 and G_2:

$\{(x,i),(x',i')\} \in E(G_1 \,\Box\, G_2)$ if and only if $x = x'$ and $\{i,i'\} \in E(G_2)$, or $\{x,x'\} \in E(G_1)$ and $i = i'$.

Other names: product, cartesian product, cartesian sum (a misleading name in view of Proposition 3.1).

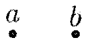
If $G_1 =$ and $G_2 =$ then

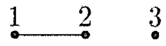
$G_1 \,\Box\, G_2$:  .

The *complete product* $G_1 * G_2$ of G_1 and G_2:

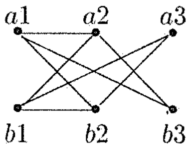
$$\{(x,i),(x',i')\} \in E(G_1 \times G_2) \text{ if } \{x,x'\} \in E(G_1) \text{ and } i = i', \\ \text{or } x = x' \text{ and } \{i,i'\} \in E(G_2), \\ \text{or } x \neq x' \text{ and } i \neq i'.$$

Other name: join product.

If $G_1 =$ 

and $G_2 =$ 

then

$G_1 * G_2$: 

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Theorem 2.5. (a) *The box product $G_1 \,\Box\, G_2$ together with the mapping $\tau = \mathrm{id}_{V(G_1)\times V(G_2)} : V(G_1) \times V(G_2) \longrightarrow G_1 \,\Box\, G_2$ is the tensor product of G_1 and G_2 in* **Gra**.

(b) *The box product $G_1 \,\Box\, G_2$ together with the mapping $\tau = \mathrm{id}_{V(G_1)\times V(G_2)} : V(G_1) \times V(G_2) \longrightarrow G_1 \,\Box\, G_2$, is the tensor product of G_1 and G_2 in* **EGra**.

(c) *The complete product $G_1 * G_2$ together with the mapping $\tau = \mathrm{id}_{V(G_1)\times V(G_2)} : V(G_1) \times V(G_2) \longrightarrow G_1 * G_2$ is the tensor product of G_1 and G_2 in* **CGra**.

Proof. (a): It is clear that τ is tensorial. For any graph H and any tensorial mapping $\xi : V(G_1) \times V(G_2) \longrightarrow H$ it is obvious that the morphism $\xi^{\otimes} : G_1 \,\Box\, G_2 \longrightarrow H$ required in the definition of the tensor is given by $\xi^{\otimes}\big((x_1,x_2)\big) = \xi\big((x_1,x_2)\big)$.

If $\{(x_1,x_2),(x_1',x_2')\} \in E(G_1 \Box G_2)$, then $\{x_1,x_1'\} \in E(G_1)$ and $x_2 = x_2'$, say. But then, $\{\xi(x_1,x_2),\xi(x_1',x_2)\} \in E(H)$, since ξ is tensorial. This means that $\xi^{\otimes} \in \mathbf{Gra}(G_1 \Box G_2, H)$.

(b): Analogous to (a).

(c): If $\{\tau((x_1,x_2)),\tau((x_1',x_2))\} \in E(G_1 * G_2)$, i.e., $\{(x_1,x_2),(x_1',x_2)\} \in E(G_1 * G_2)$, then it follows from the definition of the complete product that $\{x_1,x_1'\} \in E(G_1)$. Thus, $\tau(\ ,x_2)$ belongs to **CGra**. Similarly, $\tau(x_1,\ )$ belongs to **CGra**. Hence τ is tensorial.

It is clear that $\xi^{\otimes} = \xi$ for any tensorial mapping $\xi : V(G_1)\times V(G_2) \longrightarrow H$, $H \in$ **CGra**.

Let $\{\xi^{\otimes}((x_1,x_2)),\xi^{\otimes}((x_1',x_2'))\} \in E(H)$, i.e., $\{\xi((x_1,x_2)),\xi((x_1',x_2'))\} \in E(H)$. If $x_1 = x_1'$ then $\{x_2,x_2'\} \in E(G_2)$ since $\xi(x_1,\ )$ is tensorial, or if $x_2 = x_2'$ then $\{x_1,x_1'\} \in E(G_1)$ since $\xi(\ ,x_2)$ is tensorial. Thus in these cases $\{(x_1,x_2),(x_1',x_2')\} \in E(G_1 * G_2)$ by the definition of $G_1 * G_2$. Finally, if $x_1 \neq x_1'$ and $x_2 \neq x_2'$, then $\{(x_1,x_2),(x_1',x_2')\} \in E(G_1 * G_2)$ again by the definition of $G_1 * G_2$. □

By an argument similar to the proof of Corollary 2.2 one obtains the following

Corollary 2.6. *In* **SGra** *and* **SEgra** *tensor products do not exist in general.*

It is a straightforward observation that graph products and tensor products give covariant functors. For example, the box product defines for $G \in$ **Gra** the functor

$$\begin{array}{rccccc}
G\Box - : & \mathbf{Gra} & \longrightarrow & & \mathbf{Gra} & \\
& H_1 & \longmapsto & & G \;\Box\; H_1 & \\
& & & & & (x,y_1) \\
& \big\downarrow \varphi & \longmapsto & G\Box\varphi := & \big\downarrow & \big\downarrow \\
& & & & & (x,\varphi(y_1)) \\
& H_2 & \longmapsto & & G \;\Box\; H_2 & .
\end{array}$$

3. Tensor functors

It is known from other categories that tensor functors are left adjoint to certain Hom-functors (cf. for example [12]).

We define three graph operations which give functors right adjoint to the tensor functors (i.e., functors defined by tensor products in categories **Gra**, **EGra** and **CGra**). Although these constructions are different we use a common name "diamond product" for all of them and distinguish them adding words showing for which graph category they generate the right adjoint to the tensor functor. The diamond product in **Gra** has been investigated in [11].

Construction 3.1. (a) The *diamond product* $G \Diamond H$ of two graphs G and H in **Gra** is defined by

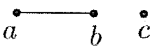
$$V(G \Diamond H) = \mathbf{Gra}(G, H), \text{ the set of graph homomorphisms from } G \text{ to } H,$$
$$E(G \Diamond H) = \Big\{\{\alpha, \beta\} \mid \{\alpha(x), \beta(x)\} \in E(H) \text{ for all } x \in G\Big\}.$$

(b) The *diamond product* $G \Diamond H$ of two graphs G and H in **EGra** is defined by

$$V(G \Diamond H) = \mathbf{EGra}(G, H), \text{ the set of egamorphisms from } G \text{ to } H,$$
$$E(G \Diamond H) = \Big\{\{\alpha, \beta\} \mid \{\alpha(x), \beta(x)\} \in E(H) \text{ for all } x \in G\Big\}.$$

(c) The *diamond product* $G \Diamond H$ of two graphs G and H in **CGra** is defined by

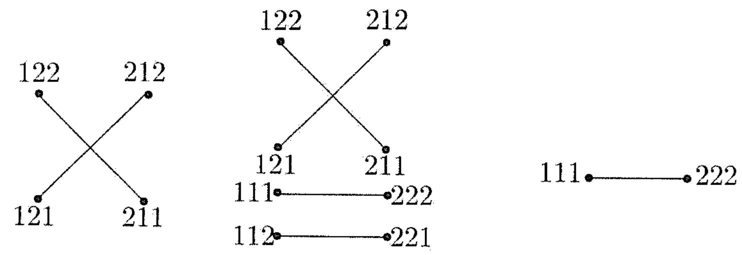
$$V(G \Diamond H) = \mathbf{CGra}(G, H), \text{ the set of comorphisms from } G \text{ to } H,$$
$$E(G \Diamond H) = \Big\{\{\alpha, \beta\} \mid \exists x \in G \text{ such that } \{\alpha(x), \beta(x)\} \in E(H)\Big\}.$$

For $G =$ 

and $H =$ 

we get $G \Diamond H$

in **Gra**: in **EGra**: in **CGra**:



Here the vertex ijk denotes the morphism taking a to i, b to j and c to k, $i, j, k \in \{1, 2\}$.

The diamond products define covariant functors in the respective categories. For example, for **Gra** one has:

$$
\begin{array}{llcl}
G \diamond - : & \mathbf{Gra} & \longrightarrow & \mathbf{Gra} \\
& H_1 & \longmapsto & G \diamond H_1 \\
& \Big\downarrow \varphi & \longmapsto & G \diamond \varphi := \quad \Big\downarrow \; \overset{\alpha}{\underset{\varphi\alpha}{\Big\downarrow}} \\
& H_2 & \longmapsto & G \diamond H_2 \, .
\end{array}
$$

Recall the definition of a natural transformation of functors and of the related freeness from [8] or [12] or any other book on categories and functors. Both definitions are formulated explicitly in the proof of the first part of the following theorem.

Theorem 3.2. (a) *The box functor $G \square -$ is left adjoint to the diamond functor $G \diamond -$ in* **Gra**,

$$(G \square -) \dashv (G \diamond -) \; .$$

(b) *The box functor $G \square -$ is left adjoint to the diamond functor $G \diamond -$ in* **EGra**,

$$(G \square -) \dashv (G \diamond -) \; .$$

(c) *The complete functor $G * -$ is left adjoint to the diamond functor $G \diamond -$ in* **CGra**,

$$(G * -) \dashv (G \diamond -) \; .$$

Proof. (a): We have to show that

(1) there exists a natural transformation

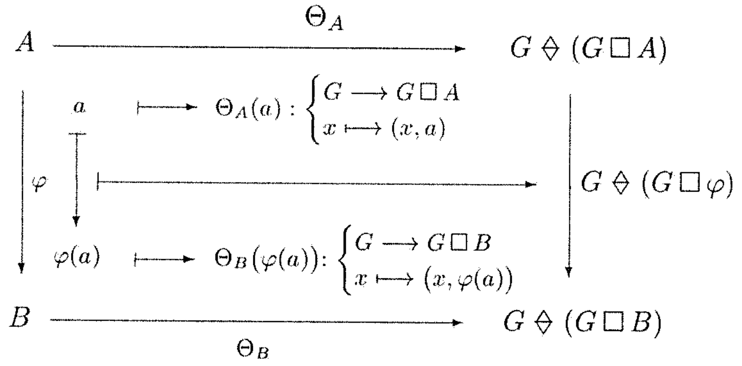
$$\Theta : \mathrm{Id}_{\mathbf{Gra}}(-) \longrightarrow (G \diamond -)(G \square -) = G \diamond (G \square -)$$

where $\mathrm{Id}_{\mathbf{Gra}}(-)$ denotes the identity functor on **Gra** and

(2) for every $A \in \mathbf{Gra}$ the pair $(\Theta_A, G \square A)$ is $(G \diamond -)$-free over A.

Proof of (1): 1. Consider the following rectangle which contains the definition of

$\Theta_A(a)$ for $A \in \mathbf{Gra}$ and $a \in A$:



This diagram is commutative for any morphism $\varphi : A \longrightarrow B$ in **Gra** as the following computation for all $a \in A$ and all $x \in G$ shows:

$$\begin{aligned} \big((G \diamond (G \square \varphi)\big)\big(\Theta_A(a)\big)(x) &= \big(G \diamond (\mathrm{id}_G \square \varphi)\big)\big(\Theta_A(a)\big)(x) \\ &= \big((\mathrm{id}_G \square \varphi) \circ \Theta_A(a)\big)(x) \\ &= (\mathrm{id}_G \square \varphi)(\Theta_A(a)(x)) = (\mathrm{id}_G \square \varphi)(x, a) \\ &= (x, \varphi(a)) = (\Theta_B(\varphi(a)))(x) \ . \end{aligned}$$

2. Since

$$\big\{\Theta_A(a)(x), \Theta_A(a)(x')\big\} = \big\{(x, a), (x', a)\big\} \in E(G \square A) \ ,$$

for $\{x, x'\} \in E(G)$ one has $\Theta_A(a) \in G \diamond (G \square A) = \mathbf{Gra}(G, G \square A)$.

If $\{a, a'\} \in E(A)$ then for all $x \in G$ we get

$$\big\{\Theta_A(a)(x), \Theta_A(a')(x)\big\} = \big\{(x, a), (x, a')\big\} \in E(G \square A)$$

by the definition of $G \square A$ and thus

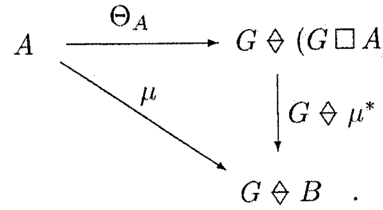
$$\big\{\Theta_A(a), \Theta_A(a')\big\} \in E\big(G \diamond (G \square A)\big) \ .$$

Thus Θ_A belongs to **Gra** and so we have that Θ is a natural transformation.

Proof of (2): 1. Take $A \in \mathbf{Gra}$. To show that $(\Theta_A, G \square A)$ is $(G \diamond -)$-free over A, for every $B \in \mathbf{Gra}$ and every $\mu : A \longrightarrow G \diamond B$ in **Gra** define

$$\mu^* : \begin{cases} G \square A & \longrightarrow B \\ (x, a) & \longmapsto \mu(a)(x). \end{cases}$$

Then the following triangle is commutative



Indeed, for $a \in A$ and $x \in G$ we have

$$\begin{aligned}((G \between \mu^*)(\Theta_A(a))(x) &= (\mu^* \circ \Theta_A(a))(x) = \mu^* \circ ((\Theta_A(a))(x)) \\ &= \mu^*((x,a)) = \mu(a)(x) .\end{aligned}$$

2. Assume $\{(x,a),(x',a'\} \in E(G \square A)$. If $\{x,x'\} \in E(G)$ and $a = a'$ then $\{\mu^*((x,a)),\ \mu^*((x',a))\} = \{\mu(a)(x), \mu(a)(x')\} \in E(B)$ since $\mu(a) \in$ **Gra**(G,B). If $\{a,a'\} \in E(A)$ and $x = x'$ then $\{\mu^*((x,a)),\ \mu^*((x,a'))\} = \{\mu(a)(x), \mu(a')(x)\} \in E(B)$ since $\{\mu(a), \mu(a')\} \in E(G \between B)$. Thus $\mu^* \in$ **Gra**$(G \square A, B)$. So we have proved that the pair $(\Theta_A, G \square A)$ is $(G \between -)$-free over A.

(b): Analogous to (a).

(c): We follow the scheme of the proof of (a).

(1): 1. The definition of the mapping $\Theta_A : A \longrightarrow (G \between -)(G * -)$ for $A \in$ **CGra** and the proof of commutativity of the corresponding diagram are similar to those of (a).

2. If $\{\Theta_A(a)(x), \Theta_A(a)(x')\} = \{(x,a),(x',a)\} \in E(G \square A)$ then the definition of the complete product implies $\{x,x'\} \in E(G)$. Thus $\Theta_A(a) \in G \between (G * A) =$ **CGra**$(G, G * A)$.

If $\{\Theta_A(a), \Theta_A(a')\} \in E(G \between (G * A))$, i.e., there exists $x \in V(G)$ such that $\{\Theta_A(a)(x), \Theta_A(a')(x)\} = \{(x,a),(x,a')\} \in E(G * A)$, then the definition of the complete product implies $\{a,a'\} \in E(G)$. Thus Θ_A belongs to **CGra**, and we have that Θ is a natural transformation.

(2): 1. The definition of the mapping $\mu^* : G * A \longrightarrow B$ for $\mu : A \longrightarrow G \between B$ in **CGra** and the proof of commutativity of the corresponding triangle are similar to those of (a).

2. If $\{\mu^*((x,a)), \mu^*((x',a'))\} = \{\mu(a)(x), \mu(a')(x'))\} \in E(B)$ then $a = a'$ implies $\{x,x'\} \in E(G)$ since $\mu(a)$ belongs to **CGra**, and $x = x'$ implies $\{a,a'\} \in E(A)$ since in this case $\{\mu(a), \mu(a')\} \in E(G \between B)$ and μ belongs to **CGra.** Thus in these cases $\{(x,a),(x',a')\} \in E(G * A)$ by the definition of $G * A$. Finally, if $x \neq x'$ and $a \neq a'$ then $\{(x,a),(x',a')\} \in E(G * A)$ again by the definition of $G * A$. Thus $\mu^* \in$ **CGra**$(G * A, B)$ and we have that $(\Theta_A, G * A)$ is $(G \between -)$-free over A. □

4. Right adjoints to product functors

In this section we present three more graph operations which give functors right adjoint to the product functors (i.e., functors defined by products in categories **Gra**, **EGra** and **CGra**). Again, although these constructions are different we use a common name "power product" for all of them and distinguish them by adding words showing for which graph category they generate the right adjoint to the product functor.

The power product in **Gra** has been investigated in [11].

Construction 4.1. (a) The *power product* $G \downarrow H$ of two graphs G and H in **Gra** is defined by

$$V(G \downarrow H) = \mathrm{Map}(G, H), \text{ the set of mappings from } G \text{ to } H\,,$$

$$E(G \downarrow H) = \Big\{ \{\alpha, \beta\} \,\big|\, \alpha \neq \beta,\ \{\alpha(x), \beta(x')\} \in E(H) \text{ for all } \{x, x'\} \in E(G) \Big\}.$$

(b) The *power product* $G \downarrow H$ of two graphs G and H in **EGra** is defined by

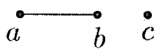
$$V(G \downarrow H) = \mathbf{EGra}(G, H)\,,$$

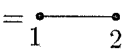
$$E(G \downarrow H) = \Big\{ \{\alpha, \beta\} \,\big|\, \{\alpha(x), \beta(x')\} \in E(H) \text{ for all } \{x, x'\} \in E(G) \text{ and } \{\alpha(x), \beta(x)\} \in E(H) \text{ for all } x \in G \Big\}.$$

(c) The *power product* $G \downarrow H$ of two graphs G and H in **CGra** is defined by

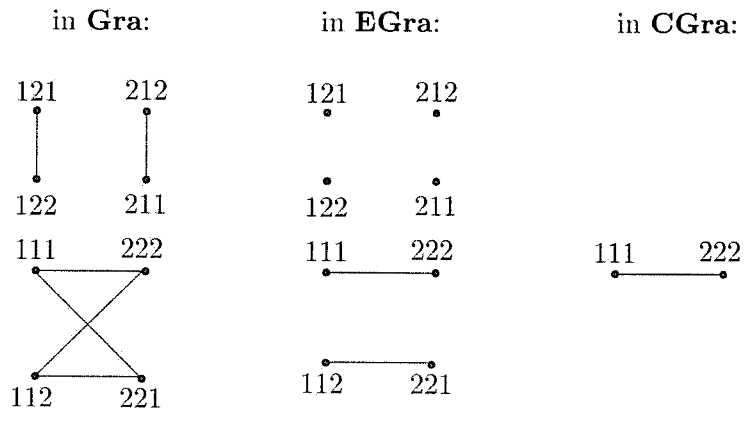
$$V(G \downarrow H) = \mathbf{CGra}(G, H)\,,$$

$$E(G \downarrow H) = \Big\{ \{\alpha, \beta\} \,\big|\, \exists x, x' \in G\,,\ \{\alpha(x), \beta(x')\} \in E(H)\,, \{x, x'\} \notin E(G) \Big\}.$$

For $G =$ 

and $H =$ 

we get $G \downarrow H$



The power products define covariant functors in the respective categories by a similar rule as diamond functors.

Theorem 4.2. (a) *The cross functor* $G \times -$ *is left adjoint to the power functor* $G \downarrow -$ *in* **Gra**,

$$(G \times -) \dashv (G \downarrow -) .$$

(b) *The box-cross functor* $G \boxtimes -$ *is left adjoint to the power functor* $G \downarrow -$ *in* **EGra**,

$$(G \square -) \dashv (G \downarrow -) .$$

(c) *The disjunction functor* $G \vee -$ *is left adjoint to the power functor* $G \downarrow -$ *in* **CGra**,

$$(G \vee -) \dashv (G \downarrow -) .$$

Proof. The proof of the cases (a) and (b) follows the scheme of the proof of (a) of Theorem 3.2. We present the proof of (c).

(1): 1. The definition of the mapping $\Theta_A : A \longrightarrow (G \downarrow -)(G \vee -)$ for $A \in$ **CGra** and the proof of commutativity of the corresponding diagram are similar to those of (a) of Theorem 3.2.

2. If $\{\Theta_A(a)(x), \Theta_A(a)(x')\} = \{(x,a),(x',a)\} \in E(G \vee A)$ then the definition of the disjunction implies $\{x,x'\} \in E(G)$. Thus $\Theta_A(a) \in G \downarrow (G \vee A) = \mathbf{CGra}(G, G \vee A)$.

If $\{\Theta_A(a), \Theta_A(a')\} \in E(G \downarrow (G \vee A)$, i.e., there exist $x, x' \in V(G)$ such that $\{\Theta_A(a)(x), \Theta_A(a')(x')\} = \{(x,a),(x',a')\} \in E(G \vee A)$ but $\{x,x'\} \notin E(G)$, then the definition of the disjunction implies $\{a,a'\} \in E(G)$. Thus Θ_A belongs to **CGra**, and we have that Θ is a natural transformation.

(2): 1. The definition of the mapping $\mu^* : G \vee A \longrightarrow B$ for $\mu : A \longrightarrow G \downarrow B$ in **CGra** and the proof of commutativity of the corresponding triangle are similar to those of (a) of Theorem 3.2.

2. If $\{\mu^*((x,a)), \mu^*((x',a'))\} = \{\mu(a)(x), \mu(a')(x'))\} \in E(B)$ then $\{x,x'\} \notin E(G)$ implies $\{\mu(a), \mu(a')\} \in E(G \downarrow B)$ by the definition of the power product in **CGra**. Since μ belongs to **CGra**, the latter implies $\{a,a'\} \in E(A)$. Thus $\{(x,a),(x',a')\} \in E(G \vee A)$ by the definition of $G \vee A$. Then $\mu^* \in \mathbf{CGra}(G \vee A, B)$ and we have that $(\Theta_A, G \vee A)$ is $(G \downarrow -)$-free over A. □

References

[1] A. Buldas, *An algebraic approach to the structure of graphs*, Dissertation, TTÜ Kirjastus, Tallinn, 1999.

[2] R.T. Bumby and D.M. Latch *Categorical constructions in graph theory*, Internat. J. Math. Math. Sci. **9** (1986), 1–16.

[3] K. Čulik, *Zur Theorie der Graphen*, Časopis Pěst. Mat. **83** (1958), 133–155.

[4] H. Dierks, *Der Einsatz von Theorem-Beweisern am Beispiel der Graphentheorie*, Diplomarbeit, Oldenburg, 1997.

[5] W. Dörfler and H. Music, *A bibliographical survey of products of graphs*, Preprint, 1980.

[6] P. Hell and J. Nesetril, *Cohomomorphisms of graphs and hypergraphs*, Math. Nachr. **87** (1979), 53–61.

[7] R. L. Hemminger, *The group of an X-join of graphs*, J. Combin. Theory **5** (1968), 408–418.

[8] H. Herrlich and G. Strecker, *Category Theory*, Allyn and Bacon, Boston, 1973, and later editions.

[9] W. Imrich, *Assoziative Produkte von Graphen*, Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II **180** (1972), 203–239.

[10] W. Imrich, A. Pultr, *Tensor products in the category of graphs*, Comment. Math. Univ. Carolinae **32** (1991), 619–639.

[11] H. Jürgens, *Funktoren auf Graphen*, Diplomarbeit, Oldenburg, 1998.

[12] M. Kilp, U. Knauer and A.V. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, 2000.

[13] A. Pultr, *Tensor products in the category of graphs*, Comment. Math. Univ. Carolinae **11** (1970), 619–639.

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