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On α-nuclearity and total accessibility for some tensor norms α

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ABSTRACT. For a tensor norm α on the class of all Banach spaces there is a natural way to define the Banach operator ideal N_α associated with the tensor norm. We give sufficient conditions for an operator to be in N_α if its second adjoint possesses this property. We show one way to get from this the negative answers to some questions of A. Defant and K. Floret: their tensor norms g_∞, w_1 and w_∞ are not totally accessible.

The question goes back to A. Grothendieck [3] of whether an operator between Banach spaces is nuclear if its adjoint (or, what is here the same, its second adjoint) is nuclear (answered in negative by T. Figiel and W.B. Johnson [2]). The corresponding questions concerning some other operator ideals, such as the ideals of p-nuclear operators, or operators factoring through the spaces l^p etc. have been considered, for example, in [5], [4], [6], [7], [8], [9]. In those works some sufficient conditions were given for the corresponding questions to have the positive answers, and in all of these conditions an assumption of the kind "either X^* or Y^{***} has some approximation property" appeared.

Thus it is rather natural to state a general question of the above-mentioned character. Namely, if α is a tensor norm on the class of all algebraic tensor product of Banach spaces and N_α is the operator ideal related to α of, let us say, α-nuclear operators (for the definitions see below), then *in which cases it is true that for given Banach spaces X and Y and an operator $T : X \to Y$ it follows from the inclusion $T^{**} \in \mathrm{N}_\alpha(X^{**}, Y^{**})$ the α-nuclearity of the operator T (from X to Y) itself?*

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Below (see the Theorem) we will give some general sufficient conditions (similar to those which were mentioned above) for a positive answer to the last question. The results of [4], [6], [7], [8], [9] show that, in general, the conditions are sharp and likely seem to be necessary.

We use then our Theorem to give negative answers to some questions, posed in the book [1] (we recall these questions after some preliminary definitions and notation).

Recall [1; 12.1] that a *tensor norm* α on the class B of all Banach spaces assigns to each pair (E, F) of Banach spaces E and F a norm $\alpha(\cdot; E, F)$ on the algebraic tensor product $E \otimes F$ (shorthand: $E \otimes_\alpha F$ and $E \widetilde{\otimes}_\alpha F$ for the completion) such that the following two conditions are satisfied:

(1) α is *reasonable*, i.e. $\varepsilon \leq \alpha \leq \pi$, where ε and π are, respectively, the smallest (injective) and the greatest (projective) Grothendieck tensor norms;

(2) α satisfies the *metric mapping property*: if $T_i : E_i \to F_i$, then

$$||T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2|| \leq ||T_1||\,||T_2||.$$

If α is a tensor norm then α^t denotes the *transposed norm* of α, i.e. $\alpha^t(\sum x_j \otimes y_j) := \alpha(\sum y_j \otimes x_j)$.

For a tensor norm α we denote by N_α the operator ideal generated by α: for every pair of Banach spaces X, Y the space $\mathrm{N}_\alpha(X, Y)$ is the factor space of the tensor product $X^* \widetilde{\otimes}_\alpha Y$ via the canonical mapping $X^* \widetilde{\otimes}_\alpha Y \to \mathrm{L}(X, Y)$ with the corresponding factor norm.

To formulate the above-mentioned questions of A. Defant and K. Floret, we need some additional information.

Given a tensor norm α on the class B, we define (see [1; 12.4]) for Banach spaces E and F and for $z \in E \otimes F$

$$\overrightarrow{\alpha}(z; E, F) := \inf\{\alpha(z; M, N) \,|\, M \subset E, N \subset F, z \in M \otimes N\}$$

where the infimum is taken over all finite dimensional subspaces $M \subset E$ and $N \subset F$, and

$$\overleftarrow{\alpha}(z; E, F) := \sup\{\alpha(Q_K^E \otimes Q_L^F(z); E/K, F/L) \,|\, K \subset E, L \subset F\}$$

where the supremum is taken over all closed finite codimensional subspaces $K \subset E$ and $L \subset F$ (here $Q_K^E : E \to E/K$ is the canonical mapping).

It is known (cf. [1; Prop. 12.4]) that $\overrightarrow{\alpha}$ and $\overleftarrow{\alpha}$ are tensor norms on B with

$$\varepsilon \leq \overleftarrow{\alpha} \leq \alpha \leq \overrightarrow{\alpha} \leq \pi.$$

A tensor norm α is called *finitely generated* if $\alpha = \overrightarrow{\alpha}$ and *cofinitely generated* if $\alpha = \overleftarrow{\alpha}$.

Examples. The tensor norms g_p and w_p, where $1 \le p \le \infty$, are finitely generated (cf. [1; 12.5 and 12.7]). For our purposes it is enough only to recall that for $z \in E \otimes F$, where E and F are Banach spaces, and for $1 \le p \le \infty$

$$w_p(z;E,F) = \inf\left\{ \sup_{\|x'\|\le 1} \left(\sum_{i=1}^{n} |\langle x', x_i\rangle|^p \right)^{1/p} \sup_{\|y'\|\le 1} \left(\sum_{i=1}^{n} |\langle y', y_i\rangle|^{p'} \right)^{1/p'} \;\middle|\; z = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

(in the cases where $p = 1$ or $p = \infty$ evident modifications must be made).

Note that, by definitions from [1], $g_\infty = w_\infty$ and $w_{p'} = w_p^t$ for all $p \in [1, \infty]$. Let us remark that g_∞ is the tensor norm associated with ∞-nuclear operators, w_∞ — with compactly c_0-factored operators and w_1 — with compactly l_1-factored operators.

A tensor norm α is called *totally accessible* if $\overrightarrow{\alpha} = \overleftarrow{\alpha}$, i.e. if α is finitely and cofinitely generated.

Some of the questions of A. Defant and K. Floret [1; 21.12] are as follows: *are the last three norms totally accessible?*

Below we answer the questions in the negative.

Let α be a finitely generated tensor norm. A Banach space E is said to have the *α-approximation property* (cf. [1; 21.7]) if for all Banach spaces F the natural mapping $F\widetilde{\otimes}_\alpha E \to F\widetilde{\otimes}_\varepsilon E$ is injective.

We shall use Proposition 21.7,(2), from [1]:

(*) *If α is totally accessible, then each Banach space has the α-approximation property.*

Theorem. *Let α be a finitely generated tensor norm. If*

(1) *the space X^* has the α^t-approximation property*

or

(2) *the space Y^{****} has the α-approximation property*

or

(3) *the space Y^{**} has the α-approximation property and the space Y^{***} has the α^t-approximation property,*

then any operator from X into Y, whose second adjoint is an N_α-operator, itself belongs to the space $\mathrm{N}_\alpha(X,Y)$.

Remark. The idea to consider the third conjugate space Y^{***} belongs to Eve Oja (it was used in some of the above-mentioned papers). The appearance in the formulation of the Theorem of a fourth conjugate space seems to be interesting but perhaps confusing.

Proof of Theorem. Suppose that there exists an operator $T \in \mathrm{L}(X,Y)$ such that $T \notin \mathrm{N}_\alpha(X,Y)$, but $\pi_Y T \in \mathrm{N}_\alpha(X,Y^{**})$. Since either X^* has the α^t-approximation property, or Y^{**} has the α-approximation property, we have $\mathrm{N}_\alpha(X,Y^{**}) = X^*\widetilde{\otimes}_\alpha Y^{**}$. Therefore, the operator $\pi_Y T$ can be identified with a tensor element $t \in X^*\widetilde{\otimes}_\alpha Y^{**}$; in addition, by the choice of T, $t \notin X^*\widetilde{\otimes}_\alpha Y$ (here $X^*\widetilde{\otimes}_\alpha Y$ is considered as a subspace of the space $X^*\widetilde{\otimes}_\alpha Y^{**}$). Hence, there exists an operator $U \in \left(X^*\widetilde{\otimes}_\alpha Y^{**}\right)^* \subset \mathrm{L}(Y^{**},X^{**})$ such that trace $U \circ t =$ trace $(t^* \circ (U^*|_{X^*})) = 1$ and trace $U \circ \pi_Y \circ z = 0$ for every $z \in X^*\widetilde{\otimes}_\alpha Y$. From the last it follows, in particular, that

$$U|_{\pi_Y(Y)} = 0 \text{ and } \pi_Y^* U^*|_{X^*} = 0. \tag{1}$$

Indeed, if $x' \in X^*$ and $y \in Y$, then

$$\langle U\pi_Y\, y, x'\rangle = \langle y, \pi_Y^*\, U^*|_{X^*} x'\rangle = \text{trace } (U\pi_Y)\circ(x'\otimes y) = 0.$$

Evidently, the tensor element $U \circ t$ generates the operator $U\pi_Y T$ which is, by the previous reasoning, identically equal to zero.

If the space X^* has the α^t-approximation property, then $X^*\widetilde{\otimes}_\alpha X^{**} = \mathrm{N}_\alpha(X,X^{**})$ and so the tensor element $U \circ t$ is zero, in contradiction with the equality trace $U \circ t = 1$.

Let now Y^{****} have the α-approximation property, or Y^{***} have the α^t-approximation property. In this case $V := (U^*|_{X^*}) \circ T^* \circ \pi_Y^*$:

$$Y^{***} \xrightarrow{\pi_Y^*} Y^* \xrightarrow{T^*} X^* \xrightarrow{U^*|_{X^*}} Y^{***}$$

uniquely determines a tensor element $t_0 \in Y^{****}\widetilde{\otimes}_{\alpha^t} Y^{***}$. Let us take any representation $t = \sum \alpha_n x_n' \otimes y_n''$ for t as an element of the space $X^*\widetilde{\otimes}_\alpha Y^{**}$. Denoting for the simplicity the operator $U^*|_{X^*}$ by U_* and recalling that $\pi_Y T = t$, we get:

$$\begin{aligned} Vy''' &= U_*\ (T^*\pi_Y^*\, y''') = U_*\ ((T^*\pi_Y^*\pi_{Y^*})\,\pi_Y^*\, y''') = \\ &= U_*\ ((\pi_Y T)^*\,\pi_{Y^*})\,\pi_Y^*\, y''') = U_*\ \left(\left(\sum \alpha_n y_n''\otimes x_n'\right)\ \pi_{Y^*}\,\pi_Y^*\, y'''\right) = \\ &\qquad = U_*\ \left(\sum \alpha_n \langle y_n'', \pi_Y^*\, y'''\rangle\, x_n'\right) = \sum \alpha_n \langle \pi_Y^{**} y_n'', y'''\rangle\, U_* x_n'. \end{aligned}$$

Thus, the operator V (or the element t_0) has, in the space $Y^{****}\widetilde{\otimes}_{\alpha^t} Y^{***}$, the representation

$$V = \sum \alpha_n \pi_Y^{**}(y_n'') \otimes U_*(x_n').$$

Therefore (see (1)),

$$\begin{aligned}\text{trace } t_0 = \text{trace } V &= \sum \alpha_n \langle \pi_Y^{**}(y_n''), U_*(x_n') \rangle = \sum \alpha_n \langle y_n'', \pi_Y^* U_* x_n' \rangle \\ &= \sum 0 = 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}Vy''' &= U_* (\pi_Y T)^* y''' = U_* \circ t^*(y''') = \\ &= U_* \left(\sum \alpha_n \langle y_n'', y''' \rangle x_n' \right) = \sum \alpha_n \langle y_n'', y''' \rangle U_* x_n',\end{aligned}$$

whence $V = \sum \alpha_n y_n'' \otimes U_* x_n'$. Therefore,

$$\text{trace } t_0 = \text{trace } V = \sum \alpha_n \langle y_n'', U_* x_n' \rangle = \sum \alpha_n \langle U y_n'', x_n' \rangle = \text{trace } U \circ t = 1.$$

The obtained contradiction completes the proof of the theorem. □

Corollary 1. *If α is a finitely generated tensor norm, and if there exists a non-N_α-operator in Banach spaces whose second adjoint is an N_α-operator, then there exists a Banach space both without the α^t-approximation property, and without the α-approximation property.*

Proof. The assertion follows immediately from the previous theorem: to get one space without α- and α^t-approximation properties it is enough to sum by type l^2 two spaces obtained with the help of the Theorem. □

Corollary 1 and $(*)$ immediately yield

Corollary 2. *If α is a finitely generated tensor norm, and if there exists a non-N_α-operator in Banach spaces whose second adjoint is an N_α-operator, then neither the tensor norm α, nor the tensor norm α^t is totally accessible.*

Because there are operators, whose second adjoints factor compactly through l^1 (respectively, through c_0), but which do not factor through l^1 (respectively, through c_0) themselves (see, [5], [6], [9]), we get from Corollary 2

Corollary 3. *The tensor norms g_∞, w_1 and w_∞ are not totally accessible.*

Addendum. We thank the referee who pointed out to us that a part of Corollary 3 was proved also by Frank Oertel (to be published).

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