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Spectral systems with the one-way spectral mapping property

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ABSTRACT. For Banach algebras we characterize spectral systems which satisfy the one-way spectral mapping property. We show that these spectral systems are determined by projective systems of multiplicative linear functionals of finitely generated subalgebras and, using this, we derive several consequences concerning multiplicative extensions of multiplicative linear functionals of subalgebras.

1. Introduction

As defined in [18], a *subspectrum* on a complex unital Banach algebra A is a map σ which assigns a non-empty compact subset $\sigma(\mathbf{a}) \subset \mathbb{C}^n$ to every commuting n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n \quad (n \in \mathbb{N})$ in such a way that

(I) $\sigma(\mathbf{a}) \subset \prod_{k=1}^{n} \sigma_A(a_k) \quad$ ($\sigma_A(a_k)$ stands for the spectrum of a_k in A),

(II) $\mathbf{p}(\sigma(\mathbf{a})) = \sigma(\mathbf{p}(\mathbf{a}))$

for each m-tuple of polynomials $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ in n indeterminates.

In [18] W. Żelazko gave a functional representation of subspectra in terms of maximal ideal spaces of maximal commutative subalgebras. Namely, he proved that if σ is a subspectrum defined on a complex unital Banach algebra A and B is any maximal commutative subalgebra of A, then there exists a compact subset $\Delta(\sigma, B)$ in the maximal ideal space of B such that

$$\sigma(a_1, a_2, \ldots, a_n) = \{(\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n)) : \Lambda \in \Delta(\sigma, B)\}$$

for all $(a_1, a_2, \ldots, a_n) \in B^n$.

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In the present paper we shall consider the spectral systems that possess the one-way spectral mapping property. In particular, we establish a relation between the spectral systems with the one-way spectral mapping property and projective systems of multiplicative linear functionals of finitely generated subalgebras (Section 3, Theorem 1). Then we derive several corollaries concerning multiplicative extensions of multiplicative linear functionals of subalgebras. Finally, in Section 4 we compare our description of spectral system with different axiomatic approaches developed in [5,7,8,13,16].

2. Preliminaries

Throughout this paper, all algebras are assumed to be associative, unital and over the complex field $\mathbb{C}$.

Let A be a Banach algebra. The set of all n-tuples $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ $(n = 1, 2, \ldots)$ of elements of A will be denoted by A_∞ and A_{com} will be the set of all commuting n-tuples in A_∞. For any $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$, $[\mathbf{a}] = [a_1, a_2, \ldots, a_n]$ is the subalgebra of A generated by the elements $a_1, a_2, \ldots, a_n$ and the unit e and, in addition, in what follows we assume that the set A_∞ is directed by :

$$\mathbf{a} \leq \mathbf{b} \iff [\mathbf{a}] \subset [\mathbf{b}] \qquad (\mathbf{a}, \mathbf{b} \in A_\infty).$$

Further, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ then by $(\mathbf{a} - \lambda)$ we denote the n-tuple $(a_1 - \lambda_1 e, a_2 - \lambda_2 e, \ldots, a_n - \lambda_n e)$ and by $I_A^l(\mathbf{a})$ (resp. $I_A^r(\mathbf{a})$) the left (resp. right) ideal generated in A by the n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. We also set $P(\mathbf{a}) = I_{[\mathbf{a}]}^l(\mathbf{a})$ and if E is any family of subsets of A and $\mathbf{a} \in A_\infty$ then

$$\sigma_E(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : P(\mathbf{a} - \lambda) \subset U \text{ for some } U \in E\}.$$

Furthermore, the set of all non-zero multiplicative linear functionals (not necessarily continuous) on a subalgebra B of A, endowed with the weak*-topology, will be denoted by $\operatorname{Hom} B$, $\ker\Lambda$ will stand for the kernel of the functional Λ and when $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$, we shall let $\hat{\mathbf{a}}$ denote the Gelfand transform of $\mathbf{a}$, i.e.

$$\hat{\mathbf{a}}(\Lambda) = (\Lambda(a_1), \Lambda(a_2), \ldots, \Lambda(a_n))$$

for every Λ in $\operatorname{Hom} A$. Also, by $\hat{A}$ we will denote the algebra $\{\hat{a} : a \in A\}$.

Finally, by a *spectral system* we mean in the sequel a mapping σ which assigns to every n-tuple $\mathbf{a} = (a_1, a_2, ..., a_n)$ in A^n a subset $\sigma(\mathbf{a})$ of $\mathbb{C}^n$ (possibly empty) such that $\sigma(\Theta_A) = \{0\}$ and for each $\mathbf{a} \in A_\infty$

(III) $\sigma(\mathbf{a}, e) = \{(\lambda, 1) : \lambda \in \sigma(\mathbf{a})\}$;

and a spectral system σ is said to have the *one-way spectral mapping property* on A if

(IV) $\mathbf{p}(\sigma(\mathbf{a})) \subset \sigma(\mathbf{p}(\mathbf{a}))$

for every n-tuple $\mathbf{a} \in A_\infty$ and m-tuple of polynomials $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ in n indeterminates.

3. Spectral systems with the one-way spectral mapping property

In this section we show that if σ is a spectral system defined on a Banach algebra A then σ has the one-way spectral mapping property if and only if it is defined by a projective system of multiplicative linear functionals of finitely generated subalgebras. So, our main result can be stated as follows:

Theorem 1. *Let A be a Banach algebra and let σ be a spectral system on A such that $\sigma(\mathbf{a})$ is non-empty for every $\mathbf{a} \in A_\infty$. The following assertions are equivalent:*

(a) *σ possesses the one-way spectral mapping property,*

(b) *there is a projective system $\{\Delta_{\mathbf{a}} \subset \mathrm{Hom}[\mathbf{a}],\ \pi_{\mathbf{a}}^{\mathbf{b}},\ A_\infty\}$ such that $\sigma(\mathbf{a}) = \hat{\mathbf{a}}(\Delta_{\mathbf{a}})$ for every $\mathbf{a} \in A_\infty$,*

(c) *there is a family E of subsets of A such that $\sigma(\mathbf{a}) = \sigma_E(\mathbf{a})$ for every $\mathbf{a}$ in A_∞.*

Proof. (a)$\Rightarrow$(b). Take any $\mathbf{a} \in A_\infty$ and put

$$\Delta_{\mathbf{a}} = \{\Lambda \in \mathrm{Hom}[\mathbf{a}] : \hat{\mathbf{a}}(\Lambda) \in \sigma(\mathbf{a})\}.$$

Since σ has the one-way spectral mapping property, it is easy to see that $\hat{\mathbf{a}}(\Delta_{\mathbf{a}}) = \sigma(\mathbf{a})$ (see [2], Theorem 1). Denote now for any $\mathbf{a} \le \mathbf{b}$ $(\mathbf{a}, \mathbf{b} \in A_\infty)$ by $\pi_{\mathbf{a}}^{\mathbf{b}}$ the restriction map $\Delta_{\mathbf{b}} \to \Delta_{\mathbf{a}}$. If $\mathbf{a} \in A^m$, $\mathbf{b} \in A^n$ are such that $\mathbf{a} \le \mathbf{b}$ then $[\mathbf{a}] \subset [\mathbf{b}]$ and, therefore, $\mathbf{a} = \mathbf{p}(\mathbf{b}) + \alpha e$ for some $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{C}^m$ and m-tuple $\mathbf{p} = (p_1, p_2, ..., p_m)$ of polynomials in n indeterminates. Moreover, if $\Lambda \in \Delta_{\mathbf{b}}$ then $\pi_{\mathbf{a}}^{\mathbf{b}}(\Lambda) \in \Delta_{\mathbf{a}}$ since

$$\begin{aligned}\hat{\mathbf{a}}(\Lambda) &= (\Lambda(p_1(\mathbf{b})) + \alpha_1, \Lambda(p_2(\mathbf{b})) + \alpha_2, ..., \Lambda(p_m(\mathbf{b})) + \alpha_m)\\ &= (p_1(\hat{\mathbf{b}})(\Lambda) + \alpha_1, p_2(\hat{\mathbf{b}})(\Lambda) + \alpha_2, ..., p_m(\hat{\mathbf{b}})(\Lambda) + \alpha_m)\\ &= \mathbf{p}(\hat{\mathbf{b}}(\Lambda)) + \alpha \in \sigma(\mathbf{p}(\mathbf{b})) + \alpha \subset \sigma(\mathbf{a}).\end{aligned}$$

So, the family $\{\Delta_{\mathbf{a}}, \pi_{\mathbf{a}}^{\mathbf{b}}, A_\infty\}$ constitutes a projective system such that $\sigma(\mathbf{a}) = \hat{\mathbf{a}}(\Delta_{\mathbf{a}})$ for any $\mathbf{a} \in A_\infty$.

(b)$\Rightarrow$(c). Put $E = \{\ker\Lambda : \Lambda \in \Delta_{\mathbf{a}}, \mathbf{a} \in A_\infty\}$ and let $\lambda \in \sigma(\mathbf{a})$ for some $\mathbf{a} \in A_\infty$. Then there is $\Lambda \in \Delta_{\mathbf{a}}$ such that $\lambda = \hat{\mathbf{a}}(\Lambda)$, so that $P(\mathbf{a} - \lambda) \subset \ker\Lambda$. We conclude that $\sigma(\mathbf{a}) \subset \sigma_E(\mathbf{a})$. On the other hand, if λ is in $\sigma_E(\mathbf{a})$ then there are $\mathbf{b} \in A_\infty$ and $\Lambda \in \Delta_{\mathbf{b}}$ such that $P(\mathbf{a} - \lambda) \subset \ker\Lambda \subset [\mathbf{b}]$. Now $\mathbf{a} \le \mathbf{b}$ and $\pi_{\mathbf{a}}^{\mathbf{b}}(\Lambda) \in \Delta_{\mathbf{a}}$. Hence $\lambda = \hat{\mathbf{a}}(\Lambda) \in \sigma(\mathbf{a})$.

(c)$\Rightarrow$(a). Let $\mathbf{a} = (a_1, a_2, ..., a_n) \in A^n$ and let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ be in $\sigma(\mathbf{a}) = \sigma_E(\mathbf{a})$. Then there is U in E such that $P(\mathbf{a} - \lambda) \subset U$. Moreover, for any polynomial p in n indeterminates there are elements $b_1, b_2, ..., b_n$ in $[\mathbf{a}]$ such that

$$p(\mathbf{a}) - p(\lambda) = \sum_{k=1}^{n} b_k(a_k - \lambda_k e)$$

(see, for example, [1]). Hence $p(\mathbf{a}) - p(\lambda) \in P(\mathbf{a} - \lambda)$ for every polynomial p in n indeterminates. It readily follows that, if $\mathbf{p} = (p_1, p_2, ..., p_m)$ is any m-tuple of polynomials in n indeterminates then $p_i(\mathbf{a}) - p_i(\lambda) \in P(\mathbf{a} - \lambda)$ $(i = 1, 2, ..., n)$, so that $P(\mathbf{p}(\mathbf{a}) - \mathbf{p}(\lambda)) \subset P(\mathbf{a} - \lambda) \subset U$. Thus $\mathbf{p}(\lambda) \in \sigma_E(\mathbf{p}(\mathbf{a})) = \sigma(\mathbf{p}(\mathbf{a}))$. So, σ possesses the one-way spectral mapping property. □

For two spectral systems σ and $\tilde{\sigma}$ defined on a Banach algebra A we shall write $\sigma \leq \tilde{\sigma}$ if $\sigma(\mathbf{a}) \subset \tilde{\sigma}(\mathbf{a})$ for any $\mathbf{a}$ in A_∞.

Corollary 1. ([3], Corollary 2.2). *The maximal spectral system σ_{max} on a Banach algebra A possessing the one-way spectral mapping property is given for every $\mathbf{a}$ in A_∞ by $\sigma_{max}(\mathbf{a}) = \hat{\mathbf{a}}(\text{Hom}[\mathbf{a}])$.*

Recall now that the *joint approximate point spectrum* $\pi(\mathbf{a})$ for an arbitrary n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ (not necessarily commuting) is defined to be the set of all those $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ for which there is a sequence $(z_j) \subset A$ with $\|z_j\| = 1$ such that either $\lim_j \|(a_k - \alpha_k e)z_j\| = 0$ $(k = 1, 2, \ldots, n)$ or $\lim_j \|z_j(a_k - \alpha_k e)\| = 0$ $(k = 1, 2, \ldots, n)$. Following W. Żelazko [19] let us denote for a commutative Banach algebra A by Ł(A) the set of all those $\Lambda \in$Hom$A$ for which kerΛ is an ideal in A consisting of joint topological divisors of zero. In other words, if A is a commutative Banach algebra then $\Lambda \in$ Ł(A) if and only if there is a net $(z_j) \subset A$ with $\|z_j\| = 1$ such that $\lim_j \|az_j\| = 0$ for every $a \in$ kerΛ. It is well known that Ł(A) is a closed subset of Hom$A$ and $\Gamma(A) \subset$ Ł(A), where $\Gamma(A)$ stands for the Shilov boundary of A [17].

Furthermore, following A. McIntosh and A. Pryde [9], let us denote by $\gamma(\mathbf{a})$ the *spectral set*

$$\gamma(\mathbf{a}) = \{(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n : 0 \in \sigma_A(\sum_{k=1}^{n}(a_k - \alpha_k e)^2)\}$$

of an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_\infty$. As it is established in [4,15], for commuting n-tuples $\mathbf{a} \in A_\infty$ the spectral set $\gamma(\mathbf{a})$ is always non-empty. In addition, for commuting n-tuples of elements with real spectrum some of the important joint spectra and the spectral set coincide [4,10,14]. Denoting now by $cl[\mathbf{a}]$ $(\mathbf{a} \in A_\infty)$ the closure of $[\mathbf{a}]$ in A and by $\hat{\sigma}$ the spectral system on A defined by $\hat{\sigma}(\mathbf{a}) = \hat{\mathbf{a}}(\text{Hom}(cl[\mathbf{a}]))$, we easily have from Theorem 1 the following corollary.

Corollary 2. *Let A be a Banach algebra. The following are equivalent for an n-tuple $\mathbf{a} \in A_{com}$:*

(a) $\hat{\sigma}(\mathbf{a}) \subset \mathbb{R}^n$,

(b) $\hat{\sigma}(\mathbf{a}) = \pi(\mathbf{a}) = \gamma(\mathbf{a})$.

Proof. (a)$\Rightarrow$(b). Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an n-tuple in A_{com} such that $\hat{\sigma}(\mathbf{a}) \subset \mathbb{R}^n$. Then all the elements $a_k \quad (k = 1, 2, \ldots, n)$ are with real spectrum and, by the well-known properties of Banach algebras, $\hat{\mathbf{a}}(\mathrm{L}(cl[\mathbf{a}])) = \hat{\mathbf{a}}(\mathrm{Hom}(cl[\mathbf{a}])) \subset \mathbb{R}^n$. So, $\hat{\sigma}(\mathbf{a}) \subset \pi(\mathbf{a}) \subset \hat{\sigma}(\mathbf{a})$. Besides, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ then

$$\lambda \in \gamma(\mathbf{a}) \iff 0 \in \sigma_A\Big(\sum_{k=1}^{n} (a_k - \lambda_k)^2\Big) \iff \lambda = \hat{\mathbf{a}}(\Lambda) \text{ for some } \Lambda \in \mathrm{Ł}(cl[\mathbf{a}]).$$

(b)$\Rightarrow$(a). Clear. □

We say that a spectral system σ on A is *bounded* if the set $\sigma(\mathbf{a})$ is bounded for every $\mathbf{a} \in A_\infty$.

Now we are ready to derive some corollaries concerning the multiplicative extensions of multiplicative linear functionals of subalgebras.

Corollary 3. *Let A be a Banach algebra and let σ be a bounded spectral system on A with the one-way spectral mapping property. The following assertions are equivalent:*

(a) $cl\,\sigma(\mathbf{a})$ *is non-empty for any* $\mathbf{a}$ *in* A_∞,

(b) *there is* Λ *in* $\mathrm{Hom}\, A$ *such that* $\hat{\mathbf{a}}(\Lambda) \in cl\,\sigma(\mathbf{a})$ *for each* $\mathbf{a} \in A_\infty$.

Proof. Since σ has the one-way spectral mapping property, there is a projective system $\{\Delta_{\mathbf{a}}, \pi_{\mathbf{a}}^{\mathbf{b}}, A_\infty\}$ with $\sigma(\mathbf{a}) = \hat{\mathbf{a}}(\Delta_{\mathbf{a}})$ for each $\mathbf{a} \in A_\infty$. It is easy to see that the mapping $\Lambda \to \hat{\mathbf{a}}(\Lambda)$ is a homeomorphism of $\Delta_{\mathbf{a}}$ onto $\sigma(\mathbf{a}) \quad (\mathbf{a} \in A_\infty)$ [2]. So, the family $\{cl\Delta_{\mathbf{a}}, \pi_{\mathbf{a}}^{\mathbf{b}}, A_\infty\}$ is a projective system of non-void compact Hausdorff topological spaces. Hence there is a non-empty compact subset $F \subset \prod cl\Delta_{\mathbf{a}} \quad (\mathbf{a} \in A_\infty)$ such that $\pi_{\mathbf{a}}^{\mathbf{b}}(f(\mathbf{b})) = f(\mathbf{a})$ for each $\mathbf{a} \leq \mathbf{b}$ and $f \in F$. It readily follows that there is a multiplicative linear functional Λ on A such that $\hat{\mathbf{a}}(\Lambda) \in cl\,\sigma(\mathbf{a})$ for every $\mathbf{a} \in A_\infty$. □

The *left* (resp. *right*) *Harte joint spectrum* $\sigma_A^l(\mathbf{a})$ (resp. $\sigma_A^r(\mathbf{a})$) of an n-tuple $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A^n$ with respect to A is defined to be the set of all those $\lambda \in \mathbb{C}^n$ such that $I_A^l(\mathbf{a} - \lambda) \neq A$ (resp. $I_A^r(\mathbf{a} - \lambda) \neq A$) and the *Harte joint spectrum* $\sigma_A(\mathbf{a})$ of $\mathbf{a}$ with respect to A is the set $\sigma_A^l(\mathbf{a}) \cup \sigma_A^r(\mathbf{a})$. Note that $\sigma_A(a)$ is the usual spectrum of $a \in A$ and $r_A(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$ is the spectral radius of a in A. In addition, the Harte joint spectrum is bounded and possesses the one-way spectral mapping property [1].

Corollary 4. *Let B be a commutative Banach algebra, let A be a subalgebra of B sharing the identity of B and let $\Lambda \in \operatorname{Hom} A$. Then Λ has a multiplicative linear extension to the algebra B if and only if $\hat{\mathbf{a}}(\Lambda) \in \sigma_B(\mathbf{a})$ for every $\mathbf{a}$ in A_∞.*

An *extension* (resp. *isomorphic extension*) of a commutative Banach algebra A is a commutative unital Banach algebra B together with a unital isomorphism (resp. topological isomorphism) of A into B.

As it is shown in [6], isomorphic extensions coincide with isometric extensions. That is if B is an isomorphic extension of A then it can be normed in a such way that it becomes an isometric extension of A and the norm is equivalent of the old one.

The *cortex* $\operatorname{cor}A$ of A is the set of all functionals in $\operatorname{Hom}A$ which extend to members of $\operatorname{Hom}B$ for each isometric extension B of A. Now, as it is well known, $\Gamma(A) \subset \mathrm{L}(A) = \operatorname{cor}(A)$ [12].

A Banach algebra A is said to have the *spectral extension property* if $r_A(a) = r_B(a)$ for any a in A and extension B of A, and the *multiplicative Hahn-Banach property* if every multiplicative linear functional in $\operatorname{Hom}A$ has a multiplicative linear extension to every extension B of A [11]. From Corollary 4 one can deduce the following corollaries concerning the multiplicative Hahn-Banach property.

Corollary 5. ([11], Theorem 1). *Let A be a semisimple commutative Banach algebra. The following are equivalent:*

(a) *the algebra A has the spectral extension property,*

(b) *every $\Lambda \in \Gamma(A)$ has a multiplicative linear extension to every commutative extension B of A.*

Proof. We only need to prove (a)$\Rightarrow$(b). To this end, let B be any commutative extension of A and let $\Lambda \in \Gamma(A)$. Now, r_A is a norm on A and, clearly, $\Lambda \in \Gamma(D)$, where D is the completion of $\hat{A}$ in the supremum norm. Hence $\hat{\mathbf{a}}(\Lambda) \in \pi(\hat{\mathbf{a}})$ for any $\mathbf{a} \in A_\infty$ and if $a_1, a_2, \ldots, a_n$ in A then there is a sequence (x_k) in A such that $r_A((a_j - \Lambda(a_j)e)x_k)$ tends to zero for each $j = 1, 2, \ldots, n$ and $0 < \alpha \leq r_A(x_k) \quad (k = 1, 2, \ldots)$. So,

$$r_B\Big(\sum_{j=1}^{n}(a_j - \Lambda(a_j)e)x_k b_j\Big) \leq \sum_{j=1}^{n} r_A((a_j - \Lambda(a_j)e)x_k) r_B(b_j) \to 0 \quad (k \to \infty)$$

for any $b_1, b_2, \ldots, b_n$ in B. We conclude that $\hat{\mathbf{a}}(\Lambda) \in \sigma_B(\mathbf{a})$ for any $\mathbf{a}$ in A_∞ and, by Corollary 4, Λ has a multiplicative linear extension to B. □

Corollary 6. ([11], Theorem 3). *A semisimple commutative Banach algebra A has the multiplicative Hahn-Banach property if and only if it possesses the spectral extension property and* $\operatorname{Hom} A = \Gamma(A)$.

4. Regularities and spectral subsets

According to [7] (see also [16]), a *generalized joint spectrum* $\tilde{\sigma}$ is a spectral system with the property (I) for all $\mathbf{a} \in A_\infty$ and with the one-way spectral mapping property (IV) for all commuting n-tuples $\mathbf{a} \in A_{com}$.

Moreover, a linear subspace $E \subset A$ of a Banach algebra A is called *spectral* if it does not contain invertible elements and $P(\mathbf{a}) \subset E$ for all $\mathbf{a} \in E_{com}$. The set of all spectral linear subspaces in A is denoted by $E(A)$. Clearly all ideals in A and subalgebras consisting of noninvertible elements are spectral subsets. But there exist spectral subspaces that are not subalgebras [7, p. 133].

In [7] A. Martinez and A. W. Wawrzyńczyk described generalized joint spectra which satisfy the one-way spectral mapping property in terms of spectral linear subspaces and established that there exists a correspondence between generalized joint spectra and linear spectral subspaces. From Theorem 1 we directly obtain the following result.

Theorem 2. (cf. [7], Proposition 2.3). *A generalized joint spectrum $\tilde{\sigma}$ satisfies the one-way spectral mapping property for all $\mathbf{a} \in A_\infty$ if and only if there exists a subset $U \subset E(A)$ consisting of subalgebras of A such that* $\tilde{\sigma}(\mathbf{a}) = \sigma_U(\mathbf{a}) \quad (\mathbf{a} \in A_\infty)$.

In [5] V. Kordula and V. Müller, in [8] M. Mbektha and V. Müller and in [13] V. Müller described wide classes of spectra and joint spectra using the concepts of regularities and joint regularities.

A *joint regularity* $R \in A$ is a subset of A_{com} such that:

(a) if $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \in A_{com}$ is such that $\sum_{i=1}^n x_i y_i = e$ then $(x_1, x_2, \ldots, x_n) \in R \quad (n \in \mathbb{N})$;

(b) $(x_1, x_2, \ldots, x_n, x_{n+1}) \in A_{com}$ and $(x_1, \ldots, x_n) \in R$ then the $(n+1)$-tuple $(x_1, x_2, \ldots, x_n, x_{n+1}) \in R \quad (n \in \mathbb{N})$;

(c) if $(x_0 - \lambda, x_1, \ldots, x_n) \in R$ for every $\lambda \in \mathbb{C}^n$ then $(x_1, x_2, \ldots, x_n) \in R$.

The spectral system associated with the regularity R is defined by $\tilde{\sigma}_R(\mathbf{a}) = \{\lambda \in \mathbb{C}^n : \mathbf{a} - \lambda \notin R\}$ for each commuting n-tuple $\mathbf{a} \in A_{com}$. Moreover, a spectral system σ is said to have the *spectral mapping property for commuting n-tuples* if it satisfies the property (II) for every $\mathbf{a} \in A_{com}$.

The following theorem gives us a correspondence between joint regularities and spectral systems with spectral mapping property for commuting n-tuples (see [7,13]).

Theorem 3. *Let σ be a spectral system with the one-way spectral mapping property on a Banach algebra A and let $\sigma(\mathbf{a}) \neq \emptyset$ for every $\mathbf{a} \in A_\infty$. Then the following are equivalent:*

(a) *σ has the spectral mapping property for commuting n-tuples,*

(b) *there is a projective system $\{\Delta_\mathbf{a} \subset \mathrm{Hom}[\mathbf{a}], \pi_\mathbf{a}^\mathbf{b}, A_\infty\}$ with $\sigma(\mathbf{a}) = \hat{\mathbf{a}}(\Delta_\mathbf{a})$ for any $\mathbf{a} \in A_\infty$ such that $\pi_\mathbf{a}^\mathbf{b}$ is onto for any $\mathbf{a}, \mathbf{b} \in A_{com}$ $(\mathbf{a} \leq \mathbf{b})$,*

(c) *there is a joint regularity R in A such that $\tilde{\sigma}_R(\mathbf{a}) = \sigma(\mathbf{a})$ for any $\mathbf{a} \in A_{com}$.*

Proof. (a)⇒(b). Clear by Theorem 1.

(b)⇒(c). Put $R = \{\mathbf{a} \in A_{com} : \mathbf{0} \notin \sigma(\mathbf{a})\}$. Now, as it is easily to be seen, R is a joint regularity in A and for every $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in A_{com}$ and $\lambda \in \mathbb{C}^n$ we have

$$\lambda \in \sigma(\mathbf{a}) \iff \mathbf{0} \in \sigma(\mathbf{a} - \lambda) \iff (\mathbf{a} - \lambda) \notin R \iff \lambda \in \tilde{\sigma}_R(\mathbf{a}).$$

(c)⇒(a). Let $\mathbf{a}$ be a commuting n-tuple and $\mathbf{p} = (p_1, p_2, \ldots, p_m)$ be an m-tuple of polynomials in n indeterminates. Take any $\lambda \in \sigma(\mathbf{p}(\mathbf{a})) = \sigma(p_1(\mathbf{a}), \ldots, p_m(\mathbf{a}))$. Then $\lambda \in \tilde{\sigma}_R(\mathbf{p}(\mathbf{a}))$ and, since $(\mathbf{p}(\mathbf{a}) - \lambda) \notin R$, there is an n-tuple $\mu \in \mathbb{C}^n$ such that $(\mathbf{a} - \mu, \mathbf{p}(\mathbf{a}) - \lambda) \notin R$. Now $(\mu, \lambda) \in \sigma(\mathbf{a}, \mathbf{p}(\mathbf{a}))$ and, by Theorem 1, there is Λ in $\mathrm{Hom}[\mathbf{a}]$ with $\mu = \hat{\mathbf{a}}(\Lambda) \in \sigma(\mathbf{a})$. Consequently, $\mathbf{p}(\mu) = \mathbf{p}(\hat{\mathbf{a}}(\Lambda)) = \widehat{\mathbf{p}(\mathbf{a})}(\Lambda) = \lambda$. □

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