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Tauberian theorems for Bonsall core in sequence spaces

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ABSTRACT. The authors investigate cores $K(x)$ of sequences x that are defined by different Bonsall functionals π. Two general Tauberian core theorems are proved. These are applied to special functionals π and matrix methods A. Some Tauberian conditions are generated with the theorems.

1. Preliminaries

The concept of the core of a sequence $x = (\xi_k)$ of complex numbers has been defined by Knopp in 1930 (see [3], chpt. VI). His definition is equivalent to the following: the core of x is the set of all complex numbers t for which

$$\mathrm{Re}(\alpha t) \leq \limsup_k \mathrm{Re}(\alpha \xi_k) \ \ \forall \alpha \in \mathbb{C}$$

(here $\mathrm{Re}(t)$ is the real part of t).

In 1953 Bonsall (see [1]) generalized the concept and defined the core for an element x of a vector space X over the field $\mathbb{K}$ (here $\mathbb{K}$ is $\mathbb{C}$ or $\mathbb{R}$) as the set

$$K(x) := \{t \in \mathbb{K} \,|\, \mathrm{Re}(\alpha t) \leq \pi(\alpha x) \ \ \forall \alpha \in \mathbb{K}\}, \tag{1}$$

where π is an arbitrarily fixed functional on X with range $[-\infty, +\infty]$ such that

1) $\pi(x+y) \leq \pi(x) + \pi(y)$,
2) $\pi(\alpha x) = \alpha \pi(x) \ \ \forall \alpha > 0$.

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This functional is called the Bonsall functional. It follows from (1) that

$$K(x) = \{t \in \mathbb{K} \mid -\pi(-\alpha x) \leq \mathrm{Re}(\alpha t) \leq \pi(\alpha x) \ \ \forall \alpha \in \mathbb{K}\}.$$

Central problems in core theory deal chiefly with inclusion relations between different cores. Let $K_1(x)$ and $K_2(y)$ be two cores defined by Bonsall functionals π_1 and π_2, respectively. It follows directly from the definition of core that if

$$\pi_1(\alpha x) \leq \pi_2(\alpha y) \ \ \forall \alpha \in \mathbb{K},$$

then

$$K_1(x) \subset K_2(y).$$

Due to the possibility of empty cores it is obvious that the converse implication is not always true. Therefore, the problems of core inclusions are closely connected with the investigation of the validity of inequalities of type

$$\pi_1(Ax) \leq \pi_2(Bx) \ \ \forall x \in X,$$

where A and B are operators in X.

Let

$$c_\pi := \{x \in X \mid K(x) \text{ is a singleton, } \pi(\alpha x) \in \mathbb{R} \quad \forall \alpha \in \mathbb{K}\}$$

(the set of all π-convergent elements) and let

$$c_{\pi 0} := \{x \in c_\pi \mid \pi(x) = 0\}$$

(the set of all π-null elements). The sets c_π and $c_{\pi 0}$ are linear subspaces of X. The functional π is additive and homogeneous over $\mathbb{R}$ on c_π (see [1]).

Proposition 1. *For every $x \in c_{\pi 0}$ and $y \in X$,*

$$\pi(x+y) = \pi(y) \tag{2}$$

and

$$K(x+y) = K(y). \tag{3}$$

Proof. Let $x \in c_{\pi 0}$ and $\alpha \in \mathbb{K}$. As $c_{\pi 0}$ is a vector space, $\pi(\alpha x) = 0$. For any $y \in X$ we get that

$$\pi(\alpha y) \leq \pi(\alpha x + \alpha y) + \pi(-\alpha x) = \pi(\alpha x + \alpha y) \leq \pi(\alpha x) + \pi(\alpha y) \leq \pi(\alpha y),$$

i.e.,

$$\pi(\alpha(x+y)) = \pi(\alpha y) \ \ \forall \alpha \in \mathbb{K},$$

and (2) is true. The equality of cores (3) follows now from (1). □

Corollary 2. *If $x - y \in c_{\pi 0}$, then $\pi(x) = \pi(y)$ and $K(x) = K(y)$.*

Let ω be the set of all sequences $x = (\xi_k)$, where $\xi_k \in \mathbb{K}$, $k \in \mathbb{N}$ and $\mathbb{N} = \{0, 1, 2, ...\}$. Each linear subspace of ω is called a sequence space. In the sequel we shall investigate the cores in sequence spaces. The following subsets of ω are obviously sequence spaces:

$$m := \{x \in \omega | \sup_k |\xi_k| < \infty\}$$

(the set of bounded sequences),

$$c_0 := \{x \in \omega | \lim \xi_k = 0\}$$

(the set of null sequences) and

$$f_0 := \left\{ x \in \omega \,\middle|\, \lim_m \frac{1}{m+1} \sum_{k=n}^{n+m} \xi_k = 0 \text{ (uniformly for } n) \right\}$$

(the set of almost null sequences).

What follows are some well-known Bonsall functionals in ω.

1) The functional

$$\pi_1(x) := \limsup_k \mathrm{Re}\xi_k$$

that defines Knopp core $K_1(x)$ in ω (cf. [3]). The set of π_1-null elements is c_0, i.e., $c_{\pi_1 0} = c_0$.

2) The functional

$$\pi_2(x) := \limsup_m \left(\sup_n \frac{1}{m+1} \sum_{k=n}^{n+m} \mathrm{Re}\xi_k \right)$$

that defines the almost convergence core (or Lorentz core) $K_2(x)$ in ω (cf. [7]). The set of π_2-null elements is f_0, i.e., $c_{\pi_2 0} = f_0$.

It is easy to see that

$$\pi_2(x) \le \pi_1(x) \;\; \forall x \in \omega,$$

and therefore

$$K_2(x) \subset K_1(x) \;\; \forall x \in \omega,$$

and $f_0 \supset c_0$.

3) Let now $x = (\xi_k)$ be a real sequence. Let $\delta(M)$ be the natural density of a set M of positive integers, i.e.,

$$\delta(M) := \lim_n \frac{1}{n} \text{ card } \{k \in M \mid k \leq n\},$$

where card U denotes the cardinality of a set U. A sequence x is statistically convergent to a, denoted st-lim $x = a$, if for every $\varepsilon > 0$,

$$\delta(\{k \mid |\xi_k - a| \geq \varepsilon\}) = 0.$$

Let M_x denote the set

$$M_x := \{t \in \mathbb{R} \mid \delta\{k \mid \xi_k > t\} \neq 0\}.$$

Note that the statement $\delta(M) \neq 0$ means that $\delta(M) > 0$ or M does not have natural density. The notion of the statistical limit superior of x is given by

$$\text{st-limsup}\, x := \begin{cases} \sup M_x, & \text{if } M_x \neq \emptyset, \\ -\infty, & \text{if } M_x = \emptyset. \end{cases}$$

It is easy to check that $\pi_3(x) :=$ st-limsup x is a Bonsall functional. The core that is defined by $\pi_3(x)$ is called the statistical core of x (for the statistical core see [4]).

Let $\lambda = (\lambda_k) \in \omega$ and $\lambda x = (\lambda_k \xi_k)$. Denote for $X \subset \omega$

$$\lambda X := \{\lambda x \mid x \in X\}.$$

Let A and $A\lambda$ be the matrix methods that are determined by matrices $A = (\alpha_{nk})$ and $A\lambda = (\alpha_{nk}\lambda_k)$, respectively (here $\alpha_{nk} \in \mathbb{K}$). Let ω_A denote the application domain of A. If the inverse operator of A exists, then it is denoted by A^{-1}. Denote

$$\|A\| := \sup_n \sum_{k=0}^{\infty} |\alpha_{nk}| \leq \infty.$$

Let $\sum = (\sigma_{nk})$ be the summation matrix, i.e.,

$$\sigma_{nk} = \begin{cases} 1, & \text{if } k \leq n, \\ 0, & \text{if } k > n. \end{cases}$$

For $\sum^{-1} = (\hat{\sigma}_{nk})$ we have

$$\hat{\sigma}_{nk} = \begin{cases} 1, \text{ if } k = n, \\ -1, \text{ if } k = n-1, \\ 0, \text{ if } k \neq n, k \neq n-1. \end{cases}$$

It means that

$$\Sigma x = (\sum_{k=1}^{n} \xi_k) \quad \text{and} \quad \Sigma^{-1} x = (\xi_n - \xi_{n-1}).$$

It is obvious that $\omega = \omega_\Sigma = \omega_{\Sigma^{-1}}$.

Let $\lambda \in \omega$, $\lambda_k > 0$, $k \in \mathbb{N}$ and $\lambda^{-1} = (\lambda_k^{-1})$. A classical Tauberian theorem is as follows.

If x is summable by A and if $\lambda^{-1}x \in c_0$ then x is summable also by $\sum$. (The condition $\lambda^{-1}x \in c_0$ is called a Tauberian condition.)

That is why we call a more general theorem which states that

$$K(\Sigma x) \subset K(Ax) \;\; \textit{or} \;\; K(\Sigma x) = K(Ax) \;\; \textit{for every} \;\; x \in X \subset \omega$$

a Tauberian core theorem. The condition $x \in X \subset \omega$ in a Tauberian (core) theorem is called a Tauberian condition.

2. A Tauberian theorem induced by the sequence λ

We will prove a Tauberian core theorem where the Tauberian condition is determined by a sequence λ.

Theorem 3. *Let the Bonsall functional π be such that $c_0 \subset c_{\pi 0}$ and let $\lambda = (\lambda_k) \in \omega$. Suppose for $A = (\alpha_{nk})$ that*

$$\lim_n \alpha_{nk} = 1 \quad \forall k \in \mathbb{N}. \tag{4}$$

If

$$\|(A - \Sigma)\lambda\| < \infty, \tag{5}$$

then

$$\pi(Ax) = \pi(\Sigma x) \quad \forall x \in \lambda c_0$$

and

$$K(Ax) = K(\Sigma x) \quad \forall x \in \lambda c_0.$$

Proof. Let $G = (g_{nk}) := (A - \Sigma)\lambda$, i.e.,

$$g_{nk} = \begin{cases} (\alpha_{nk} - 1)\lambda_k, & \text{if} \quad k \leq n, \\ \alpha_{nk}\lambda_k, & \text{if} \quad k > n. \end{cases}$$

As for every $k \in \mathbb{N}$

$$\lim_n g_{nk} = \lim_n (\alpha_{nk} - 1)\lambda_k = 0$$

and

$$\sup_n \sum_k |g_{nk}| < \infty,$$

the matrix method G is regular on c_0 (see [2], p. 44). Therefore

$$(A - \Sigma)(\lambda y) \in c_0 \subset c_{\pi 0} \ \ \forall y \in c_0.$$

By (5) A is defined on λc_0 and consequently

$$(A - \Sigma)x \in c_{\pi 0} \ \ \forall x \in \lambda c_0.$$

Due to Corollary 2 the statement of theorem follows. □

Let now $A = (R, P_n) = (\alpha_{nk})$ be a Riesz matrix determined by a sequence $(p_k) \in \omega$, i.e.,

$$\alpha_{nk} = \begin{cases} 1 - \dfrac{P_{k-1}}{P_n}, & \text{if} \quad k \leq n, \\ 0, & \text{if} \quad k > 0. \end{cases} \tag{6}$$

Here $P_{-1} = 0$ and $P_n = \sum\limits_{k=0}^{n} p_k \neq 0, n \in \mathbb{N}$.

Corollary 4. *If the Bonsall functional π is such that $c_0 \subset c_\pi$ and if the Riesz matrix $A = (R, P_n)$ and the sequence $\lambda = (\lambda_k)$ satisfy the conditions*

$$\lim_n |P_n| = \infty, \tag{7}$$

$$\sup_n \frac{1}{|P_n|} \sum_{k=1}^{n} |p_k| < \infty, \tag{8}$$

$$|\lambda_k| \leq M \left| \frac{p_k}{P_{k-1}} \right|, \quad k = 1, 2, \ldots, \tag{9}$$

then

$$\pi(Ax) = \pi(\Sigma x) \quad \textit{and} \quad K(Ax) = K(\Sigma x)$$

for every $x \in \lambda c_o$.

Proof. The statement follows from Theorem 3 because in this case $(g_{nk}) = (A - \sum)\lambda$ is defined by

$$g_{nk} = \begin{cases} -\dfrac{P_{k-1}}{P_n}\lambda_k, & \text{if} \quad k \leq n, \\ 0, & \text{if} \quad k > 0, \end{cases}$$

and therefore

$$\|(A-\Sigma)\lambda\| = \sup_n \sum_{k=0}^{n} \left| \frac{P_{k-1}}{P_n} \lambda_k \right| .$$

The condidtion (4) is fulfilled due to (7), and (5) is fulfilled due to (8) and (9). □

Note, for comparison, that a well-known result states as follows.

If $p_n > 0$, $P_n \to \infty$, and if x is summable by $A = (R, P_n)$ to ξ with $\left| \frac{P_n}{p_n} \xi_n \right| \leq M$, $n \in \mathbb{N}$, then x is summable to ξ also by Σ (see [8], p. 103).

3. A Tauberian theorem induced by the matrix method B

We will prove a Tauberian core theorem where the Tauberian condition is determined by a sequence λ and a matrix B.

Theorem 5. *Let π be an arbitrary Bonsall functional and let A be a matrix method that has the property*

$$A\Sigma^{-1} : c_{\pi 0} \to c_{\pi 0}. \tag{10}$$

Let $\lambda \in \omega$ and let B be a normal matrix method. If $C = B^{-1} - \Sigma^{-1}$ satisfies the condition

$$C : c_{\pi 0} \to \lambda c_{\pi 0} \tag{11}$$

and

$$\pi(\Sigma x) \leq \pi(Ax) \ \ \forall x \in \lambda c_{\pi 0} \cap \omega_A, \tag{12}$$

then

$$K(\Sigma x) \subset K(Ax) \ \ \forall x \in c_{\pi 0 B} \cap \omega_A \tag{13}$$

(here $c_{\pi 0 B} = \{x \in \omega | Bx \in c_{\pi 0}\}$).

Proof. Suppose that $x \in c_{\pi 0 B} \cap \omega_A$, i.e., there exists an element $y \in c_{\pi 0}$ such that $Bx = y$. Consequently,

$$Ax = AB^{-1}y + A\Sigma^{-1}y - A\Sigma^{-1}y.$$

As (10) holds, we get using Proposition 1 that

$$\pi(Ax) = \pi(AB^{-1}y - A\Sigma^{-1}y) = \pi(ACy).$$

It means that

$$\pi(Ax) = \pi(ACy) \ \ \forall x \in c_{\pi 0 B} \cap \omega_A. \tag{14}$$

As $Bx = y$, we get

$$\Sigma Cy = \Sigma(B^{-1} - \Sigma^{-1})y = \Sigma x - y.$$

By Proposition 1, it follows that

$$\pi(\Sigma Cy) = \pi(\Sigma x - y) = \pi(\Sigma x) \;\; \forall x \in c_{\pi 0B}. \tag{15}$$

Due to (11), $Cy \in \lambda c_{\pi 0}$ and thus, as (12) holds, we obtain

$$\pi(\Sigma Cy) \leq \pi(ACy) \;\; \forall Cy \in \lambda c_{\pi 0} \cap \omega_A.$$

Therefore, by (14) and (15) we have

$$\pi(\Sigma x) \leq \pi(Ax) \;\; \forall x \in c_{\pi 0B} \cap \omega_A. \tag{16}$$

If $x \in c_{\pi 0B} \cap \omega_A$, then $\alpha x \in c_{\pi 0B} \cap \omega_A$ (for every $\alpha \in \mathbb{K}$), and consequently (16) implies

$$\pi(\alpha \Sigma x) \leq \pi(\alpha Ax) \;\; \forall \alpha \in \mathbb{K}.$$

Due to (1), the inclusion (13) follows. □

Remark 6. Notice that if $\hat{c} \subset c_{\pi 0}$ is an arbitrary linear subspace of $c_{\pi 0}$, then Theorem 5 remains true if one replaces the condition (12) with

$$\pi(\Sigma x) \leq \pi(Ax) \;\; \forall x \in \lambda \hat{c} \cap \omega_A$$

and simultaneously (13) with

$$K(\Sigma x) \subset K(Ax) \;\; \forall x \in \hat{c}_B \cap \omega_A,$$

where $\hat{c}_B := \{x \in \omega | Bx \in \hat{c}\}$. Note also that Theorem 5 remains true if one replaces the relations " $\leq$ " and " $\subset$ " in formulae (12) and (13) simultaneously with the relations " $\geq$ " and " $\supset$ ", respectively.

In the sequel suppose that $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ are sequences of strictly positive real numbers and let r be a fixed real number.

Let $B = (b_{nk})$ be defined as follows:

$$b_{nk} = \begin{cases} \left(\dfrac{\mu_n}{\mu_k}\right)^r, & \text{if} \quad k \leq n, \\ 0, & \text{if} \quad k > n. \end{cases} \tag{17}$$

Therefore

$$c_{\pi 0B} = \left\{ x \in \omega \middle| \left(\mu_n^r \sum_{k=1}^{n} \left(\frac{1}{\mu_k}\right)^r \xi_k \right) \in c_{\pi 0} \right\}.$$

If $\pi = \pi_1$, then $c_{\pi 0 B} = c_{0B}$, where

$$c_{0B} = \left\{ x \in \omega \,\middle|\, \lim_n \mu_n^r \sum_{k=1}^{n} \left(\frac{1}{\mu_k} \right)^r \xi_k = 0 \right\}.$$

The inverse operator $B^{-1} = (\hat{b}_{nk})$ is given by

$$\hat{b}_{nk} = \begin{cases} 1, & \text{if } k = n, \\ -\dfrac{\mu_n^r}{\mu_{n-1}^r}, & \text{if } k = n-1, \\ 0, & \text{if } k \neq n \text{ and } k \neq n-1. \end{cases}$$

A sequence space X is called solid if

$$\{(\eta_k) \in \omega \mid \exists (\xi_k) \in X \ \forall k \in \mathbb{N} : |\eta_k| \leq |\xi_k|\} \subset X.$$

It is well known that X is solid if and only if $mX \subset X$ (cf. [2], p. 342), where

$$mX := \{(\alpha_k \xi_k) \mid (\alpha_k) \in m, (\xi_k) \in X\}.$$

For example, c_0 is solid but f_0 is not. A sequence space X is called shift invariant if

$$X = \{x = (\xi_k) | \ Sx \in X, Sx = (\xi_{k+1})\}.$$

The spaces c_0 and f_0 are shift invariant.

Lemma 7. *Let the Bonsall functional π be such that $c_0 \subset c_{\pi 0} \subset m$, where $c_{\pi 0}$ is shift invariant. Let $C = B^{-1} - \Sigma^{-1}$, where $B = (b_{nk})$ is defined by* (17).

1) *If C satisfies* (11), *then*

$$\sup_n \left| 1 - \left(\frac{\mu_n}{\mu_{n-1}} \right)^r \right| \frac{1}{\lambda_n} < \infty \tag{18}$$

holds.

2) *If*

$$\lim_n \left| 1 - \left(\frac{\mu_n}{\mu_{n-1}} \right)^r \right| \frac{1}{\lambda_n} = 0 \tag{19}$$

or

$$\left[1 - \left(\frac{\mu_n}{\mu_{n-1}} \right)^r \right] \frac{1}{\lambda_n} = K \quad \forall n \in \mathbb{N} \tag{20}$$

(*where K is a constant*), *then C has the property* (11).

3) *If the space* $c_{\pi 0}$ *is solid, then* (18) *is necessary and sufficient for* C *to have* (11).

Proof. In this case the matrix method $C = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} 1 - \left(\dfrac{\mu_n}{\mu_{n-1}}\right)^r, & \text{if } k = n-1, \\ 0, & \text{if } k \neq n-1. \end{cases} \tag{21}$$

If $y = (\eta_n) = Cx$, then

$$\eta_n = \left[1 - \left(\frac{\mu_n}{\mu_{n-1}}\right)^r\right] \xi_{n-1} = \lambda_n \left[1 - \left(\frac{\mu_n}{\mu_{n-1}}\right)^r\right] \frac{1}{\lambda_n} \xi_{n-1}. \tag{22}$$

Now, if (19) holds, then it is evident that $x \in c_{\pi 0} \subset m$ implies $Cx \in \lambda c_0 \subset \lambda c_{\pi 0}$.

As the space $c_{\pi 0}$ is shift invariant, the property (11) follows trivially from (20) due to (22).

If the space $c_{\pi 0}$ is solid, then it is obvious that (18) is sufficient for (11).

Let $\nu = (\nu_k)$, where $\nu_k = \dfrac{1}{\lambda_{k+1}}$. It follows from (21) that the operator C has the property (11) if and only if the operator $C\nu$ enjoys the property $C\nu : c_{\pi 0} \to c_{\pi 0}$. Therefore, as $c_0 \subset c_{\pi 0} \subset m$, it is necessary for (11) that $\|C\nu\| < \infty$, i.e., (18) is valid (see, e.g., [2], p. 42).

This completes the proof. □

Examples 8. The condition (18) is satisfied for the following $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$. Here $r \geq 1$ is an integer.

a) $\lambda_n = \mu_n = \dfrac{1}{n+1}$;

b) $\lambda_n = \mu_n = \dfrac{1}{\ln(n+2)}$;

c) $\lambda_n = \dfrac{1}{\ln(n+2)}$ and $\mu_n = \dfrac{1}{n+1}$;

d) $\lambda_n = \dfrac{p_n}{P_n}$ and $\mu_n = \dfrac{1}{P_n}$

where $\left(\dfrac{p_n}{P_n}\right) \in c_0$, $p_n > 0$ and $P_n = \sum_{k=0}^{n} p_k \quad \forall n \in \mathbb{N}$.

We will show for the case of d), that λ and μ enjoy the property (19), if $r > 1$, and (20), if $r = 1$.

Indeed, if $r > 1$, then

$$\left|1-\left(\frac{\mu_n}{\mu_{n-1}}\right)^r\right|\frac{1}{\lambda_n}=\left|1-\left(\frac{P_{n-1}}{P_n}\right)^r\right|\frac{P_n}{p_n}=$$

$$=\left|\frac{(P_{n-1}+p_n)^r-P_{n-1}^r}{P_n^r}\right|\frac{P_n}{p_n}=$$

$$=\frac{1}{P_n^{r-1}p_n}\sum_{k=1}^{r}\binom{r}{k}P_{n-1}^{r-k}p_n^k=$$

$$=\sum_{k=1}^{r}\binom{r}{k}\left(\frac{P_{n-1}}{P_n}\right)^{r-k}\frac{p_n}{P_n}.$$

As $\left(\frac{p_n}{P_n}\right)\in c_0$ we get that $\lim\limits_n \frac{P_{n-1}}{P_n}=1$ and therefore (19) is true.

If $r = 1$, then (20) holds with $K = 1$.

Let $A = (R, P_n) = (\alpha_{nk})$ be a Riesz matrix determined by the formula (6). It is easy to check that in this case the matrix $A\Sigma^{-1}$ is the matrix of Riesz means $(R, p_n) = (a_{nk})$, where

$$a_{nk}=\begin{cases}\frac{p_k}{P_n}, & \text{if } k\le n,\\ 0, & \text{if } k>n.\end{cases}$$

If the sequence (p_n) that generates (R, p_n) is monotone and if $\left(\frac{p_n}{P_n}\right)\in c_0$, $P_n\to\infty$, then (R, p_n) transfers regularily every almost convergent sequence into a convergent sequence, and therefore (R, p_n) is of type $f_0\to c_0$ (for Riesz means see [2], p. 112–126).

We will give some well-known sequences that generate Riesz means (R, p_n) of type $f_0\to c_0$.

a) $p_n=\frac{1}{n+1}$ (this Riesz method is called the logarithmic means),

b) $p_n=\binom{n+\alpha}{n}$, $\alpha>-1$,

c) $p_n=e^{n^\beta}$, $0<\beta<1$.

Proposition 9. *Let π be the Bonsall functional that determines Lorentz core, i.e., $\pi=\pi_2$. Let $A=(R,P_n)$ be a Riesz method defined by (6). Suppose that the sequence (p_n) is monotone, $P_n\to\infty$, $p_n>0$, $n\in\mathbb{N}$, and $\left(\frac{p_n}{P_n}\right)\in c_0$. Let r be an integer, $r\ge 1$. Then*

$$K_2(\Sigma x)=K_2(Ax)$$

for every x from the set

$$\left\{ x = (\xi_k) \,\middle|\, \lim_n \frac{1}{P_n^r} \sum_{k=1}^{n} P_k^r \xi_k = 0 \right\}. \tag{23}$$

Proof. Due to $A\Sigma^{-1} : f_0 \to f_0$ the condition (10) of Theorem 5 is fulfilled. Suppose that $\lambda_n = \frac{p_n}{P_n}$ and $\mu_n = \frac{1}{P_n}$. Then by Lemma 7 and Examples 8, d), the matrix C defined by (21) has the property (11). Obviously the set (23) is the space c_{0B} for $B = (b_{nk})$ defined by (17). By Corollary 4

$$\pi_2(Ax) = \pi_2(\Sigma x) \quad \forall x \in \lambda c_0.$$

The assertion follows now from Theorem 5 and Remark 6. □

Let L: $[0, \infty) \to (0, \infty)$ be a slowly varying function, i.e.,

$$\lim_{\alpha\to\infty} \frac{L(\alpha t)}{L(\alpha)} = 1 \ \forall t > 0.$$

Let $\delta > -1$ be a fixed number and let (p_n) be a sequence of numbers, where $p_n > 0$, $n \in \mathbb{N}$. Moreover, suppose that

$$\lim_n \frac{(n+1)^\delta L(n)}{p_n} = 1. \tag{24}$$

In this case

$$\lim_n \frac{(n+1)^{\delta+1} L(n)}{(\delta+1)P_n} = 1, \tag{25}$$

and

$$\lim_n \frac{P_{n-1}}{P_n} = 1 \tag{26}$$

(for (25) see [5], Lemma A, and for (26) see [6], Lemma 3).

Proposition 10. *Let $A = (R, P_n)$ be a Riesz method defined by (6). Suppose that $p_n > 0$, $n \in \mathbb{N}$, and let (24) be satisfied. Let r be an integer, $r \geq 1$. Then*

$$K_1(\Sigma x) = K_1(Ax)$$

for every x from the set

$$\left\{ x = (\xi_k) \,\middle|\, \left(\frac{1}{(n+1)^r} \sum_{k=0}^{n} (k+1)^r \xi_k \right) \in c_0 \right\}. \tag{27}$$

Proof. Denote $\lambda_n = \mu_n = \dfrac{1}{n+1}$ and note that $c_{\pi 0} = c_0$ and the set (27) is the space c_{0B} for $B = (b_{nk})$ defined by (17). Due to Examples 8, a), our $\lambda = (\lambda_k)$ and $\mu = (\mu_k)$ enjoy the property (18) and therefore, by Lemma 7, the operator C is of type $c_0 \to \lambda c_0$.

Note that from (24) and (25) we get that

$$\lim_n \frac{(\delta+1)P_n}{(n+1)p_n} = 1, \tag{28}$$

and evidently there exists $K > 0$ such that

$$p_n \geq K\frac{1}{n+1} \quad \forall n \in \mathbb{N}.$$

Consequently, $P_n \to \infty$ and thus $A\Sigma^{-1} = (R, p_n)$ is of type $c_0 \to c_0$, i.e., satisfies the condition (10) of Theorem 5 (for (R, p_n) see [2], p. 113). Moreover, it follows from (26) and (28) that there exists $M > 0$ such that

$$\frac{1}{n+1} \leq M\frac{p_n}{P_{n-1}} \quad \forall n \in \mathbb{N}.$$

The assertion follows now immediately from Theorem 5 and Corollary 4. □

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