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On commutativity of rings with conditions involving nilpotent elements and the Jacobson radical

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Abstract. Let R be an associative ring with unity 1, N the set of nilpotents, J the Jacobson radical of R and $n > 1$ be a fixed integer. We prove that if R is $n(n+1)$-torsion free and satisfies the identity $(xy)^n = y^n x^n$ for all $x, y \in R \setminus (N \cup J)$, then R is commutative.

1. Introduction

Throughout, R represents an associative ring with centre Z. We denote the commutator ideal by C and the Jacobson radical by J. The totality of all nilpotent elements will be denoted by N and, for any pair of elements $x, y \in R$, the commutator $xy - yx$ by $[x, y]$.

A well-known result due to Herstein [4] asserts that a ring satisfying the identity $(xy)^n = x^n y^n$ for all $x, y \in R$, where $n > 1$ is a fixed integer, must have a nil commutator ideal. Later, Awtar [2] proved that any ring with unity in which there exists a fixed positive integer $n > 1$ such that no prime $p \leq n$ is a zero divisor and $(xy)^n = x^n y^n$ for all $x, y \in R$ turns out to be commutative. Further, Abu-Khuzam [1] proved that if a ring with unity is $n(n-1)$-torsion free and satisfies the identity $(xy)^n = x^n y^n$ for all $x, y \in R$, then R is necessarily commutative. Motivated by these observations we prove the following:

Theorem. *Let $n > 1$ be a fixed integer and R be an $n(n+1)$-torsion free ring with unity 1. If R satisfies the identity*

$$(xy)^n = y^n x^n \text{ for all } x, y \in R \setminus (N \cup J), \qquad (*)$$

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then R is commutative.

We begin our discussion with the following known results which are pertinent for the development of the proof of the above theorem. Lemma 1 and Lemma 2 are proved in [3] and [6] respectively, whereas the proof of Lemma 3 can be found in [7].

Lemma 1. *Let R be a ring satisfying an identity $f(X) = 0$, where $f(X)$ is a polynomial in non-commuting indeterminates, its coefficients being integers with highest common factor one. If there exists no prime p for which the ring of 2×2 matrices over $\mathrm{GF}(p)$ satisfies $f(X) = 0$, then R has a nil commutator ideal and the nilpotent elements form an ideal.*

Lemma 2. *If $x, y \in R$ and $[x, [x, y]] = 0$, then $[x^m, y] = mx^{m-1}[x, y]$ for all positive integers m.*

Lemma 3. *Let R be a ring and $f : R \longrightarrow R$ be a function such that $f(x+1) = f(x)$ holds for all $x \in R$. If for some positive integer m, $x^m f(x) = 0$ for all $x \in R$, then necessarily $f(x) = 0$.*

Now, we observe that a ring R satisfying the identity $(xy)^n = y^n x^n$ for a fixed integer $n > 1$ also satisfies $(yx)^{n+1} = y^{n+1}x^{n+1}$. Indeed,

$$\begin{aligned} y^{n+1}x^{n+1} &= y(y^n x^n)x \\ &= y(xy)^n x \\ &= y(xy\ xy \ldots xy)x \\ &= (yx)^{n+1} \end{aligned}$$

Thus, in view of [1, Theorem], cited above, we obtain:

Lemma 4. *Let R be a ring with unity 1. If R is $n(n+1)$-torsion free and satisfies the identity $(xy)^n = y^n x^n$ for all $x, y \in R$, then R is commutative.*

2. Proof of the Theorem

Let U denote the set of units in R. For any $u, v \in U$, the hypothesis $(*)$ gives

$$(uvu^{-1})^n = u^{-n}v^n u^n.$$

Thus,

$$[u^{n+1}, v^n] = 0 \text{ for all } u, v \in U. \tag{1}$$

This readily yields that $[u^{n(n+1)}, v^n] = 0$ and $[u^{n(n+1)}, v^{n+1}] = 0$, hence

$$[u^{n(n+1)}, v] = 0 \text{ for all } u, v \in U. \tag{2}$$

Now, suppose $x \in N \cup J$, so that $1 - x \in U$. Then (2) gives

$$[u^{n(n+1)}, x] = 0 \text{ for all } u \in U,\ x \in N \cup J. \tag{3}$$

If $ux \in N \cup J$, then (3) gives

$$[u^{n(n+1)}, x] = 0 \text{ for all } u \in U,\ ux \in N \cup J. \tag{4}$$

Suppose next that $x \notin N \cup J$ and $ux \notin N \cup J$. Then, using the hypothesis, we get

$$[u^{n(n+1)}, x^n] = 0 \text{ for all } u \in U,\ x \notin N \cup J,\ ux \notin N \cup J. \tag{5}$$

Combining (3), (4) and (5), we obtain

$$[u^{n(n+1)}, x^n] = 0 \text{ for all } u \in U,\ x \in R. \tag{6}$$

Now, let u be an arbitrary unit in U and let

$$m = n(n+1). \tag{7}$$

Then, by (6) we get

$$[u^m, x^m] = 0 \text{ for all } u \in U,\ x \in R. \tag{8}$$

Next, we shall show that

$$(x^{m-1}u^m x)^{mn} = u^{m^2n}x^{m^2n} \text{ for all } u \in U,\ x \in R. \tag{9}$$

To this end, we distinguish two cases :

Case 1. $x^{m-1} \in N \cup J$ or $u^m x \in N \cup J$. This implies

$$1 - x^{m-1} \in U \text{ or } 1 - u^m x \in U.$$

Comparison of (2) and (7) gives

$$[1 - x^{m-1}, u^m] = 0 \text{ or } [1 - u^m x, u^m] = 0.$$

Hence, in either case we get $[x^{m-1}, u^m] = 0$. Combining this fact with (8), we obtain

$$(x^{m-1}u^m x)^{mn} = u^{m^2n}x^{m^2n}.$$

Case 2. $x^{m-1} \notin N \cup J$ and $u^m x \notin N \cup J$.
We distinguish two sub-cases:

Case A. $x \in N \cup J$. By an application of (3) and (7), we get $[u^m, x] = 0$ and hence $(x^{m-1}u^m x)^n = (x^m u^m)^n = u^{mn}x^{mn}$. Thus $(x^{m-1}u^m x)^{mn} = u^{m^2n}x^{m^2n}$, which proves (9).

Case B. $x \notin N \cup J$. Applying the hypothesis $(*)$ twice together with (6) and (7) we obtain $(x^{m-1}u^m x)^n = u^{mn}x^{mn}$ which yields

$$(x^{m-1}u^m x)^{mn} = u^{m^2n}x^{m^2n}.$$

Thus (9) is proved in all cases. Also for any $x \in R,\ u \in U$,

$$(x^{m-1}u^m x)^{mn} = x^{m-1}u^m x^m u^m x^m \dots u^m x = x^{m-1}(u^m x^m)^{nm-1}u^m x.$$

Using (8), we get

$$(x^{m-1}u^m x)^{mn} = x^{nm^2-1}u^{nm^2}x. \tag{10}$$

Combining (9) and (10), we get

$$x^{nm^2-1}u^{nm^2}x = u^{nm^2}x^{nm^2} = x^{nm^2}u^{nm^2}$$

by (8). Hence,

$$x^{nm^2-1}[u^{nm^2}, x] = 0 \text{ for all } x \in R,\ u \in U. \tag{11}$$

Replacing x by $1+x$ in (11) and using Lemma 3, we obtain

$$[u^{nm^2}, x] = 0 \text{ for all } x \in R,\ u \in U. \tag{12}$$

Let $a \in N$. Then there exists a minimal positive integer p_0 such that

$$[a^p, x] = 0 \text{ for all integers } p \geq p_0. \tag{13}$$

We claim that $p_0 = 1$. Suppose not; then by (12)

$$[(1 + a^{p_0-1})^{nm^2}, x] = 0 \text{ for all } x \in R.$$

Thus, in view of (13) we obtain $nm^2[a^{p_0-1}, x] = 0$ for all $x \in R$. Hence R is nm^2-torsion free, so we get $[a^{p_0-1}, x] = 0$. This contradicts the minimality of p_0. Hence (13) gives

$$[a, x] = 0 \text{ for all } x \in R.$$

Therefore,

$$N \subseteq Z. \tag{14}$$

Next, we show that

$$x[x^{n+1}, y^n] \in Z \text{ for all } x \notin N \cup J,\ y \notin N \cup J. \tag{15}$$

Let $x \notin N \cup J$ and $y \notin N \cup J$, then by the hypothesis $(xy)^n = y^n x^n$. Now $(xy)^n x = x(yx)^n$. This implies that $y^n x^{n+1} = x^{n+1}y^n$ and hence

$$x[x^{n+1}, y^n]x = 0,$$

which gives

$$(x[x^{n+1}, y^n])^2 = 0.$$

Hence $x[x^{n+1}, y^n] \in N \subseteq Z$, by (14), which proves (15). Now, suppose that $x \in N \cup J$. Then $1 - x \in U$, and (12) implies

$$[(1-x)^{nm^2}, y] = 0 \text{ for all } y \in R,\ x \in N \cup J. \tag{16}$$

Similarly, for $y \in N \cup J$ we have $1 - y \in U$, and hence by (12) we get

$$[x, (1-y)^{nm^2}] \text{ for all } x \in R,\ y \in N \cup J. \tag{17}$$

Combining (15), (16) and (17), we readily verify that the following holds in R:

$$[[x[x^{n+1}, y^n], (1-x)^{nm^2}], (1-y)^{nm^2}] = 0 \text{ for all } x, y \in R. \tag{18}$$

Now (18) is a polynomial identity which is satisfied by all the elements of R. Further, (18) is not satisfied by any 2×2 matrix ring over GF(p) for any prime p, as a consideration of $x = e_{22}$ and $y = e_{21} + e_{22}$ shows. An application of Lemma 1 gives that C is nil, and hence by (14)

$$C \subseteq N \subseteq Z. \tag{19}$$

Recall that $[u^{nm^2}, x] = 0$ for all $x \in R$, $u \in U$. Using (19) together with Lemma 2 gives

$$nm^2 u^{nm^2-1}[u, x] = 0 \text{ for all } x \in R,\ u \in U.$$

Since R is nm^2-torsion free and u is a unit, $[u, x] = 0$ for all $u \in U$ and $x \in R$, which in turn implies that

$$U \subseteq Z. \tag{20}$$

Suppose $x \in N \cup J$. Then $1 - x \in U \subseteq Z$. Thus,

$$x \in Z,\ (xy)^n = y^n x^n \text{ for all } y \in R,\ x \in N \cup J. \tag{21}$$

Similarly, if $y \in N \cup J$, then $1 - y \in U \subseteq Z$. Thus,

$$y \in Z,\ (xy)^n = y^n x^n \text{ for all } x \in R,\ y \in N \cup J. \tag{22}$$

Combining (21), (22) and $(*)$, we obtain

$$(xy)^n = y^n x^n \text{ for all } x, y \in R. \tag{23}$$

Hence, R is commutative by Lemma 4.

3. Corollaries

As a consequence of the above theorem, we can derive the following:

Corollary 1. *Let $n > 1$ be a fixed integer and R be an $n(n+1)$-torsion free ring with unity 1. If R satisfies the identity*

$$(xy)^n = y^n x^n \text{ for all } x, y \in R \setminus N,$$

then R is commutative.

Corollary 2. *Let $n > 1$ be a fixed integer and R be an $n(n+1)$-torsion free ring with unity 1. If R satisfies the identity*

$$(xy)^n = y^n x^n \text{ for all } x, y \in R \setminus J,$$

then R is commutative.

We provide an example which shows that if we replace the condition of $n(n+1)$-torsion freeness in the hypothesis of our theorem by n-torsion freeness or $(n+1)$-torsion freeness, then R may be badly non-commutative.

Example. *Let*

$$R = \left\{ aI + B \mid B = \begin{pmatrix} 0 & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{pmatrix}, \ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c, d \in \mathrm{GF}(3) \right\}.$$

It can easily be verified that $(xy)^2 = y^2x^2$ *and* $(xy)^3 = y^3x^3$. *So, with* $n = 2$, R *is* n*-torsion free and* $(xy)^n = y^nx^n$, *moreover* R *is not commutative. Also with* $n = 3$, R *is* $n+1$*-torsion free and* $(xy)^n = y^nx^n$, *but* R *is not commutative.*

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